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Solution of an optimal reservoir capacity problem under probabilistic constraints

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Solution of an optimal reservoir capacity problem under probabilistic constraints

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Abstract. We formulate and solve probabilistic constrained stochastic programming problems, where we prescribe lower and upper bounds for k-out-of-n and consecutive-k-out-of-n reliabilities in the form of probabilistic constraints. The practical problem we are dealing with is mentioned in a paper by Prékopa, Szántai, Zsuffa (2010), where four optimization problems are formulated in connection with water resource problem. However, solutions are offered for three of them and it is the fourth one which is the starting point of our paper. The problem is to determine the optimal capacity of a water release, or pump station, to satisfy the demand for irrigation, i.e., a reliability constraint where the reliability is one of the above-mentioned type. For the non-consecutive type reliability problem, normal and gamma distributions are used for inflow and demand values, respectively. By using the property of standard gamma distribution, reliability constraint is written up as an equation which can then be solved by simulation. For the k-consecutive case, different probability bounds are used in order to solve the reliability equation. To create lower and upper bounds for the reliability constraint, the discrete binomial moment problem is used, which are indeed LP's are constructed. S_1, S_2, S_3 sharp lower bounds, Hunter's upper bound and Cherry tree upper bound are calculated to obtain desired probability level for the reliability constraint. Bi-section algorithm is later applied to find the optimal water reservoir capacity level.

Keywords: Probabilistic modeling, Optimization, Bounding, Bi-section algorithm

1 Introduction

In probabilistic constrained stochastic programming, problems we look at the joint probability of a finite number of stochastic constraints and impose a lower bound on it, chosen by ourselves. This ensures that the system we are looking at has a prescribed level of probability. The joint occurrence of constraints, or events, depending on a decision vector is however, only one type of Boolean function of events among those that appear in reliability theory. In this paper, we formulate and solve probabilistic constrained stochastic programming problems, where we prescribe lower bounds for k-out-of-n and consecutive-k-out-of-n reliabilities in the form of probabilistic constraints.

The problem comes from water resources problem applied in agriculture. It is widely accepted view and practical experience that a plant can survive a given number of dry days, which depends on the plant. If n is the total number of days until harvest and the maximum number of dry days the plant can survive is $k-1$, then we want to ensure the possibility of irrigation in any k consecutive days which means we have a k-out-of-n reliability. The problem is not as to calculate the mentioned reliability but to optimize with respect to a decision variable subject to the constraint that the k-out-of-n reliability holds true on a prescribed level, near 1 in practice. We also look at probabilistic constrained problems, where the reliability is of a weaker type, it is of k-out-of-n type.

The practical problem we are dealing with is mentioned in a paper by Prékopa, Szántai, Zsuffa (2010), where four optimization problems are formulated in connection with water resource problem. However, solutions are offered for three of them and it is the fourth one which is the starting point of our paper. The problem is to determine the optimal capacity of a water release, or pump station, to satisfy the demand for irrigation, i.e., a reliability constraint where the reliability is one of the above-mentioned type.

Even though the original problem comes from an application in water resources, the type of problem we are dealing with has many other applications. For example, we can determine the optimal safety cash reserve of a bank or safety stocks, in general, in inventory control systems. It is also novel, from the point of model construction, and enrich the collection of these stochastic programming models that have immediate and wide applications.

Formulas are available to compute probability of various Boolean function of events. For the probability of at least k-out-of-n ($P_{(k)}$) and exactly k-out-of-n ($P_{[k]}$) we have:

$$P_{(k)} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i-1}{k-1} S_k$$

$$P_{[k]} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} S_k$$

where $S_1, \ldots, S_n$ are the bionomial moments of the random variable equal to the number of events that occur. However, in practice we cannot compute all $S_1, \ldots, S_n$if n is large then we apply bionomial procedures to approximate the probabilities.

In order to create lower and upper bounds for Boolean functions of events arranged in a finite sequence, a simple and frequently efficient method is the one provided by the discrete binomial moment problems. These are LP's, where the right-hand side numbers are some of the binomial moments $S_1, S_2, \ldots$. Since S_k is the sum of joint probabilities of k-tuples of events, these LP's are called aggregated problems. Better bounds can be obtained if we use the individual probabilities in the sums of all S_k binomial moments that turn up in the aggregated problem. However, the LP's based on these called the disaggregated problems, have huge sizes, in general, and we may not be able to solve them (see Prékopa, Vizvári, Regös, 1998). Bounding probabilities of Boolean function of events has an extensive literatutre. The first upper and lower bounds are given by Bonferroni (1937) and Boole (1854), respectively. However, they are weak and rarely useful in practice. Sharp S_1, S_2, S_3 lower bounds were proposed by Dawson and Sankoff (1967) and S_1, S_2, S_3 lower and upper bounds by Kwerel (1975a, 1975b). Prékopa (1988, 1989, 1990, 1995) generalized these results and gave formulas as well as dual type algorithms to obtain the bounds. See also Boros and Prékopa (1989) for a collection of formulas. We will use this in our paper but we also use bounds where most sums of probabilities (as in $S_1, \ldots, S_n$) but individual proabilities are used. Hunter (1976) gives a solution for an upper bound which is going to be used for the solution of the k-out-of-n type of problem. In Bukszár, Prékopa (2001), a third order upper bound by using graphs called cherry trees are presented. These are graphs that are recursivley generated by connecting the new vertex into two already existing vertices. Cherry tree bounds also corresponds to a dual feasible bases however they are always as good as or better than Hunter's upper bound. In this paper, S_1, S_2, S_3 sharp lower bounds, Hunter and Cherry tree upper bounds are taken into consideration. Bi-section method is then applied to the model for obtaning the optimal capacity level while satisfying the reliability constraints.

2 Formulation of the problem

There will be two problems modeled in this section. In the first model, k dry periods out of n periods are permitted while obtaining the optimal water reservoir capacity. In the second model, same objective function is solved while observing at most consecutive k dry periods out of n periods are allowed. Following notation is used for both problems:

ξ_i	inflow in the i^{th} period, normally distributed random variable
γ_i	demand in the i^{th}period, gamma distributed random variable
δ_i	rain amount in the i^{th} period, normally distributed random variable
m	capacity of the water reservoir
k	number of permitted dry days
p	probability level of reliability

First Model. At most k dry periods permitted
Problem to be solved:

$$\begin{gathered} \min m \\ \text{subject to} \\ P\{\min\left((\xi_i, m) + \delta_i \geq x_i\gamma_i\right), \ \ i = 1, \dots, n, \ \ x_1 + \cdots + x_n \geq n - k\} \geq p \\ x_i \in \{0, 1\} \ \ i = 1, \dots, n \\ 0 \leq m \leq M \end{gathered} \tag{1}$$

Second Model. At most k consecutive dry periods permitted
Problem to be solved:

$$\begin{gathered} \min m \\ \text{subject to} \\ P\{\min\left((\xi_i, m) + \delta_i \geq x_i\gamma_i\right), \ \ i = 1, \dots, n, \ \ x_i + \cdots + x_{i+k-1} \geq 1, \ \ i = 1, \dots, n - k + 1\} \geq p \\ x_i \in \{0, 1\} \ \ i = 1, \dots, n \\ 0 \leq m \leq M \end{gathered} \tag{2}$$

3 Mathematical properties of the reservoir system design model

In this section we prove a convexity theorem for problem (1) and (2) where there are no discrete variables. The convexity statement is based on the theory of multivariate logconcave measures and functions. In order to make the paper self contained we recall some facts from logconcavity. First we present two definitions.

A function $f(x) \geq 0$, $x \in R^n$ is logconcave if for every $x, y \in R^n$ and $0 < \lambda < 1$ we have

$$f(\lambda x + (1 - \lambda)y) \geq (f(x))^{\lambda}(f(y))^{1-\lambda}) \tag{3}$$

A probability measure P is the Borel subsets of R^n is logconcave Prékopa(1971, 1973a) if for every convex subsets A, B of R^n and $0 < \lambda < 1$ we have

$$f(\lambda A + (1 - \lambda)B) \geq f(A))^{\lambda}(f(B))^{1-\lambda}$$

A simple consequence of the second definition is that the c.d.f., corresponding to a logconcave probability measure, is logconcave (as a point function). The basic theorem of logconcave measure is the following:

Theorem 3.1. *(Prékopa, 1971, 1973a). If the probability measure P is generated by a logconcave p.d.f., then P is a logconcave measure.*

Another theorem that we use in connection with problem (1) is the following:

Theorem 3.2. *(Prékopa, 1972). If $g_1(x,y),\dots,g_r(x,y)$ are concave functions in R^{n+q}, where $x \in R^n$, $y \in R^q$ and $\xi \in R^n$ is a random variable that has logconcave distribution, then the function*

$$P(g_i(\xi,y) \geq 0,\ \ i = 1,\dots,r)$$

is a logconcave function of $y \in R^q$.

A consequence of the above theorem is

Theorem 3.3. *If the joint p.d.f. of the random variables $\xi_i, \delta_i, \gamma_i$, $i = 1,\dots,n$ is logconcave, then for every fixed x,*

$$P(\min(\xi_i,m) + \delta_k \geq x_i\gamma_i,\ \ i = 1,\dots,n)$$

Proof. Theorem 3.3 ensures the logconcavity of the joint distribution of the random variables $\xi_i, \delta_i, \gamma_i,\ \ i = 1,\dots,n$. On the other hand, if we look fo a moment at $\xi_i, \delta_i, \gamma_i,\ \ i = 1,\dots,n$ as deterministic variables, then we can see that the functions

$$\min(\xi_i,m) + \delta_i - x_i\gamma_i,\ \ i = 1,\dots,n$$

are concave in all these variables and n. By Theorem 3.2 the assertion follows.

4 Solution of the problem

4.1 Solution of the Model (1)

First we present a method to find an upper bound for the optimal solution of model where there are no discrete variables. For the case of I.I.D. $\gamma_1,\dots,\gamma_n$, where each has gamma distribution with p.d.f.:

$$\frac{\lambda^{\vartheta} z^{\vartheta-1} e^{-\lambda z}}{\tau(\vartheta)}, z > 0$$

we can obtain an upper bound for M_{opt}. For simplicity we assumed that $(\gamma_1,\dots,\gamma_n)$ is independent of $(\xi_1,\dots,\xi_n,\delta_1,\dots,\delta_n)$.

First we mention that the random variables $\lambda\gamma_1,\dots,\lambda\gamma_n$ have standard gamma distribution, i.e. distribution with p.d.f. (4), where $\lambda = 1$. The second observation is that the following relations hold:

$$\begin{aligned} & P(\min(\xi_i,m) + \delta_i \geq x_i\gamma_i,\ \ i = 1,\dots,n) \\ \leq\ & P(\sum_{i=1}^{n} [\min(\xi_i,m) + \delta_i] \geq \sum_{i=1}^{n} x_i\gamma_i) \\ =\ & P(\sum_{i=1}^{n} \lambda[\min(\xi_i,m) + \delta_i] \geq \sum_{i=1}^{n} x_i\lambda\gamma_i) \end{aligned} \tag{4}$$

The distribution of $\sum_{i=1}^{n} x_i \lambda \gamma_i$ is the same as the distribution of $\lambda \gamma_1 \sum_{i=1}^{n} x_i$. In fact, the sum of independent standard gamma random variables is also a standard gamma random variable and the (ϑ)parameter of the sum is the sum of the parameter of the terms. Thus, we can replace $\sum_{i=1}^{n} x_i \lambda \gamma_i$ by $\lambda \gamma_1 \sum_{i=1}^{n} x_i$ for the last line in (4). On the other hand, it is prescribed that $\sum_{i=1}^{n} x_i \geq n-k$, hence we obtain the inequality

$$\begin{aligned} & P \quad (\min(\xi_i, m) + \delta_i \geq x_i \gamma_i, \; i = 1, \ldots, n) \\ \leq & P \quad \left(\sum_{i=1}^{n} \lambda \left[\min(\xi_i, m) + \delta_i \right] \geq (n-k) \lambda \gamma_1 \right). \end{aligned} \tag{5}$$

Inequality (5) implies that the optimum value of the problem

$$\begin{gathered} \min m \\ \text{subject to} \\ P \left(\sum_{i=1}^{n} \lambda \left[\min(\xi_i, m) + \delta_i \right] \geq (n-k) \lambda \gamma_1 \right) \geq p \\ 0 \leq m \leq M \end{gathered}$$

is an upper bound for the optimum value of problem (1). On the other hand, if there exists a feasible m in problem (6), then, due to the monocity of the constraining function in the first constraint, the optimal solution of problem (6) can simply be obtained by the solution of the equation:

$$P \left(\sum_{i=1}^{n} \lambda \left[\min(\xi_i, m) + \delta_i \right] \geq (n-k) \lambda \gamma_1 \right) = p. \tag{6}$$

4.2 Solution of the Model (2)

The problem (2) can be solved in multiple ways however we will use bounding techniques. For the sake of computational easiness, we will ignore δ_i. In order to apply bounding methodology the reliability constraint in model (2) will be re-written as follows:

$$\begin{gathered} P\{\min\left((\xi_i, m) \geq \gamma_i x_i\right), \; i = 1, \ldots, n, \; x_i + \cdots + x_{i+k-1} \geq 1, \; i = 1, \ldots, n-k+1\} \geq p \\ x_i \in \{0, 1\} \; i = 1, \ldots, n \end{gathered}$$

The inequality (7) can also be expressed as;

$$\begin{gathered} P\{\min(\xi_i \geq \gamma_i x_i, m \geq \gamma_i x_i), \; i = 1, \ldots, n, \; x_i + \cdots + x_{i+k-1} \geq 1, \; i = 1, \ldots, n-k+1\} \geq p \\ x_i \in \{0, 1\} \; i = 1, \ldots, n \end{gathered}$$

The inequality (7) claims that, minimum of the inflow or the capacity of the reservoir should be greater than or equal to the demand with probability p therefore the condition for having non-dry land for k-consecutive periods will be satisfied. In order to solve this inequality, we will consider each k periods starting from the first period, (first period + $k-1$ periods), second period (second period+ $k-1$ periods) etc. individually and then will consider the intersection of these $n-k+1$ events. The intersection of these events will ensure that at least one period from first day until k^{th} period and from second period until $k+1^{th}$ period will have enough water for the land. When we consider total number of events l, the land will never be dry for the k-consecutive periods out of n total periods.

The event A_l means at least one out of these k periods there will be sufficient water for the land. On the other hand, complementary event $\overline{A}_l$ represents that there will not be sufficient water for the land during any k periods. Below event represents when $k=7$ periods:

$$A_1 = \ (\xi_i \geq \gamma_i, \ M \geq \gamma_i) \geq 1, \ i = 1, \dots, 7$$

which implies that at least 1 period out of 7 periods, land will gave suffcient water, below the $\overline{A}_1$ event:

$$\overline{A}_1 = \ (\xi_i \geq \gamma_i, \ M \geq \gamma_i) < 1, \ i = 1, \dots, 7$$

which implies that none of the periods out of 7 periods, land will have sufficient water. Below the A_2 event:

$$A_2 = \ (\xi_i \geq \gamma_i, \ M \geq \gamma_i) \geq 1, \ i = 2, \dots, 8$$

which implies that at least 1 period out 7 periods (from second period until the eigth period), land will have sufficient water, below the $\overline{A}_2$ event:

$$\overline{A}_2 = \ (\xi_i \geq \gamma_i, \ M \geq \gamma_i) < 1, \ i = 2, \dots, 8$$

which implies that none of the periods out of 7 periods, land will have sufficient water.

The pattern to create the event A_l's, is to start from the l^{th} period and consider until $(l+k-1)^{th}$ period where k is the desired number of consecutive non-dry periods. There will be total $n-k+1$ number of events where n is the total number of periods that will be taken into consideration. As descibed, below represents the last event:

$$A_{n-k+1} = \ (\xi_i \geq \gamma_i, \ M \geq \gamma_i) \geq 1, \ i = n-k+1, \dots, n-k+7$$

below the $\overline{A}_{n-k+1}$ event:

$$\overline{A}_{n-k+1} = \ (\xi_i \geq \gamma_i, \ M \geq \gamma_i) < 1, \ i = n-k+1, \dots, n-k+7$$

which implies that none of the periods out of 7 periods, land will have sufficient water.

Now, the probability of intersection of all the events, which ensures the non-dry land condition for k-consecutive periods can be written as follows:

$$P(A_1 \cap \dots \cap A_{n-k+1}) \geq p \tag{7}$$

The purpose of expressing the reliability constraint as intersection of the specially defined events is to apply bounding techniques while finding the optimum value of the capacity m.

4.3 Sharp bounds on the probability

Solving the equation (7) is not practically easy and managable therefore well known bounds for the union of events will be used to define a lower and upper bound to the desired probability calculation which meanwhile will be used to determine the optimal capacity of the water reservoir.

In order to calculate the lower and upper bounds for the reliability constraint, the definition and description of the bounds will be presented. The same notation and definition as Prékopa [11] is used in below sections. Since all most known bounds are for the union of the events, later, we will explain the conversion of the union of the events into intersection of the events for the water reservoir problem.

4.3.1 Lower bounds, S_1, S_2, S_3 given

Sharp lower bound that is given by Prékopa [11]

$$P(A_1 \cup \cdots \cup A_n) \geq \frac{i+2n-1}{(i+1)n} S_1 - \frac{2(2i+n-2)}{i(i+1)n} S_2 + \frac{6}{i(i+1)} S_3 \tag{8}$$

where

$$i = 1 + \left\lfloor \frac{-6S_3 + 2(n-2)S_2}{-2S_2 + (n-1)S_1} \right\rfloor$$
$$S_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_j} \cap \cdots \cap A_{i_k}), \; k = 1, \ldots, n.$$

4.3.2 Hunter's upper bound

Let $A_1, \ldots, A_n$ be arbitrary events in an arbitrary probability space. Hunter (1976) gave an upper bound for $P(A_1 \cup \cdots \cup A_n)$ by the use of S_1 and the individual probabilities $P(A_i \cap A_j), \; 1 \leq i < j \leq n$. Hunter's upper bound is given by;

$$P(A_1 \cup \cdots \cup A_n) \leq S_1 - \sum_{(i,j) \in T} P(A_i \cap A_j). \tag{9}$$

The second term on the right hand side in (9) is the weight of the spanning tree T. The best bound of this type is obtained when we choose the maximum weight spanning tree T^*. The maximum spanning tree can be found by Kruskal's (1956) algorithm.

4.3.3 Cherry tree upper bound

A third order upper bound is presented on the probability of the union of a finite number of events, by means of graphs called cherry trees. These are graphs that we construct recursively in such a way that every time we pick a new vertex and connect it with two already existing vertices. If the latters are always adjacent, we call the cherry tree a t-cherry tree. A cherry tree has a weight that provides us with the upper bound on the union. A cherry tree bound can be identified as a feasible solution to the dual of the Boolean probability bounding problem. A t-cherry tree bound can be identified as the objective function value of the dual vector corresponding to a dual feasible basis in the Boolean problem. This enables us to improve on the bound algorithmically, if we use the dual method of linear programming.

Definition (Bukszár, Prékopa 2001)
We define a cherry tree recursively in the following manner:
(i) An adjacent pair of vertices constitutes the only cherry tree that has exactly two vertices.
(ii) From a cherry tree we can obtain another cherry tree by adding a new vertex and two new edges, connecting the new vertex with two already existing vertices. These two edges constitute a cherry.
(iii) If V is the set of vertices, ξ the set of edges and ε the set of cherries obtained that way then we call the triple $\Delta = (V, \xi, \varepsilon)$ a cherry tree.
Theorem 4.1. *For any cherry tree $\Delta = (V, \xi, \varepsilon)$ with $V = 1, \dots, n$ we have*

$$\begin{gathered} P(A_1 \cup \dots \cup A_n) \leq \sum_{i=1}^{n} p(A_i) - w(\Delta) = S_1 - w(\Delta) \\ \textit{where} \\ w(\Delta) = \sum_{\{i,j\} \in \xi} P(A_i \cap A_j) - \sum_{(i,j,k) \in \xi} P(A_i \cap A_j \cap A_k) \end{gathered} \tag{10}$$

Proof of the Theorem (4.1) can be found in Ref. [3].

4.4 Our lower and upper bounds

In the above section, sharp lower and upper bounds that are widely used in the probability theory are defined and formulated. There are two proposed upper bounds for the union of the events; Hunter and Cherry tree upper bounds which are both dual feasible bases for the Boolean bounding problem. However, all of these bounds calculate upper and lower range for the probability of the union of the events. Since, the main interest of water reservoir problem is to find a lower and upper bound for the intersection of the events that are defined in section 2, the following conversion is needed:

$$P(A_1 \cap \dots \cap A_{n-k+1}) = 1 - P(\overline{A}_1 \cup \dots \cup \overline{A}_{n-k+1}) \tag{11}$$

If the lower bound for the union of the events is defined by equation (8), then we will have the following formulationf for the lower bound:

$$\frac{i + 2n - 1}{(i+1)n} S_1 - \frac{2(2i + n - 2)}{i(i+1)n} S_2 + \frac{6}{i(i+1)} S_3 = LB_1.$$

Altough we will calculate both Hunter and Cherry tree upper bounds, our computational experince showed that cherry tree bounds are always better than or equal to Hunter upper bound so that, we will define the upper bound for the union of the events with the equation (10) and rename the right hand side as UB_1. Then we will have the following:

$$S_1 - w(\Delta) = UB_1 \tag{12}$$

The ranges for the union of the events can be rewritten as follows:

$$LB_1 \leq P(\overline{A}_1 \cup \cdots \cup \overline{A}_{n-k+1}) \leq UB_1 \tag{13}$$

If we rewrite the union of all the events in terms of the intersection of complementary events and write the range (13), we will obtain:

$$LB_1 \leq 1 - P(A_1 \cap \cdots \cap A_{n-k+1}) \leq UB_1 \tag{14}$$

After manipulating the equation (14), we will have the lower and upper bounds for the interesction of the events as follows:

$$1 - UB_1 \leq P(A_1 \cap \cdots \cap A_{n-k+1}) \leq 1 - LB_1 \tag{15}$$

4.5 Bi-section method

After calculating a lower and an upper bound for the intersection of the events with above equations, the interval of the probability for the reliability constraint will be used to obtain the optimal reservoir capacity, m. The m value will be bi-sected until the p_{value} is observed in one-decimal accuracy in the probability bounding range.

To do so, bi-section algorithm will be used. The bi-section method in mathematics, is a root-finding method which repeatedly bisects an interval then selects a subinterval in which a root must lie for further processing. In our case the root that we would like to find is the p_{value}. (see Wood, 1989)

4.6 Summary of the steps for the solution of model 2

Here we summarize the solution steps.

Step 0
ξ and η are randomly generated in Matlab. They are taken from normal distribution. m is fixed to some reasonable fixed number at first subject to adjust during the bi-section algorithm.

Step 1
$\overline{A}_i$,$\overline{A}_{ij}$, $\overline{A}_{ijk}$ are calculated in order to calculate the S_1, S_2, S_3.

Step 2
S_1, S_2, S_3 are calculated.

Step 3
Lower and upper bounds are calculated with the equation 8 and equation 9 for the event:

$$P(\overline{A}_1 \cup \cdots \cup \overline{A}_n)$$

Step 4
Transformation of the intersection of the events from union of the events is done as follows:

$$LB_1 \leq P(\overline{A}_1 \cup \cdots \cup \overline{A}_{n-k+1}) \leq UB_1$$
$$LB_1 \leq 1 - P(A_1 \cap \cdots \cap A_{n-k+1}) \leq UB_1$$
$$1 - UB_1 \leq P(A_1 \cap \cdots \cap A_{n-k+1}) \leq 1 - LB_1.$$

Step 5
Bi-section algorithm is applied.Three possibilities can occur during the bi-section algorithm:

- If p is larger than Upper Bound, then pick larger m
- If p is smaller than Lower Bound, then pick smaller m
- If p is in between Lower and Upper Bound, try picking smaller/larger m
  - If large m works keep bi-section into the same direction
  - If large m doesnt work, do bi-section in other direction.

Step 6
If p is in between one decimal digit of lower and upper bound STOP, else go to *step 0* and change m.

5 Illustrative Example

In this example, a total period of 8 weeks (56 days) is considered. In the first formulation of the problem, any 7 days of dryness is permitted with a probability level of 90%. For the sake of computational easiness, rain amount is considered with the inflow . Distribution of the inflow is normal and demand is considered to be gamma distribution in the first formulation. In the second formulation of the problem, we used normal distribution for each day's inflow and demand distribution. Dryness in 7 consecutive days is forbidden with a probability level of 90%.

Model (1)

$$\begin{gathered} \min m \\ \text{subject to} \\ P\{\min\left((\xi_i, M) \geq x_i\eta_i\right),\ \ i=1,\ldots,56\ \ x_1+\cdots+x_{56}\geq 7\}\geq 0.90 \\ x_i\in\{0,1\}\ \ i=1,\ldots,56 \\ 0\leq m\leq M \end{gathered} \tag{16}$$

Solution of the model (1) is basically solving the below equation:

$$P\left(\sum_{i=1}^{56}\lambda\left[\min(\xi_i, m)\right]\geq(49)\lambda\gamma_1\right)=0.9 \tag{17}$$

where ξ_i is normally distributed random variable with parameters $(200, 30)$ and γ_1 is a gamma distribution with parameters $\lambda = 20, \vartheta = 10$. The equation (17) is solved by coding a simulator in JAVA and running the equation 1,000 times to get an accurate most observed minimum value for the capacity value m. The most frequently observed m value is 162 therefore we can say that with the above distributed inflow and demand variables, the upper bound of the minimum capacity for the reservoir can be approximated as 162, while maintaining at most 7 days of dry periods with a probabilty level of 90% (a precision of 0.01).

Model (2)

$$\begin{gathered} \min m \\ \text{subject to} \\ P\{\min\left((\xi_i, M) \geq x_i\eta_i\right),\ \ i=1,\ldots,56\ \ x_i+\cdots+x_{i+6}\geq 1\ \ i=1,\ldots,50\}\geq 0.90 \\ x_i\in\{0,1\}\ \ i=1,\ldots,56 \\ 0\leq m\leq M \end{gathered} \tag{18}$$

where the reliability constraint of the problem is equivilant to:

$$P\{\min(\xi_k\geq\eta_k, M\geq\eta_k),\ \ i=1,\ldots,56\ \ x_i+\cdots+x_{i+6}\geq 1\ \ i=1,\ldots,50\}\geq 0.90$$

Distribution of the mean and standard deviation for inflow and demand can be found in the Table 2, Appendix A. In order to start to bi-section algorithm, we picked the initial capacity value m as 180. And then we applied bi-section algorithm based on the intervals of lower and upper bounds of the probability value. When we applied the steps in the section 4, we find the results given in Table 1.

We obtained an interval that contains our desired p_{value} when the capacity value, m, equals 151,4. The results clearly indicate that, with given inflow and demand distributions, the land will not be dry for 7 consecutive days with the probability level of 0.90 when the capacity of the reservoir is 151,4.

Table 1: Bi-section Algorithm Results

Steps	Capacity, M	S_1, S_2, S_3 Lower Bound	Hunter Upper Bound	Cherry Tree Bound	p_{value}	Comment
1	180	0.973505	14.45638	12.345356	0.9	p<LB, pick smaller M
2	158.8	0.879283	3.664441	3.5678921	0.9	LB<p<UB, try decreasing M
3	153.4	0.890781	1.708109	1.684544	0.9	LB<p<UB, try decreasing M
4	152.1	0.893874	1.195414	1.194523	0.9	LB<p<UB, try decreasing M
5	151.3	0.895434	0.937376	0.937283	0.9	LB<p<UB, try increasing M
6	151.7	0.894653	1.06553	1.064428	0.9	LB<p<UB, try decreasing M
7	151.5	0.895043	1.00159	1.000967	0.9	LB<p<UB, try decreasing M
8	151.4	0.895541	0.919803	0.91789	0.9	STOP!

6 Conclusion

This paper was motivated by research work in stochastic programming aimed at solving problems in which probabilities are associated with a large number of events. The several bounding techniques are taken into account and the most efficient and suitable ones for the water reservoir problem is used to define new solutions for k-out-of-n and consecutive k-out-of-n reliabilities in the form of probabilistic constraints. An upper bound for k-out-of-n is developed by the use of well known property of standard gamma distribution. Lower and upper bounds are used to develop an algorithm for consecutive k-out-of-n type reliability constraints. S_1, S_2, S_3 sharp lower, Hunter and Cherry tree upper bounds are discussed and bi-section algorithm is used to optimize the capacity value of the reservoir.

Numerical examples are presented for both models. Solution for the k-out-of-n type reliability is simulated in Java and we obtained an approximate result for the case of normal and gamma distribution of inflow and demand respectively. For the solution of consecutive k-out-of-n type, S_1, S_2, S_3 sharp lower, Hunter's upper and Cherry tree bounds are calculated then the minimum reservoir capacity is obtained by bi-section algorithm. Our solution technique is novel in the sense that real-life restrictions for water engineering is taken into account. Our approach can also be used not only for water reservoir systems but also in any type of reliability theory applications such as finance, power, communication, traffic system reliability problems.

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Table 2: Inflow ξ and Demand η Distributions

Inflow (ξ) Distribution		Demand (η) Distribution	
Mean	Standard Deviation	Mean	Standard Deviation
198	17	149	38
186	50	165	49
117	13	204	35
186	2	181	32
162	14	199	42
206	72	178	56
206	17	177	32
163	91	173	59
209	13	227	58
165	108	202	53
186	45	194	48
211	52	189	41
147	63	181	51
201	62	204	39
192	93	204	28
182	12	182	45
225	79	183	41
224	75	179	41
258	31	180	36
198	5	144	32
235	62	157	32
213	54	186	33
218	40	204	47
179	8	225	44
223	98	142	50
226	44	174	41
160	56	175	45
192	40	250	33
219	49	189	28
139	11	187	58
197	44	170	43
175	25	188	47
170	80	190	47
208	60	187	49
195	5	174	35
223	29	209	38
183	10	201	25
203	7	157	43
199	78	191	35
218	34	187	38
206	41	188	30
189	2	181	21
232	17	208	42
238	52	172	62
234	48	178	53