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On rank-profiles of stable matchings [a]

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On rank-profiles of stable matchings [1]

Endre Boros, Liliya Fedzhora, Vladimir Gurvich, Steven Jaslar

Abstract. We study the quality of stable matchings from the individuals' viewpoint. To each matchings we associate its *rank-profile* describing the individuals' satisfaction with the matching. We provide a complete and computationally efficient characterization of the rank-profiles that can arise from men-optimal, women-optimal, and arbitrary stable matchings. We also study *uniquely stable* rank-profiles, that is, for which there exists a stable matching problem that has only one stable matching and this matching has this particular rank-profile. We give some necessary and some sufficient conditions for unique stability and show that characterizations of men-optimal and women-optimal rank-profiles is reduced to the characterization of uniquely stable rank-profiles. Our characterizations imply that the set of all stable rank-profiles is monotone, unlike the sets of men-optimal, women-optimal, and uniquely stable rank-profiles. We also show that both stable and uniquely stable matchings may be highly disadvantageous for all participating individuals, simultaneously. Namely, we show that there are stable and even uniquely stable rank-profiles in which no individual gets a better partner than his/her middle choice. Furthermore, this result is sharp, since a stable matching in which all individuals get a partner ranked below their middle choice cannot be stable. Finally, we demonstrate an "instability of stable matchings" from a quality point of view.

Key Words: stable matching, preference list, rank-profile, Gale-Shapley algorithm and theorem

1 Introduction

In this paper we consider the *stable matching problem* in which the members of two groups of individuals (*men* and *women*) have strict rankings of the individuals of the other group. A *matching* is called *stable* if no men-women couple prefer one another more than their respective partners in the matching.

The celebrated result by [9] shows that a stable matching always exists, regardless of the individuals' rankings, and can be found algorithmically efficiently. Furthermore, there are unique group-optimal stable matchings (men-optimal and women-optimal), in which all members of the group simultaneously are paired with their "best possible" partners, in the sense that no individual gets a more preferred partner in any other stable matchings. These properties, the equilibrium nature of stable matchings, the coincidence of individual and group optimality, and the algorithmic tractability make stable matching based models very attractive in social sciences and economics. In fact, a very large number of recent publications utilize stable matchings for various applications; see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 13, 14, 21, 19, 18, 17, 20].

Naturally, the quality and fairness of stable matchings based models received a lot of attention. A natural parameter to consider in such investigations is the rank of the matched partner in the individual's preference list. For instance, [12] considered the so called *egalitarian* stable matchings in which the sum of the ranks of the partners of all individuals in a stable matching is minimized, and showed that such an egalitarian stable matching can also be obtained in polynomial time. [15] solved the so called *minimum regret* stable matching problem, in which the maximum rank of any partner (or in other words, the unhappiness of the least satisfied individual) is minimized.

Most of the studies in the literature focus on optimization and algorithmic issues, arising in the course of finding a specific, extremal (optimal, according to some objective) stable matching *for a given preference list*. It is however equally interesting to understand how "bad" or "good" stable matching solutions could be *over all possible preference lists* (in a worst case sense, even if we optimize for each particular instance). More precisely, we are interested in the individuals' satisfaction, in the worst scenario, that is, we consider the maximum happiness of the happiest individual in a stable matching, when the maximization takes place over all stable matchings of a given instance, and ask how unhappy such a "most happy" individual can be? For instance, is it possible that in a men-optimal stable matching no man is coupled with his most preferred woman partner? We show that unfortunately this is quite possible, and that even more so, there are infinitely many instances when even the happiest individual in the community is pretty-pretty unhappy.

More generally, we associate a so called rank-profile to each matching, describing the level of satisfaction of all individuals with their partners within the given matching, and provide a computationally efficient characterization of the rank-profiles that can arise from the men-optimal, women-optimal, and arbitrary stable matchings. We also show that characterizing the men-optimal and women-optimal stable rank-profiles is equivalent, in some sense, with the characterizing rank-profiles that arise from the stable matchings instances that have a unique stable matching. We study the structure of rank-profiles which correspond to such unique stable matchings and demonstrate several interesting their properties.

Finally, we demonstrate "an instability of the stable matchings" by exhibiting instances for which the addition of just one more couple can perfectly spoil the life of a happy community.

2 Main Definitions and Results

We consider the stable matching problems involving n men and n women, where n is a given positive integer. Let $\mathbf{N} = \{1, \ldots, n\}$ denote the set of indices, while $\mathbf{M} = \{m_1, \ldots, m_n\}$ and $\mathbf{W} = \{w_1, \ldots, w_n\}$ denote the sets of men and women, respectively. A stable matching problem is described by the set of individual preferences, that is, by a set of $2n$ mappings:

$$r_w : \mathbf{M} \to \mathbf{N} \text{ for } w \in \mathbf{W} \quad \text{and} \quad r_m : \mathbf{W} \to \mathbf{N} \text{ for } m \in \mathbf{M}. \tag{1}$$

For example, we say that man $m \in \mathbf{M}$ is of *rank* $j \in \mathbf{N}$ for woman $w \in \mathbf{W}$ if $r_w(m) = j$ and that woman $w \in \mathbf{W}$ *prefers* man $m \in \mathbf{M}$ to man $m' \in \mathbf{M}$ if $r_w(m) < r_w(m')$. We consider the most traditional model, in which the preference lists contain no ties, that is, all $2n$ mappings r_w, $w \in \mathbf{W}$ and r_m, $m \in \mathbf{M}$ are bijections. In what follows, a stable matching problem is given by a *preference list* $\sigma = \{r^\sigma_w \mid w \in \mathbf{W}\} \cup \{r^\sigma_m \mid m \in \mathbf{M}\}$ consisting of $2n$ bijections, as in (1), and we shall denote by Σ_n the set of all possible preference lists of men $\mathbf{M}$ and women $\mathbf{W}$. When it is unambiguous, which preference list is meant, we will omit the upper index σ and refer simply by r_w and r_m to the corresponding preferences.

A set $\pi \subseteq \mathbf{M} \times \mathbf{W}$ is called a *matching (or pairing)*, if for each man $m \in \mathbf{M}$ there is a unique woman $w \in \mathbf{W}$ such that $(m, w) \in \mu$ and for each woman $w \in \mathbf{W}$ there is a unique man $m \in \mathbf{M}$ such that $(m, w) \in \pi$). If $(m, w) \in \pi$, we say that w is m's *partner*, and m is w's *partner* in the matching π.

Given a preference list $\sigma \in \Sigma_n$, and a matching $\pi \subseteq \mathbf{M} \times \mathbf{W}$, we say that a pair $(m, w) \in \mathbf{M} \times \mathbf{W}$ is a *breaking couple* for π, if m and w mutually prefer each other to their current partners in π, that is, if $r^\sigma_w(m) < r^\sigma_w(m')$ and $r^\sigma_m(w) < r^\sigma_m(w')$ for $m' \in \mathbf{M}$ and $w' \in \mathbf{W}$ for which $(m, w') \in \pi$ and $(m', w) \in \pi$.

A matching π is called *stable*, if there is no breaking couple for it. The celebrated result of [9] claims that every preference list has a stable matching. (In fact, it may have many, and those form a lattice; see Section 3 for more details). Let us denote by $\Pi(\sigma)$ the set of all stable matchings of a preference list $\sigma \in \Sigma_n$.

Given a preference list $\sigma \in \Sigma_n$ and a matching $\pi \subseteq \mathbf{M} \times \mathbf{W}$, let us associate to them two vectors $k(\pi, \sigma) \in \mathbf{N}^{\mathbf{M}}$ and $\ell(\pi, \sigma) \in \mathbf{N}^{\mathbf{W}}$ defined for $m \in \mathbf{M}$ and $w \in \mathbf{W}$ by

$$\begin{aligned} k_m(\pi, \sigma) &= j \quad \text{if } r_m(w) = j \text{ for the woman } w \in \mathbf{W} \text{ for which } (m, w) \in \pi, \\ \ell_w(\pi, \sigma) &= j \quad \text{if } r_w(m) = j \text{ for the man } m \in \mathbf{M} \text{ for which } (m, w) \in \pi. \end{aligned}$$

Let us call the pair of vectors $(k(\pi, \sigma), \ell(\pi, \sigma))$ the *rank-profile* of the matching π with respect to the preference list σ. Let us note that these two vectors themselves carry all necessary information about the level of satisfaction of the individuals in $\mathbf{M} \cup \mathbf{W}$ with respect to the matching π, even though these vectors alone do not carry enough information to determine π or σ. To simplify notation, we refer to $k(\pi, \sigma)$ and $\ell(\pi, \sigma)$ as $k(\pi)$ and $\ell(\pi)$ whenever σ is clearly defined by the context, as $k(\sigma)$ and $\ell(\sigma)$ whenever π is clearly defined by the context, and simply by (k, ℓ) whenever both π and σ are clearly defined.

Furthermore, whenever π is not explicitly defined, we assume that $\pi = \{(m_1, w_1), (m_2, w_2), \ldots, (m_n, w_n)\}$, since for any matching we can relabel the women to obtain π in this form. Moreover, we will write simply k_i and ℓ_j instead of k_{m_i} and ℓ_{w_j}, and consequently, we consider rank-profiles as pairs of vectors $k, \ell \in \mathbf{N}^n$.

Let us denote by $\mathcal{R}_n$ the set of all rank-profiles, that is, the set of all pairs of vectors (k, ℓ), where $k \in \mathbf{N}^{\mathbf{M}}$ and $\ell \in \mathbf{N}^{\mathbf{W}}$. Let us note that not all rank-profiles correspond to stable matchings. For instance, $k = (2, 2)$ and $\ell = (2, 2)$, for $n = 2$, cannot, that is, no instance $\sigma \in \Sigma_2$ has a stable matching $\pi \in \Pi(\sigma)$ such that $k = k(\pi, \sigma)$ and $\ell = \ell(\pi, \sigma)$. This is simply because no matter how we choose a matching $\pi \in \mathbf{M} \times \mathbf{W}$ and a preference list $\sigma \in \Sigma_2$, any pair $(m_i, w_j) \in (\mathbf{M} \times \mathbf{W}) \setminus \pi$ must be a breaking couple for π in σ.

In this paper we will characterize the rank-profiles of the stable matchings. For a given preference list $\sigma \in \Sigma_n$ let us denote by $\mathcal{S}(\sigma)$ the set of rank-profiles of all stable matchings of σ,

$$\mathcal{S}(\sigma) = \{ (k(\pi, \sigma), \ell(\pi, \sigma)) \mid \pi \in \Pi(\sigma) \},$$

and let $\mathcal{S}_n$ denote the set of all *stable rank-profiles*, that is, $\mathcal{S}_n = \bigcup_{\sigma \in \Sigma_n} \mathcal{S}(\sigma)$.

Our first result characterizes stable rank-profiles.

Theorem 1 *$(k, \ell) \in \mathcal{S}_n$ if and only if*

$$\sum_{i \in I} k_i + \sum_{j \in J} \ell_j \leq n|I| + n|J| + |I \cap J| - |I||J| \tag{2}$$

holds for all subsets $I, J \subseteq \mathbf{N}$. Furthermore, the membership $(k, \ell) \in \mathcal{S}_n$ can be tested in $O(n^5)$ time, and if $(k, \ell) \in \mathcal{S}_n$, then we can construct in the same time an instance $\sigma \in \Sigma_n$ for which $k = k(\sigma)$ and $\ell = \ell(\sigma)$.

Consider first a few small examples. For instance, for $n = 3$ the rank-profile $((1, 1, 2), (3, 3, 1))$ is not stable, since inequality (2) fails for the subsets $M = \{3\}, W = \{1, 2\}$, because $2 + (3 + 3) = 8 > 7 = 3(1+2) - (1 \times 2 - 0)$. Furthermore, $((1, 2, 2), (2, 3, 3)) \notin \mathcal{S}_3$, because for the subsets $M = \{2, 3\}$, and $W = \{1, 2, 3\}$ we get $(2+2)+(2+3+3) = 12 > 11 = 3(2+3) - (2 \times 3 - 2)$. Similarly, $((1, 2, 3), (2, 1, 3)) \notin \mathcal{S}_3$, since $(2 + 3) + (2 + 3) = 10 > 9 = 3(2 + 2) - (2 \times 2 - 1)$ for $M = \{2, 3\}$, and $W = \{1, 3\}$.

Let us consider now several corollaries of Theorem 1. Two special cases of Theorem 1, corresponding to $|I| = |J| = 1$ and $|I| = |J| = n$, can be reformulated as follows.

Corollary 1 *If $(k, \ell) \in \mathcal{S}_n$ then $(k_i, \ell_j) \neq (n, n)$ whenever $i \neq j$.*

Proof. Apply (2) with $I = \{i\}$ and $J = \{j\}$. Then the left hand side of (2) is $2n$, while the right hand side is only $2n - 1$. $\square$

For example, rank-profiles $((1,2),(2,1))$ for $n = 2$ and $((1,1,3),(1,3,1))$ for $n = 3$ are not stable according to the above Corollary 1.

Corollary 2 *If* $(k,\ell) \in \mathcal{S}_n$ *then* $\sum_{i\in\mathbf{N}} k_i + \sum_{j\in\mathbf{N}} \ell_j \leq n(n+1)$.

Proof. Apply (2) with $I = \mathbf{N}$ and $J = \mathbf{N}$. $\square$

For example, the following rank-profiles are not stable:
$((1,2),(2,2))$, $((1,2,3),\ (2,2,3))$, $((2,2,2),\ (2,2,3))$.

We shall derive two more consequences of Theorem 1. Let us write $f \leq g$ for two vectors f and g, if those are labeled by the same domain, and the inequality holds componentwise.

Corollary 3 *Set* $\mathcal{S}_n$ *is anti-monotone with respect to the relation* $\leq$ *applied between rank-profiles. In other words, if* $(k,\ell) \in \mathcal{S}_n$ *and* $(k',\ell') \in \mathcal{R}_n$ *such that* $k' \leq k$ *and* $\ell' \leq \ell$ *hold, then* $(k',\ell') \in \mathcal{S}_n$.

Proof. Conditions (2) are clearly monotone in k and ℓ. $\square$

Corollary 4 *If* n *is an odd integer and* $k_i = \ell_j = \frac{n+1}{2}$ *for all* $i,j \in \mathbf{N}$ *then* $(k,\ell) \in \mathcal{S}_n$.

Proof. We need to derive from Theorem 1 that for arbitrary subsets $I, J \subseteq \mathbf{N}$ we have

$$\frac{n+1}{2}(|I| + |J|) \leq n|I| + n|J| + |I \cap J| - |I||J|.$$

Since $|I \cap J| - |I||J| \leq |I||J| - (|I| + |J|) + n$, it is enough to show that

$$\frac{n+1}{2}(|I| + |J|) \leq n(|I| + |J|) - |I||J| + |I| + |J| - n,$$

which follows readily by $|I| \leq n$ and $|J| \leq n$. $\square$

Corollary 4 implies, somewhat surprisingly, that there are instances $\sigma \in \Sigma_n$ and stable matchings $\pi \in \Pi(\sigma)$ for which all individuals are pretty unhappy. Of course, for some other stable matchings of the same instance, some individuals may be much happier. Let us return to the problem of measuring individuals' satisfaction. As one of the simplest measures of a given preference list $\sigma \in \Sigma_n$ and a stable matching $\pi \in \Pi(\sigma)$, let us introduce

$$h(\pi,\sigma) = \min\left\{\min_i k_i(\pi,\sigma),\ \ \min_j \ell_j(\pi,\sigma)\right\} \quad \text{and} \quad h(\sigma) = \min_{\pi\in\Pi(\sigma)} h(\pi,\sigma), \tag{3}$$

that is, the "happiness" of the "happiest" individual in his/her "luckiest" stable matching of σ. Clearly, if $h(\sigma) = 1$, then there is a "very happy" individual, who gets his/her best choice in some of the stable matchings of σ.

It was observed by Conway (see, for example, [15]) that for every preference list $\sigma \in \Sigma_n$ the set $\Pi(\sigma)$ forms a lattice, that is, if $\alpha, \beta \in \Pi(\sigma)$ are two arbitrary stable matchings, then there exists $\gamma, \delta \in \Pi(\sigma)$ such that

$$k_i(\gamma) = \max\{k_i(\alpha), k_i(\beta)\} \text{ and } \ell_j(\gamma) = \min\{\ell_j(\alpha), \ell_j(\beta)\};$$

$$k_i(\delta) = \min\{k_i(\alpha), k_i(\beta) \quad \text{and} \quad \ell_j(\delta) = \max\{\ell_j(\alpha), \ell_j(\beta)\}$$

for all $i, j \in \mathbf{N}$. In particular, for every preference list $\sigma \in \Sigma_n$ there exists unique *men-optimal* and *women-optimal* matchings $\pi^M, \pi^W \in \Pi(\sigma)$, for which we have

$$k(\pi^M) \geq k(\pi) \geq k(\pi^W) \quad \text{and} \quad \ell(\pi^W) \geq \ell(\pi) \geq \ell(\pi^M) \tag{4}$$

for all stable matchings $\pi \in \Pi(\sigma)$. In fact the algorithm by [9] can be executed in two natural ways, the so called men-oriented and women-oriented way, and these produce the unique matchings π^M and π^W. Thus, both of these extremal stable matchings can be obtained in $O(n^2)$ time, implying that $h(\sigma)$ can in fact be computed for every instance σ in $O(n^2)$ time, because by (3) we have

$$h(\sigma) = \min\left\{\min_i k_i(\pi^M, \sigma), \;\; \min_j \ell_j(\pi^W, \sigma)\right\}. \tag{5}$$

Of course, $h(\sigma)$ is a very weak measure of happiness of the community, since there might be many very unhappy individuals at the same time, who can get only their very low ranked choices, no matter how we choose a stable matching for σ. However, even this simple measure can be very informative; for example, if for some preference list $h(\sigma)$ has a high value, then all individuals will be unhappy to some extent no matter how we choose a stable matching for σ. Let us note that by Corollary 4 we have some preference lists $\sigma \in \Sigma_n$ for which $h(\pi, \sigma)$ is very high for some stable matching $\pi \in \Pi(\sigma)$. For the same preference list, however, we may have another stable matching $\pi' \in \Pi(\sigma)$ for which $h(\pi', \sigma)$ is much lower. It is an interesting question, how high $h(\sigma)$ can be. To investigate this, let us introduce

$$h(n) = \max_{\sigma \in \Sigma_n} h(\sigma). \tag{6}$$

In studying $h(n)$, special instances which have a unique stable matching will play an important role. Let us denote by Σ_n^* the set of all those preference lists $\sigma \in \Sigma_n$ for which $|\Pi(\sigma)| = 1$, and let

$$h^*(n) = \max_{\sigma \in \Sigma_n^*} h(\sigma). \tag{7}$$

Clearly, for $\sigma \in \Sigma_n^*$ we have $\pi^M = \pi^W = \pi$ by (3); hence, the computation of $h(\sigma)$ in (5) can be simplified for such instances. Furthermore, since $\Sigma_n^* \subseteq \Sigma_n$, we get by (6) and (7) that $h^*(n) \leq h(n)$ for all $n \in \mathbb{Z}_+$.

Our next result claims that the example in Corollary 4 is essentially tight, even for instances with a unique stable matching:

Theorem 2 *For every positive integer n we have*

$$\left\lceil \frac{n}{2} \right\rceil \geq h(n) \geq h^*(n) = \left\lfloor \frac{n}{2} \right\rfloor . \tag{8}$$

Interesting open problems are (i) to describe the instances for which $h(\sigma)$ is high (for example, $h(\sigma) > cn$ for some constant c) and (ii) to compute the value of $h(\sigma)$ for typical (say, random) instances.

Besides stable rank-profiles, it would also be very interesting to understand which rank-profiles can arise from men-optimal and/or from women-optimal stable matchings. As we have recalled earlier, for every preference list $\sigma \in \Sigma_n$ there exists a unique men-optimal stable matching $\pi^M = \pi^M(\sigma)$, and a unique women-optimal stable matching $\pi^W = \pi^W(\sigma)$. Let us denote by $(k^M(\sigma), \ell^M(\sigma))$ and $(k^W(\sigma), \ell^W(\sigma))$ respectively, the corresponding rank-profiles, and introduce

$$\mathcal{M}_n = \left\{ (k^M(\sigma), \ell^M(\sigma)) \;\middle|\; \sigma \in \Sigma_n \right\} \text{ and } \mathcal{W}_n = \left\{ (k^W(\sigma), \ell^W(\sigma)) \;\middle|\; \sigma \in \Sigma_n \right\}.$$

We say that a rank-profile (k, ℓ) is *men-optimal* (*women-optimal*) if $(k, \ell) = (k^M(\sigma), \ell^M(\sigma))$ (respectively, $(k, \ell) = (k^W(\sigma), \ell^W(\sigma))$) for some preference list $\sigma \in \Sigma_n$. As we have seen above, preference lists which have a unique stable matching play a helpful role in providing lower bounds for $h(n)$, and in the study of stable rank-profiles, in general. Let us finally define $\mathcal{U}_n = \bigcup_{\sigma \in \Sigma_n^*} \mathcal{S}(\sigma)$ and let us call $(k, \ell) \in \mathcal{U}_n$ a *uniquely stable* (US) rank-profile. It follows from the above definitions that

$$\mathcal{U}_n \subseteq \mathcal{W}_n \cap \mathcal{M}_n \subseteq \left\{ \begin{array}{c} \mathcal{W}_n \\ \mathcal{M}_n \end{array} \right\} \subseteq \mathcal{S}_n \subseteq \mathcal{R}_n.$$

We shall show that in fact all containment relations above are strict. Despite this, the characterization of the men-optimal rank-profiles $\mathcal{M}_n$ and the women-optimal rank-profiles $\mathcal{W}_n$ can be reduced to the characterization of uniquely stable rank-profiles $\mathcal{U}_n$.

3 Further notations and properties

In this section we recall several well-known properties of stable matchings from the literature, which will be instrumental in our proofs.

First of all, it will be convenient to represent an input preference list in one matrix, as in Figure 1. For a pair $m \in \mathbf{M}$ and $w \in \mathbf{W}$ the cell (m, w) contains in the upper left corner the rank $r_m(w)$ of woman w in the preference list of man m, while the lower right corner contains $r_w(m)$, the rank of m in the preference list of w. For example, for man $a \in \mathbf{M}$ and woman $x \in \mathbf{W}$ in Figure 1 we have $r_a(x) = 1$, i.e., x is a's first choice, while $r_x(a) = 2$, that

	x	y	z
a	1 / 2	**3** / **3**	2 / 2
b	**1** / **3**	2 / 1	3 / 1
c	3 / 1	1 / 2	**2** / **3**

Figure 1: An example $\sigma \in \Sigma_3$ involving three men $\mathbf{M} = \{a, b, c\}$ and women $\mathbf{W} = \{x, y, z\}$.

is, a is only the second choice of x. The bold entries in Figure 1 correspond to the matching $\pi = \{(a, y), (b, x), (c, z)\}$, which is not stable, since the pair (a, z) is a breaking couple for π.

Let us recall from [9] that in the "men-oriented" algorithm the following two stages are repeated, until a stable matching is found: each man without a current partner makes an offer to the first woman in their preference list who did not yet reject him. Each woman rejects all offers, except one man's who is ranked the highest in her preference list among those who made an offer to her. When no man is rejected, all men must have a partner, and the algorithm stops. In the "women-oriented" variant, the roles of men and women are interchanged. Even though in these algorithms there is a high degree of freedom (in what order men make offers, etc.) it is known that these procedures always converge to the same stable matching. More precisely, it was proven in [9] that:

Fact 1 *For every instance $\sigma \in \Sigma_n$ the men-oriented and the women-oriented algorithms always produce, in at most $O(n^2)$ steps respectively, the unique men-optimal $\pi^M = \pi^M(\sigma)$ and unique women-optimal $\pi^W = \pi^W(\sigma)$ stable matchings, for which $k(\pi^M) \leq k(\pi)$ and $\ell(\pi^W) \leq \ell(\pi)$ hold for all stable matchings $\pi \in \Pi(\sigma)$.* □

For instance, for the example of Figure 1 in the men-oriented version, first both a and b make offers to x, and c makes an offer to y. The offer of b is rejected, and hence he makes a second offer to y, who is his second choice. Then y has two offers, one from c in the first step, and one from b, and since she prefers b to c, she rejects c, who then makes his second offer to z. At this moment all women have exactly one offer, so nobody is rejected and the algorithm stops, outputting $\pi^M = \{(a, x), (b, y), (c, z)\}$ as the men-optimal stable matching. Analogously, the women-oriented procedure produces $\pi^W = \{(a, z), (b, y), (c, x)\}$. In fact, in this example we have only two stable matchings, $\Pi(\sigma) = \{\pi^M, \pi^W\}$. We can see that $k(\pi^M) = (1, 2, 2)$ and $\ell(\pi^W) = (1, 1, 2)$, while we have $k(\pi^W) = (2, 2, 3)$ and $\ell(\pi^M) = (2, 1, 3)$, and indeed, for these we have $k(\pi^M) \leq k(\pi^W)$ and $\ell(\pi^W) \leq \ell(\pi^M)$ (where we listed these vectors keeping the natural orders, $\{a, b, c\}$ for the men, and $\{x, y, z\}$ for the women). An immediate corollary of Fact 1 is the following:

Fact 2 *If $\pi^M(\sigma) = \pi^W(\sigma)$ for a preference list $\sigma \in \Sigma_n$ then $|\Pi(\sigma)| = 1$, that is, $\sigma \in \Sigma_n^*$.* □

Let us call a pair $(m_i, w_j) \in \mathbf{M} \times \mathbf{W}$ a *men rejection* pair if in the course of the men-oriented algorithm w rejects the offer of m at one point. Analogously, we can define *women rejection* pairs.

Fact 3 *Given an instance $\sigma \in \Sigma_n$, the men rejection pairs and the women rejection pairs are always the same in any variant of the men-oriented and women-oriented algorithms. Denoting these sets respectively by $R^M = R^M(\sigma)$ and $R^W = R^W(\sigma)$, we have*

- *for all $(m_i, w_j) \in R^M$ we have $r_{m_i}(w_j) < k_i(\pi^M)$ and $r_{w_j}(m_i) > \ell_j(\pi^M)$;*
- *for all $(m_p, w_q) \in R^W$ we have $r_{m_p}(w_q) > k_p(\pi^W)$ and $r_{w_q}(m_p) < \ell_q(\pi^W)$.*

Consequently, $R^M \cap R^W = \emptyset$.

Proof. The intrinsic property of the men-oriented algorithm, as was observed by [9], is that men make offers to women in the order of their preferences from better to weaker, while women have increasingly better and better partners, according to their preferences. Thus if a woman w rejects a man m in the man-oriented algorithm, then $r_m(w)$ is strictly better than what m ends up having in the men-optimal stable matching. On the other hand, woman w rejects man m, and hence she must have gotten a better offer; thus she will end up with a strictly better partner even in the men-optimal stable matching (which is the worst for her, by Fact 1). Combining this with the inequalities in Fact 1 we get

$$r_{m_i}(w_j) \ < \ k_i(\pi^M) \ \leq \ k_i(\pi^W) \quad \text{and} \quad r_{w_j}(m_i) \ > \ \ell_j(\pi^M) \ \geq \ \ell_j(\pi^W).$$

Analogous arguments applied to the women-oriented algorithm yields

$$r_{m_p}(w_q) \ > \ k_{m_p}(\pi^W) \ \geq \ k_{m_p}(\pi^M) \quad \text{and} \quad r_{w_q}(m_p) \ < \ \ell_{w_q}(\pi^W) \ \leq \ \ell_{w_q}(\pi^M).$$

Thus, if we have $i = p$ for some $(m_i, w_j) \in R^M$ and $(m_p, w_q) \in R^W$, then by the above inequalities we get

$$r_{m_i}(w_q) \ < \ k_{m_i}(\pi^M) \ \leq \ k_{m_p}(\pi^W) \ < \ r_{m_p}(w_q)$$

implying that $j \neq q$. Similar conclusion can be reached if $j = q$. □

For instance, returning to Figure 1 we obtain $R^M = \{(b, x), (c, y)\}$ and $R^W = \{(b, z)\}$.

The following very useful observation can be found for instance in [10].

Fact 4 *The men-oriented algorithm terminates when the last woman receives her first offer. Respectively, the women-oriented one terminates when the last man receives his first offer.*

Proof. Let us note that after woman w receives her first offer in the men-oriented algorithm, she always has a partner (she rejects only the "extra" offers, but keeps the best offer as a current partner). Thus, at the moment the last woman receives her first offer, all women have a standing offer from one of the men. But no man is offering to two different women at the same time, and the number of men is the same as the number of women, implying that at this moment no woman has multiple offers. Thus, the algorithm must terminate. Analogous arguments apply to the men-oriented version. □

4 Men-optimal, women-optimal, and uniquely stable rank-profiles

4.1 Main concepts and relations between them

In section 4 we consider men-optimal (MO), women-optimal (WO), and uniquely stable (US) rank-profiles. Given n, we denote the corresponding sets of rank-profiles by $\mathcal{M}_n, \mathcal{W}_n$ and $\mathcal{U}_n$, respectively. Let us recall that we denote the set of all rank-profiles by $\mathcal{R}_n$ and the set all stable ones by $\mathcal{S}_n$. By definition, each US rank profile is simultaneously MO and WO and each MO or WO rank-profile is stable, that is,

$$\mathcal{U}_n \subseteq \mathcal{M}_n \cap \mathcal{W}_n, \quad \mathcal{M}_n \cup \mathcal{W}_n \subseteq \mathcal{S}_n.$$

The second inclusion is strict already for $n = 3$. For example, rank-profile $((2,2,2),(2,2,2)) \in \mathcal{R}_3$ is stable, yet not uniquely stable (SYNUS), and it is neither men- nor women-optimal; see section C. We conjecture that the second containment is in fact an equality.

Conjecture 1 *If a rank-profile is both men- and women-optimal then it is US, that is,* $\mathcal{U}_n = \mathcal{M}_n \cap \mathcal{W}_n$.

This statement is not obvious, since a rank-profile can be MO for one preference list and WO for another one. However, computations for $n = 2, 3$ and 4 confirm this conjecture.

According to Theorem 1 the set of all stable rank-profiles $\mathcal{S}_n$ is monotone, that is, $(k, \ell) \in S_n$ and $(k', \ell') \leq (k, \ell)$ imply that $(k', \ell') \in S_n$. However, it is not the case with $\mathcal{M}_n, \mathcal{W}_n$, and $\mathcal{U}_n$ already for $n = 3$.

For example, we show that $((1,1,3),(2,2,1))$ is SYNUS; it is men-optimal but not women-optimal, while $((1,1,3),(2,2,3))$ is uniquely stable. Hence, $\mathcal{U}_3$ is not monotone, for example, $((1,1,3),(2,2,1)) \leq ((1,1,3),(2,2,3))$. Further we show that $((1,2,3),(2,2,2))$ is SYNUS; it is WO but not MO. Hence, $\mathcal{W}_3$ is not monotone, since $((1,1,3),(2,2,1)) \leq ((1,2,3),(2,2,2))$. By symmetry, $\mathcal{M}_3$ is not monotone either.

In general, it seems difficult to characterize $\mathcal{M}_n, \mathcal{W}_n$ or $\mathcal{U}_n$. We will obtain some sufficient (Appendix A) and some necessary (Appendix B) conditions for unique stability and also show (Theorem 3 of Appendix A) that characterization of $\mathcal{M}_n$ (as well as $\mathcal{W}_n$) is reduced to characterization of $\mathcal{U}_n$. It is an open question whether a membership in $\mathcal{M}_n, \mathcal{W}_n$ or $\mathcal{U}_n$ can be tested in polynomial time.

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APPENDIX

A Sufficient conditions

Proposition 1 *If $(k,\ell) \in \mathcal{U}_n$, then $(k',\ell') \in \mathcal{U}_{n+1}$ for all $k' = (k, k_{n+1})$ and $\ell' = (\ell, \ell_{n+1})$, where k_{n+1} and ℓ_{n+1} are arbitrary integers between 1 and $n+1$.*

Proof. Assume that $(k,\ell) \in \mathcal{U}_n$, and consider a corresponding preference list $\sigma \in \Sigma^*_n$.

To describe a preference list $\sigma' \in \Sigma^*_{n+1}$ which corresponds to the rank-profile $(k',\ell') \in \mathcal{U}_{n+1}$ we define the preferences of the individuals as follows:

$$r'_{m_i}(w_j) = r_{m_i}(w_j) \text{ for } i,j \in \mathbf{N}, \;\; r'_{w_j}(m_i) = r_{w_j}(m_i) \text{ for } i,j \in \mathbf{N},$$

$$r'_{m_i}(w_{n+1}) = n+1 \text{ for } i \in \mathbf{N}, \;\; r'_{w_j}(m_{n+1}) = n+1 \text{ for } j \in \mathbf{N},$$

$$r'_{m_{n+1}}(w_{n+1}) = k_{n+1}, \;\; r'_{w_{n+1}}(m_{n+1}) = \ell_{n+1}.$$

All the undefined preferences (of man m_{n+1} and woman w_{n+1}) can be filled in arbitrarily. Then, due to the properties of σ' constructed above, in the men-oriented procedure only man m_{n+1} makes an offer to woman w_{n+1}, and in the women-oriented procedure only woman w_{n+1} makes an offer to man m_{n+1}. Thus, in both cases (m_{n+1}, w_{n+1}) form a couple in the final stable matching, and since the preferences of the first $n-n$ individuals did not change, these stable matchings coincide, by our assumption about σ. Hence, we have $\sigma' \in \Sigma^*_{n+1}$ and has rank-profile (k',ℓ'). □

For example, $((1),(1)) \in \mathcal{U}_1$; hence $((1,2),(1,2)) \in \mathcal{U}_2$;
this in its turn, implies that $((1,2,3),(1,2,3)) \in \mathcal{U}_3$, etc., $((1,2,\ldots,n),(1,2,\ldots,n)) \in \mathcal{U}_n$.

There is an alternative way to the same conclusion. Obviously, the unique matching, $((1,1),\ldots,(n,n))$, is stable with respect to the *unanimous* instance, when all men and women have the same preference list $(1,\ldots,n)$. Hence, the corresponding rank-profile $(k^u,\ell^u) = ((1,2,\ldots,n),(1,2,\ldots,n))$ is uniquely stable. In fact, Proposition 1 implies a stronger claim:

Proposition 2 *A rank-profile $(k,\ell) \in \mathcal{U}_n$ whenever $(k,\ell) \le (k^u,\ell^u)$.*

Proof. Let us generate uniquely stable rank-profiles recursively beginning with $(k,l) = ((1),(1)) \in \mathcal{U}_1$ and applying all extensions of Proposition 1. Clearly, (k,ℓ) will be obtained in such a way if and only if $(k,\ell) \le (k^u,\ell^u)$. □

Of course, not every uniquely stable rank-profiles can be obtained in this way.

For example, rank-profiles $((1,1,3),(2,2,3)), ((1,1,3),(2,2,2)), ((1,2,2),(2,2,2)) \in \mathcal{U}_3$ but they are not majorized by $((1,2,3),(1,2,3))$.

Proposition 3 *If $(k,\ell) \in \mathcal{U}_n$ then $(k_i,\ell_j) \neq (n,n)$, $(k_i,k_j) \neq (n,n)$ and $(\ell_i,\ell_j) \neq (n,n)$ whenever $i \neq j$.*

Proof. The fact that $(k_i,\ell_j) \neq (n,n)$ for $i \neq j$ follows by Corollary 1. The other two claims are symmetric, and it is enough to show e.g. that $(\ell_i,\ell_j) \neq (n,n)$ whenever $i \neq j$. For this, assume indirectly that $\ell_i = \ell_j = n$ for some $i \neq j$, and consider the preference list $\sigma \in \Sigma_n^*$ for which (k,ℓ) is the rank-profile of the unique stable matching. In the women-oriented algorithm for σ we have both women w_i and w_j making offers to all men, implying that all men receives at least two offers. This contradicts the fact that the algorithm terminates when the last man receives his first offer. This contradiction proves the claim. □

An important relation between unique stability and men-optimality is given by the following statement.

Theorem 3 *If a rank-profile $(k,\ell) \in \mathcal{S}_{n-1}$ is men-optimal then $(k',\ell') \in \mathcal{U}_n$ whenever $\ell' = (\ell,n)$, $k' = (k,k_n)$, and $1 \leq k_n \leq n$. Moreover, for $k_n = 1$ the inverse is also true, namely, if $((k,1),(\ell,n)) \in \mathcal{U}_n$ then $(k,\ell) \in \mathcal{S}_{n-1}$ and it is men-optimal.*

Obviously, the symmetric claim for WO rank-profiles holds, too. By this Theorem, the membership test for $\mathcal{M}_{n-1}$ (or $\mathcal{W}_{n-1}$) is reduced to such a test for $\mathcal{U}_n$. There are several other corollaries.

Corollary 5 *Rank-profile $((k,k_n),(\ell,n)) \in \mathcal{R}_n$ is uniquely stable whenever $k_i = 1$ and $l_i < n$ for all $i = 1,\ldots,n-1$.*

Proof. Clearly, $(k,\ell) \in \mathcal{R}_{n-1}$ is men-optimal, since $k = (1,\ldots,1)$. Furthermore, Theorem 3 implies that $((k,k_n),(\ell,n)) \in \mathcal{U}_n$. □

Let us consider several examples. Rank-profile $((2,2),(1,1))$ is WO but not MO. Hence, $((2,2,1),(1,1,3))$ is not US. However, by corollary 5, $((2,2,3),(1,1,3)) \in \mathcal{U}_3$ (and $((2,2,2),(1,1,3)) \in \mathcal{U}_3$, too). This shows that the assumption $k_n = 1$ in the second part of Theorem 3 is essential and also that $\mathcal{U}_3$ is non monotone. Furthermore, $((1,1,3),(2,2,1))$ is men-optimal (though not US), hence $((1,1,1,3),(2,2,4,1))$ is US. In contrast, $((2,2,2),(2,2,2))$ is not men-optimal (see subsection C) and hence $((2,2,2,1),(2,2,2,4))$ is not uniquely stable (though it satisfies all necessary conditions of unique stability which will be given in subsection B). Let us also remark that $((2,2,2,2),(2,2,2,4))$, $((2,2,2,3),(2,2,2,4))$, $((2,2,2,4),(2,2,2,4)) \in \mathcal{U}_4$.

In contrast to Corollary 5, not every rank-profile majorized by $(k,\ell) = ((1,...,1,k_n),(n-1,...,n-1,n))$ is uniquely stable. For example, $((1,1,3),(2,2,1))$ is SYNUS, it is MO but not WO (see section C), though $((1,1,3),(2,2,2))$ and $((1,1,3),(2,2,3))$ are US.

In fact, all components of (k, ℓ) can be reduced without loss of the unique stability, except $\ell_n = n$ which cannot be reduced to 1.

Somewhat surprisingly, Theorem 3 implies the following "anti-monotone" property.

Corollary 6 *If* $((k, 1), (\ell, n)) \in \mathcal{U}_n$ *then* $((k, k_n), (\ell, n)) \in \mathcal{U}_n$*, where* $1 \leq k_n \leq n$ *and* $(k, \ell) \in \mathcal{R}_{n-1}$.

Proof. By Theorem 3, if $((k, 1), (\ell, n)) \in \mathcal{U}_n$ then $(k, \ell))$ is men-optimal and if $(k, \ell))$ is men-optimal then $((k, k_n), (\ell, n)) \in \mathcal{U}_n$ for any k_n between 1 and n. □

This anti-monotone property can be strengthened as follows.

Proposition 4 *If* $((k, k_n), (\ell, n)) \in \mathcal{U}_n$ *then* $((k, k_n + 1), (\ell, n)) \in \mathcal{U}_n$*, where* $1 \leq k_n < n$ *and* $(k, \ell) \in \mathcal{R}_{n-1}$.

Proof. Since we assume $((k, k_n), (\ell, n)) \in \mathcal{U}_n$, there exists a preference list $\sigma \in \Sigma_n^*$ for which both the men- and women-oriented algorithms yield the same rank-profile $((k, k_n), (\ell, n))$.

Consider then the women-oriented algorithm. In this procedure woman w_n makes n offers, and is rejected by all men m_j, $j < n$. Therefore, we must have

$$r_{m_j}(w_n) > k_j \quad \text{for all} \quad j = 1, ..., n-1. \tag{9}$$

Furthermore, man m_n must be the one who receives his first offer last, and consequently, he does not receive any other offer in this algorithm. This implies that we must also have

$$r_{w_j}(m_n) > \ell_j \quad \text{for all} \quad j = 1, ..., n-1. \tag{10}$$

Let us now choose index j such that $r_{m_n}(w_j) = k_n + 1$, and create a new preference list σ' from σ by interchanging preferences $r_{m_n}(w_j) = k_n + 1$ and $r_{m_n}(w_n) = k_n$, and leaving all other preference values unchanged.

We claim that $\sigma' \in \Sigma_n^*$, completing the proof. To see this claim, let us run first the women-oriented algorithm for σ'. Since the first $n-1$ rows of the preference table did not change, and since the inequalities (9), this algorithm will run exactly the same way as for σ.

Let us compare next the runs of the men-oriented algorithm on σ and σ'. Notice that on σ' man m_n makes now one new offer to woman w_j, which he did not make in the run on σ. This new offer may result in the rejection of some other men's offers, which were not rejected in the run on σ. This is however not the case, since if another men's offer, say m_i's, to woman w_j is rejected by this new offer of man m_n, then we must have $r_{w_j}(m_i) > r_{w_j}(m_n) > \ell_j$ by (10), implying that $i \neq j$, and hence man m_i's offer must have been rejected in the run on σ, as well (otherwise that run could not have terminated with rank-profile $((k, k_n), (\ell, n))$). Thus, both algorithms terminate the same way on σ' as on σ, proving our claim. □

It seems that this claim can be further strengthen as follows.

Conjecture 2 *If* $((k, k_n), (\ell, n)) \in \mathcal{U}_n$ *then* $((k, k'_n), (\ell, n)) \in \mathcal{U}_n$*, where* $(k, \ell) \in \mathcal{R}_{n-1}$*,* $1 \leq k_n \leq n$*, and* $2 \leq k'_n \leq n$*.*

This claim was verified by computer for $n \leq 4$. For $k_n = 1$ and, more generally, for $k_n \leq k'_n$ it follows from Theorem 3. As we already know, it does not generalize the case $1 = k'_n < k_n$.

Lemma 1 *Let* $n \geq 3$*. If* $((1, ..., 1), (\ell_1, \ell_2, ..., \ell_{n-1})) \in \mathcal{U}_{n-1}$ *such that* $\ell_j < n - 1$ *for all* $j = 1, ..., n - 1$*, and* $\ell_n < n$ *is a positive integer, then we have* $((1, ..., 1), (\ell_1 + 1, \ell_2 + 1, ..., \ell_{n-1} + 1, \ell_n)) \in \mathcal{U}_n$*.*

Proposition 5 *If* $k = (1, \ldots, 1)$ *then a rank-profile* $(k, \ell) \in \mathcal{U}_n$ *if and only if* $\ell_j = n$ *for at most one* $j \in \mathbf{N}$*.*

Proof. Corollary 6 shows that at most one of the ℓ_j values can be equal to n. If $\ell_j = n$ for some index j then Theorem 3 implies the statement. If $\ell_j < n$ for all $j = 1, .., n$ then Lemma 1 implies the claim. □

According to this proposition, rank-profiles

$$((1, 1), (2, 2)), ((1, 1, 1), (1, 3, 3)), ((1, 1, 1, 1), (1, 1, 4, 4)), \ldots, ((1, \ldots, 1, 1, 1), (1, \ldots, 1, n, n))$$

are SYNUS, while

$$((1, 1), (1, 2)), ((1, 1, 1), (2, 2, 3))((1, 1, 1, 1), (3, 3, 3, 4)), ..., ((1, \ldots, 1, 1), (n - 1, \ldots, n - 1, n))$$

are US. All rank-profiles which are majorized by the last one are US, too.

B Necessary conditions

Given a rank-profile $(k, \ell) = ((k_1, \ldots, k_n), (\ell_1, \ldots, \ell_n)) \in R_n$, let us introduce notations

$$s = \sum_{i \in \mathbf{N}} k_i + \sum_{j \in \mathbf{N}} \ell_j; \; d_k = \max_{i \in \mathbf{N}} k_i, \; d_\ell = \max_{j \in \mathbf{N}} \ell_j, \; d = \min\{d_k, d_\ell\}.$$

Proposition 6 *If* $(k, \ell) \in \mathcal{U}_n$ *then*

$$s \leq n^2 + d. \tag{11}$$

Proof. Let us consider a given uniquely stable rank-profile. Since it is men-optimal, the number of offers from men should not exceed the number of offers that women can consider. Taking into account that the algorithm terminates when the last woman gets her first offer, or, in other words, there is a woman who receives only one offer, we obtain

$$\sum_{i \in \mathbf{N}} k_i \leq \sum_{j \in \mathbf{N}} (n - \ell_j + 1) - (n - \ell_d) \quad \text{or} \quad s \leq n^2 + d_\ell.$$

Since (k, ℓ) is also a women-optimal profile, we also have $s \le n^2 + d_k$. Combining the last two inequalities completes the proof. □

For example, rank-profile $((2,2,2),(2,2,2))$ is SYNUS, since $s = 12 > 9 + 2 = n^2 + d$; similarly, $((1,3,3,3),(1,3,3,3))$ and $((2,2,3,3),(2,2,3,3))$, since $s = 20 > 16 + 3 = n^2 + d$; also $((2,2,2,2),(2,3,3,3))$ and $((1,2,2,2),(3,3,3,3))$, since $s = 19 > 16 + 2 = n^2 + d$.

In contrast, the following rank-profiles are uniquely stable (as it was verified by a computer code), and hence, they satisfy (11). In fact the equality holds in all cases:

$((1,1,1,4),(3,3,3,4))$, $((1,1,2,3),(3,3,3,3))$, $((1,1,2,4),(2,3,3,4))$, $((1,1,3,3),(2,3,3,3))$, $((1,2,3,4),(1,2,3,4))$, $((1,2,3,4),(2,2,2,4))$, $((1,3,3,3),(2,2,2,3))$, $((2,2,2,3),(2,2,3,3))$, $((2,2,2,2),(2,3,3,2))$, $((2,2,2,2),(2,2,2,4))$, $((2,2,2,3),(2,2,2,4))$, $((2,2,2,4),(2,2,2,4))$.

However, $((2,2,2,1),(2,2,2,4))$ is SYNUS, by Theorem 3, since $((2,2,2),(2,2,2))$ is not men- or women-optimal, see section C. Inequality (11) can be strengthened as follows.

Proposition 7 *If $(k, \ell) \in \mathcal{U}_n$ then*

$$s \le n^2 - n + d_k + d_\ell + \{0 \text{ or } 1\}. \tag{12}$$

where $d_k = \max_{i \in \mathbf{N}} k_i$, $d_\ell = \max_{j \in \mathbf{N}} \ell_j$.

Proof. Consider a rank-profile $(k, \ell) \in \mathcal{U}_n$. Let us proceed with the men-optimal and the women-oriented algorithms, and note which pairs are visited. We say that pair (m_i, w_j) is visited in these algorithms if at some step man m_i makes an offer to woman w_j, or vice versa.

Observe first that apart from the diagonal entries all other visited pairs are pairwise distinct. If pair (m_i, w_j) would be visited by both the man- and women-oriented algorithms, then this pair would be a breaking couple for the main diagonal. Thus, the number of the visited pairs equals $s - n$.

In the women-oriented algorithm there exists a man, say man m_i, $i \in \mathbf{N}$ who is the last to get an offer, implying that among pairs (m_i, w_j), $j \in \mathbf{N}$ the women-optimal algorithm visits only the diagonal pairs. Since the men-optimal algorithm visits exactly k_i from all (m_i, w_j), $j \in \mathbf{N}$ pairs, including the diagonal, there must be exactly $n - k_i$ unvisited pairs among (m_i, w_j), $j \in \mathbf{N}$. Repeating the same argument for the men-oriented algorithm, we conclude that there exists a woman, say woman w_j, who gets an offer last, and among all pairs (m_i, w_j), $i \in \mathbf{N}$ we must have exactly $n - \ell_j$ unvisited pairs. Consequently, we obtain

$$s - n \le n^2 - (n - k_i) - (n - \ell_j) + \{0 \text{ or } 1\};$$

we have $+1$ if $i \ne j$ (pair (m_i, w_j) is counted twice as not visited) and $+0$ otherwise. □

Given $(k, \ell) = ((k_1, \ldots, k_n), (\ell_1, \ldots, \ell_n))$, let us introduce

$$(k', \ell') = ((k_1 + 1, \ldots, k_n + 1, 1), (\ell_1 + 1, \ldots, \ell_n + 1, 1).$$

For example, if $(k, \ell) = ((2,2,2),(2,2,2))$ then $(k', \ell') = ((1,3,3,3),(1,3,3,3))$.

Proposition 8 *Inequality (11) (and (12)) holds for* (k, ℓ) *if and only if it holds for* (k', ℓ').

In its turn, inequality (12) can be strengthened in the following special case.

Proposition 9 *Given a rank-profile* $(k, \ell) \in \mathcal{U}_n$ *such that* $k_p = n$ *for some* $p \in \mathbf{N}$*, then* $s \leq n^2 + \ell_p$.

Proof. Since (k, ℓ) is uniquely stable, it is men-optimal. The men-oriented algorithm terminates when the last woman receives her first offer. Since $k_p = n$ for some $p \in \mathbf{N}$, man m_p will make offers to all women, and the last woman who receives an offer from him is woman w_p. Hence the algorithm terminates when woman w_p receives her first and only offer. Since the number of offers that men make should not exceed the number of offers that women can consider, we have the following inequality:

$$\sum_{i \in \mathbf{N}} k_i \leq \sum_{j \in \mathbf{N}} (n - \ell_j + 1) - (n - \ell_p) \quad \text{or} \quad s \leq n^2 + l_p.$$

This completes the proof. □

For example, $((1, 1, 2, 2), (2, 4, 3, 3))$ is SYNUS, since $s = 18 > 16 + 1 = n^2 + k_2$.

A much tighter upper bound for s holds for uniquely stable rank-profiles (k, ℓ) such that $k_i = 1$ or $\ell_i = 1$ for each $i \in \mathbf{N}$.

Proposition 10 *Let* a*,* b *and* n *be positive integers such that* $n = a+b$*. For a pair of vectors* $k = (k_1, \ldots, k_a, 1, \ldots, 1)$ *and* $\ell = (1, \ldots, 1, \ell_1, \ldots, \ell_b)$*, if the rank-profile* $(k, \ell) \in \mathcal{U}_n$ *then*

$$\sum_{i=1}^{a} k_i + \sum_{j=1}^{b} \ell_j \leq n^2 - (a + 1)(b + 1) + 3. \tag{13}$$

Proof. If (k, ℓ) is a uniquely stable rank-profile, then the rank-profiles (k', e) and (e', ℓ') decreased to the size of a and b accordingly are also uniquely stable, where $k' = (k'_1, \ldots, k'_a)$, $\ell = (\ell'_1, \ldots, \ell'_b)$, $e_i = e'_j = 1$ for all $i \leq a$ and $j \leq b$. Since by Proposition 3 any uniquely stable rank-profile can have at most one entry that is equal to the size of the profile, we have the following inequality:

$$\sum_{i=1}^{a} (k_i - (a - 1)) - 1 + \sum_{j=1}^{b} (\ell_j - (b - 1)) - 1 \leq ab,$$

which yields the necessary condition. □

For example, rank-profile $((1, 1, 3), (2, 2, 1))$ is SYNUS, because $(2 + 2) + 3 = 7 > 6 = 3^2 - (1 + 1)(2 + 1) + 3$. In contrast, $((1, 1, 3), (2, 2, 3))$ and $((1, 1, 3), (2, 2, 2))$ are US as we already know. Hence, $\mathcal{U}_3$ is not monotone.

Similarly, by (13), the following rank profiles are SYNUS:
$((1,1,2),(2,3,1))$, $((1,1,1),(1,3,3))$; $((1,1,1,4),(2,3,3,1))$, $((1,1,1,3),(2,3,4,1))$, $((1,1,1,3),(3,3,3,1))$, $((1,1,1,2),(3,3,4,1))$, $((1,1,1,2),(2,4,4,1))$; $((1,1,2,2),(3,4,1,1))$, $((1,1,2,3),(2,4,1,1))$, $((1,1,2,3),(3,3,1,1))$, etc.

Given positive integers a,b,c,n,n' such that $n = a+b$, $n' = a+b+c$, and a rank-profile $(k,\ell) = ((k_1,\ldots,k_a,1,\ldots,1),(1,\ldots,1,\ell_1,\ldots,\ell_b)) \in \mathcal{R}_n$, let us introduce

$$(k',\ell') = ((k_1+c,\ldots,k_a+c,1,\ldots,1),(1,\ldots,1,\ell_1+c,\ldots,\ell_b+c,1,\ldots,1)) \in \mathcal{R}_{n'}.$$

Proposition 11 *Inequality (13) holds for (k,ℓ) if and only if it holds for (k',ℓ')*

Proof. Indeed, $\sum_{i=1}^a k_i + \sum_{j=1}^b \ell_j - n^2 = \sum_{i=1}^a k'_i + \sum_{j=1}^b \ell'_j - n'^2$ □

For example, the following rank-profiles (k',ℓ') are SYNUS:

a) $((1,\ldots,1,1,1),(1,\ldots,1,n,n))$;
b) $((1,\ldots,1,1,1,n),(1,\ldots,1,n-1,n-1,1))$;
c) $((1,\ldots,1,1,n-1),(1,\ldots,1,n-1,n,1))$;
d) $((1,\ldots,1,1,1,n-2),(1,\ldots,1,n-1,n-1,n,1))$;
e) $((1,\ldots,1,1,1,n-1),(1,\ldots,1,n-1,n-1,n-1,1))$;
f) $((1,\ldots,1,1,n-2,n-2),(1,\ldots,1,n-1,n,1,1))$;
g) $((1,\ldots,1,1,1,n-2,n-1),(1,\ldots,1,n-2,n,1,1))$;
h) $((1,\ldots,1,1,1,n-2,n-1),(1,\ldots,1,n-1,n-1,1,1))$,

because the corresponding (k,ℓ) are SYNUS:

a) $((1,1),(2,2))$;
b) $((1,1,3),(2,2,1))$;
c) $((1,1,2),(2,3,1))$;
d) $((1,1,1,2),(3,3,4,1))$;
e) $((1,1,1,3),(3,3,3,1))$;
f) $((1,1,2,2),(3,4,1,1))$;
g) $((1,1,2,3),(2,4,1,1))$;
h) $((1,1,2,3),(3,3,1,1))$.

Neither (k,ℓ) nor (k',ℓ') satisfy (13).

Obviously, inequality (13) can be strengthened as follows.

Proposition 12 *Given positive integers a,b and n such that $n = a+b$ and the rank-profile $k = (k_1,\ldots,k_a,1,\ldots 1)$ and $\ell = (1,\ldots,1,\ell_1,\ldots,\ell_b)$, let $\alpha = |\{i \in \mathbf{N} \mid k_i \geq a\}|$ and $\beta = |\{j \in \mathbf{N} \mid \ell_j \geq b\}|$. If $(k,\ell) \in \mathcal{U}_n$ then*

$$\sum_{i|k_i\geq a} k_i + \sum_{j|\ell_j\geq b} \ell_j \leq ab + 2 + \alpha(a-1) + \beta(b-1). \tag{14}$$

Note that (14) turns into (13) whenever $\alpha = a$ and $\beta = b$.

C Examples of men-optimal, women-optimal, and uniquely stable rank-profiles

Example 1 *Rank-profile* $((2,2),(1,1)) \in \mathcal{S}_2$ *is women-optimal, but not men-optimal. To see this, let us consider a preference list* $\sigma \in \Sigma_2$ *for which* $((2,2),(1,1))$ *is the rank-profile of a matching – as usual, we can assume that this is the main diagonal. Then,* σ *in fact is unique, see Figure 2, since the preferences of the other individuals in each row and column are determined uniquely. Thus, it is immediate to see that the women-oriented algorithm*

2 1	1 2
1 2	2 1

Figure 2: The unique preference list corresponding to rank-profile $((2,2),(1,1))$.

will terminate on the main diagonal, while the men-oriented one does not.

Example 2 *Rank-profile* $((1,1,3),(2,2,1))$ *is men-optimal, can be seen by considering the following preference list, given in Figure 3. It is easy to see that in the men-oriented algorithm only the third man makes more than one offer, and he is rejected twice (regardless of what are the* $*$ *entries in this preference-list).*

1 2	* 1	* *
* 1	1 2	* *
1 3	2 3	3 1

Figure 3: The partial preference list showing that $((1,1,3),(2,2,1))$ is men-optimal.

Example 3 *In contrast,* $((2,2,2),(2,2,2))$ *is not men-optimal. To see this, let us assume indirectly that there exists a preference list* $\sigma \in \Sigma_3$ *in which the men-oriented algorithm terminates with* $((2,2,2),(2,2,2))$ *in the main diagonal. In this instance we cannot have three different women as the first preferences of the three men, thus, without any loss of generality (up to renaming of men and women)* σ *must look like the one in Figure 4. However, in such an example we must have a man-woman pair, who are mutually the first choices of one another, as the second man and the first woman in Figure 4. Thus, our indirect assumption cannot be true, proving hence our claim.*

2 2	1 $*$	3 $*$
1 1	2 2	3 $*$
1 3	3 $*$	2 2

Figure 4: The partial preference list proving that $((2,2,2)(2,2,2))$ is not men-optimal.

D How bad a uniquely stable matching can be

Uniquely stable matchings can be almost as bad as stable matchings: no individual gets a better partner than his/her middle choice. More precisely this is stated in the next claim.

Proposition 13 *The following rank-profiles* $(k,\ell) = ((k_1,\ldots,k_{n-1},k_n),(\ell_1,\ldots,\ell_{n-1},\ell_n)) \in \mathcal{R}_n$ *are uniquely stable:*
(i) n *is even* $k_i = l_i = n/2$ *for* $i = 1,\ldots,n-1,$ *and* $n/2 \le k_n \le n,\ n/2 \le \ell_n \le n.$
(ii) n *is odd* $k_i = (n-1)/2,\ l_i = (n+1)/2$ *for* $i = 1,\ldots,n-1,$ *and*
$(n-1)/2 \le k_n \le n,\ \ (n+1)/2 \le \ell_n \le n.$ □

Let us remark that there are other bad uniquely stable rank-profiles, too. For example, $((2,2,2,2),(2,2,3,3))$ and $((2,2,2,2),(2,3,3,2))$ for $n=4$.

E The Spoiling Effect

Some interesting examples show that a bad stable matching can be improved after the deletion of a single "spoiling couple", or in other words, a single, always quarreling couple can "spoil the life" of an entire village just by moving there, even if most villagers dislike these individuals!

For instance, $\sigma \in \Sigma_6^*$ on Figure 5 has a unique stable diagonal matching with the corresponding uniquely stable rank-profile (k,ℓ), where $k = \ell = (6,3,3,3,3,3)$. We can observe that if we delete a couple (m_1,w_1) and keep the same order of preferences for the rest we obtain the instance $\sigma' \in \Sigma_5$ with the following men- and women-optimal stable matchings

$$\pi^M = \{(m_2,w_4),(m_3,w_5),(m_4,w_6),(m_5,w_1),(m_6,w_2)\},$$

$$\pi^W = \{(m_2,w_5),(m_3,w_6),(m_4,w_1),(m_5,w_2),(m_6,w_3)\},$$

for the corresponding rank-profiles of which we have $k^M = \ell^W = (1,1,1,1,1)$.

	w_1	w_2	w_3	w_4	w_5	w_6
m_1	6 6	2 4	1 5	3 6	4 6	5 6
m_2	4 2	3 3	2 4	1 5	6 1	5 2
m_3	5 1	4 2	3 3	2 4	1 5	6 1
m_4	6 3	5 1	4 2	3 3	2 4	1 5
m_5	6 4	1 6	5 1	4 2	3 3	2 4
m_6	6 5	2 5	1 6	5 1	4 2	3 3

Figure 5: An example $\sigma \in \Sigma_6^*$, where a pair (m_1, w_1) has a spoiling effect.

F Proofs of Theorems 1 2, 3, and Lemma 1

Proof of Theorem 1. According to the simplified notations, we need to show that $(k, \ell) \in \mathcal{S}_n$ if and only if the inequality

$$\sum_{i \in I} k_i + \sum_{j \in J} \ell_j \leq n|I| + n|J| + |I \cap J| - |I||J| \tag{15}$$

holds for all subsets $I, J \subseteq \mathbf{N}$.

To see the necessity of these inequalities, let us note that none of the (m_i, w_j) pairs for $i \neq j$ can be a breaking couple in any instance in which (k, ℓ) is a stable rank-profile. Thus, either $r_{m_i}(w_j) > r_{m_i}(w_i) = k_i$ or $r_{w_j}(m_i) > r_{w_j}(m_j) = \ell_j$ must hold for all pairs $i \in I$, $j \in J$, $i \neq j$. But there are only $n - k_i$ possible values for the rank of w_j in m_i-s ranking in the first case, and there are at most $n - \ell_j$ possible values for the rank of m_i in w_j-s ranking, in the second case. Thus

$$\sum_{i \in I} (n - k_i) + \sum_{j \in J} (n - \ell_j) \geq |I||J| - |I \cap J| \tag{16}$$

must hold indeed, implying (15).

For the reverse direction, we shall show that if for a given rank-profile (k, ℓ) the inequality (15) holds for all subsets $I, J \subseteq \mathbf{N}$, then we can construct an instance $\sigma \in \Sigma_n$ for which (k, ℓ) is the rank-profile of the stable matching $\pi = \{(m_i, w_i) \mid i \in \mathbf{N}\} \in \Pi(\sigma)$.

For this let us construct a bipartite graph $G = (A \cup B, E)$. The vertex set $A = \{(m_i, w_j) \mid i, j \in \mathbf{N},\ i \neq j\}$ correspond to all couples, who are not in π, while the set

$$B = \bigcup_{i=1}^{n} \{(m_i, z) \mid z = k_i + 1, ..., n\} \ \cup \ \bigcup_{j=1}^{n} \{(w_j, z) \mid z = \ell_j + 1, ..., n\}$$

represents the available "high" rank values for each individual. The edge set is defined by

$$\begin{aligned} E &= \{\big((m_i,w_j),(m_i,z)\big) \mid z = k_i+1,...,n,\ j \in \mathbf{N}\setminus\{i\}\} \\ &\cup \{\big((m_i,w_j),(w_j,z)\big) \mid z = \ell_j+1,...,n,\ i \in \mathbf{N}\setminus\{j\}\}. \end{aligned}$$

The edge $\big((m_i,w_j),(m_i,z)\big)$ represents a potential assignment $r_{m_i}(w_j) = z$, while edge $\big((m_i,w_j),(w_j,z)\big)$ corresponds to $r_{w_j}(m_i) = z$. Clearly, if there exists a matching in this graph which maps each vertex in A to B, then this assignment of rank values can be extended to an instance (by arbitrarily assigning the unassigned rank values for each individual) in which π is a stable matching. By Hall's theorem (see [11]), there exists such a perfect $A \mapsto B$ matching if and only if for every subset $S \subseteq A$ for the neighborhood $N(S) \subseteq B$ of S we have $|S| \leq |N(S)|$. To see this, let us note that by introducing

$$I = \{i \in \mathbf{N} \mid (m_i,w_j) \in S \text{ for some } j \in \mathbf{N}\},$$

$$J = \{j \in \mathbf{N} \mid (m_i,w_j) \in S \text{ for some } i \in \mathbf{N}\},$$

and defining

$$S' = \{(m_i,w_j) \mid i \in I,\ j \in J,\ i \neq j\}$$

we have $S \subseteq S'$ and $N(S) = N(S')$. Thus, it is enough to show that we have $|S'| \leq |N(S')|$, which is the same as (16), and hence it is implied by our assumption (15), completing the proof of the first part of the statement.

To see the complexity of testing membership in $\mathcal{S}_n$, let us note that the graph $G = (A \cup B, E)$ has $O(n^2)$ vertices and $O(n^3)$ edges, and thus a maximum matching in G can be found in $O(n^5)$ time (see [16]).

If the maximum matching is of size $|A|$, that is, it matches the set A perfectly, then we can fill in the missing preference values arbitrarily, and produce an instance σ for which $(k,\ell) \in \mathcal{S}(\sigma)$. Otherwise the algorithm outputs in the same time a subset $S \subseteq A$ for which $|S| > |N(S)|$ (see [16]). Defining I, J as above, we get in this way a pair of subsets for which the corresponding inequality (15) is violated by (k,ℓ). □

Proof of Theorem 2. To see the first inequality, let us note that by definition, there must exist a stable rank-profile $(k,\ell) \in \mathcal{S}_n$ for which $k_i \geq h(n)$ and $\ell_j \geq h(n)$ hold for all indices $i,j \in \mathbf{N}$. Thus, by Corollary 2 we must have

$$2nh(n) \leq \sum_{i\in\mathbf{N}} k_i + \sum_{j\in\mathbf{N}} \ell_j \leq n(n+1)$$

implying $h(n) \leq \frac{n+1}{2}$. Since $h(n)$ is an integer, the first inequality follows readily.

As we observed earlier, the second inequality is implied by the definitions, since $\Sigma_n^* \subseteq \Sigma_n$.

Finally, to see the equality $h^*(n) = \left\lfloor \frac{n}{2} \right\rfloor$, we need to find instances $\sigma^* \in \Sigma_n^*$, for which $\min_{i,j\in\mathbf{N}}\{k_i^*,\ell_j^*\} = \left\lfloor \frac{n}{2} \right\rfloor$ holds for the unique stable rank-profile $(k^*,\ell^*) \in \mathcal{U}(\sigma^*)$. In fact we can show a slightly stronger claim, namely that the rank-profile (k^*, ℓ^*) defined by

$$\begin{aligned} k_1^* = \ell_1^* &= n, \\ k_i^* &= \left\lfloor \tfrac{n}{2} \right\rfloor \quad \text{for } i = 2, ..., n, \text{ and} \\ \ell_j^* &= \left\lceil \tfrac{n}{2} \right\rceil \quad \text{for } j = 2, ..., n. \end{aligned}$$

is uniquely stable, i.e., $(k^*, \ell^*) \in \mathcal{U}_n$. To see this we shall construct an instance $\sigma^* \in \Sigma_n^*$ for which (k^*, ℓ^*) is the only stable rank-profile. Such an example not only maximizes $h(\sigma)$ over $\sigma \in \Sigma_n^*$, but also maximizes the sum of the rank-profile values, according to Corollary 2.

To describe σ^*, we introduce $a = \left\lfloor \frac{n}{2} \right\rfloor$ and $b = \left\lceil \frac{n}{2} \right\rceil$, and define the preferences of the individuals as follows:

$$\begin{aligned}
r_{m_1}(w_1) = r_{w_1}(m_1) &= n, && \\
r_{m_i}(w_i) &= a && \text{for} \quad i = 2, ..., n, \\
r_{w_i}(m_i) &= b && \text{for} \quad j = 2, ..., n, \\
r_{m_{j-p}}(w_j) = n - r_{w_j}(m_{j-p}) &= a - p && \text{for} \quad p = 1, ..., min\{a-1, j-1\}, \\
& && \qquad j = 2, ..., n, \\
r_{m_i}(w_{i-p}) = n - r_{w_{i-p}}(m_i) &= a + p && \text{for} \quad p = 1, ..., min\{n-1-a, i-1\}, \\
& && \qquad i = 2, ..., n, \\
r_{m_p}(w_{a+p}) = n + 1 - r_{w_{a+p}}(m_p) &= n + 2 - p && \text{for} \quad p = 2, ..., n-a, \\
r_{m_{b+p}}(w_p) = n + 1 - r_{w_p}(m_{b+p}) &= p - 1 && \text{for} \quad p = 2, ..., n-b, \\
r_{m_p}(w_1) &= n && \text{for} \quad p = b+1, ..., n, \\
r_{w_p}(m_1) &= n && \text{for} \quad p = a+1, ..., n.
\end{aligned}$$

All the undefined preferences (of man m_1 and woman w_1) can be filled in arbitrarily. Figure 6 shows an example for $n = 5$, in which case $a = 2$ and $b = 3$, and where $*$ stands for the undefined preferences, which can be filled arbitrarily.

We claim that for any instance σ^* defined in this way, we have $\pi^* = \{(m_1, w_1), ..., (m_n, w_n)\}$ as the unique stable matching with rank-profile $k^* = (n, a, ..., a)$ and $\ell^* = (n, b, ..., b)$.

To see this claim, it is enough to show by Fact 2 that both the men-oriented and women-oriented versions of the Gale-Shapley algorithm produce the same $\pi^M(\sigma^*) = \pi^* = \pi^W(\sigma^*)$ stable matching.

To verify this claim, let us observe a few properties of the instances constructed by the above definitions:

$$r_{m_i}(w_1) > a \qquad \text{for all men} \qquad i = 1, ..., n \tag{17a}$$

$$r_{w_j}(m_1) > b \qquad \text{for all women} \qquad j = 1, ..., n \tag{17b}$$

$$r_{w_j}(m_i) < b \qquad \text{if and only if} \qquad r_{m_i}(w_j) > a \tag{17c}$$

	w_1	w_2	w_3	w_4	w_5
m_1	5 5	1 4	$*$ 5	$*$ 5	$*$ 5
m_2	3 2	2 3	1 4	5 1	4 2
m_3	4 1	3 2	2 3	1 4	5 1
m_4	5 $*$	4 1	3 2	2 3	1 4
m_5	5 $*$	1 5	4 1	3 2	2 3

Figure 6: An example $\sigma^* \in \Sigma_5^*$ maximizing $h(\sigma^*)$.

Let us then consider the men-oriented procedure, and observe that in this algorithm no woman w_i rejects an offer from man m_i, for $i \geq 2$. To see this, consider the first such moment when say woman w_i rejects man m_i. Then woman w_i must have an offer from another man, say $m_{i'}$ such that $r_{w_i}(m_{i'}) < b$, which by (17c) means that $r_{m_{i'}}(w_i) > a$. Since men make offers in the increasing order of their preferences, without skipping anybody in that sequence, man $m_{i'}$ must have been rejected earlier by women $w_{i'}$, or $i' = 1$. However, by (17b) $i' \neq 1$, and thus we get a contradiction with our choice of i being the first index for which woman w_i rejects man m_i. Let us then note that according to the above argument and (17a), all men m_i, $i \geq 2$ make only offers to women w_j, $j \geq 2$. Moreover, man m_1 also makes his first $n-1$ offers to the same women. Thus, until all women w_j, $j \geq 2$ rejects man m_1, there will always be a woman with at least two offers. Let us further note that according to the first choices of men, all women w_j, $j \geq 2$ will get an offer right at the beginning of the algorithm. Thus, according to Fact 4, the algorithm will terminate when man m_1 makes an offer to woman w_1, i.e., after he is rejected by all women w_j, $j \geq 2$. Since women w_j, $j > a$ reject offers from m_1 in all situations (since $r_{w_j}(m_1) = n$ for all $j > a$), man m_1 has to be rejected by woman w_2 in order to force him to make eventually an offer to w_1. Due to the diagonally symmetric distribution of rank values, woman w_2 will reject man m_1 only if she gets an offer from man m_2, which happens only after man m_2 is rejected by woman w_3, which then can happen only if man m_3 made an offer to woman w_3, etc. Thus, the algorithm will terminate, only after each man m_i made an offer to woman w_i, $i = 1, ..., n$. As we observed at the beginning, no man m_i, $i \geq 2$ makes an offer to a woman w_j with $r_{m_i}(w_j) > a$, the main diagonal is the men-optimal stable matching, proving that indeed (k^*, ℓ^*) is the men-optimal stable rank-profile.

Since our construction is perfectly symmetric, analogous arguments show that (k^*, ℓ^*) is also the women-optimal stable rank-profile, and hence by Fact 2, it is the unique stable rank-profile for instance σ^*, completing the proof of Theorem 2. □

Proof of Theorem 3. Let us prove first the second part of the statement. Assume that

$(k', \ell') \in \mathcal{U}_n$, where $k' = (k, 1)$ and $\ell' = (\ell, n)$, and let $\sigma' \in \Sigma_n^*$ be a corresponding preference list for which this rank-profile corresponds to a unique stable matching.

Let us note first that by Proposition 3 we must have $k_j < n$ and $\ell_j < n$ for all $j = 1, ..., n-1$, since $((k, k_n), (\ell, n)) \in \mathcal{U}_n$. Thus, we have $(k, \ell) \in \mathcal{S}_{n-1}$.

Let us next consider the men-optimal procedure on σ'. Since at the end man m_n has the the best possible partner, he makes only one offer in this procedure. His partner at the end is woman w_n for whom man m_n is the worst possible partner. Consequently, she receives only one offer during the men-oriented algorithm, implying that $r_{m_i}(w_n) > k_i$ must hold for all indices $i = 1, ..., n-1$. Let us now create a new preference list $\sigma \in \Sigma_{n-1}$ from σ' by keeping the relative order of the first $n-1$ men's and women's rankings. In particular, for each man m_i we have the same k_i women as the top k_i ranked candidates, as in σ', for all $i = 1, ..., n-1$. Since the women have the same relative preference orders of the first $n-1$ men, as in σ', running the men-oriented algorithm on σ will run exactly the same way as it runs on σ', and hence it will terminate with (k, ℓ) as the men-optimal rank-profile of σ.

For the first part of the statement, assume that (k, ℓ) is a men-optimal rank-profile, and consider a corresponding preference list $\sigma \in \Sigma_{n-1}^*$. To construct $\sigma' \in \Sigma_n^*$ from the σ we define the preferences of the individuals as follows:

$$
\begin{array}{llll}
r'_{m_n}(w_n) & = k_n, & & \\
r'_{w_n}(m_n) & = n, & & \\
r'_{m_i}(w_n) & = k_i + 1 & & \text{for} \quad i = 1, ..., n-1, \\
r'_{w_j}(m_n) & = n & & \text{for} \quad j = 1, ..., n-1, \\
r'_{m_i}(w_j) & = r_{m_i}(w_j) + 1 & \text{if } r_{m_i}(w_j) \geq k_i + 1 & \text{for} \quad i = 1, ..., n-1, \\
 & & & \qquad\;\; j = 1, ..., n-1, \\
r'_{m_i}(w_j) & = r_{m_i}(w_j) & \text{if } r_{m_i}(w_j) \leq k_i & \text{for} \quad i = 1, ..., n-1, \\
 & & & \qquad\;\; j = 1, ..., n-1.
\end{array}
\tag{18}
$$

All the undefined preferences (of man m_n and woman w_n) can be filled in arbitrarily. Then, due to the properties of σ' in the men-oriented procedure only man m_n can make an offer to woman w_n, since man m_i, $i \leq n-1$ will make his first k_i offers only to women w_j, $j \leq n-1$, and all offers of man m_n to these women will be rejected, because man m_n is the worst choice for all these women. Hence the men-oriented algorithm on σ' yields the men-optimal rank-profile (k', ℓ').

To show that (k', ℓ') is also women-optimal for σ', let us consider the women-oriented procedure on this preference list, and denote my π the obtained women-optimal stable matching.

Let us show first that the pair (m_n, w_n) must belong to the women-optimal stable matching. Otherwise we have a pair $(m_n, w_j) \in \pi$ for some $j < n$, and consequently, $(m_j, w_i) \in \pi$ for some $i \neq j$. Since, all men gets not better in the women-optimal matching than in the men-optimal one, we must have $r_{m_j}(w_i) > r_{m_j}(w_j)$, due to the fact that $i \neq j$. But then, couple $(m_j, w_j) \notin \pi$ would be a breaking couple for π, by $r_{w_j}(m_n) = n$, contradicting the stability of π.

Since $(m_n, w_n) \in \pi$, woman w_n must make n offers in this algorithm, and thus all offers of her to the first $n-1$ men must be rejected during the algorithm. Due to the fact that we have $r_{m_j}(w_n) = k_j + 1$ by our construction, man m_j must receive an offer from his first k_j preferences (otherwise the offer of w_n to m_j would not be rejected). On the other hand, no men can get better in the women-oriented algorithm, than in the men-oriented one, implying that man m_j cannot get an offer from his first $k_j - 1$ preferences. Thus all men end up with the same partners as in the men-oriented algorithm. □

Proof of the Lemma 1. We shall prove this claim by induction on n. The claim does not make sense for $n < 3$. Clearly, the statement holds for $n = 3$, since in this case we must have $\ell_1 = \ell_2 = 1$ and both $((1,1,1),(2,2,1))$ and $((1,1,1),(2,2,2))$ are uniquely stable rank-profiles, see section C.

Assume now that the claim is shown for all dimensions below n, and consider a preference list $\sigma \in \Sigma^*_{n-1}$ for which $((1,...,1),(\ell_1,...,\ell_{n-1})) \in \mathcal{S}(\sigma)$. We know that the women-oriented algorithm for this σ terminates when the last man receives his first offer. Without any loss of generality, we can assume that man m_{n-1} is this last man. Thus, no woman w_j, $j < n-1$ makes an offer to man m_{n-1}, and hence we must have

$$r_{w_j}(m_{n-1}) > \ell_j \quad \text{for all } j = 1,...,n-2. \tag{19}$$

Let us now create a new preference list σ' from σ by adding man m_n, woman w_n, and defining their preferences as follows:

$$\begin{aligned} r_{w_j}(m_n) &= 1 \quad \text{for all} \quad j = 1,...,n-1, \\ r_{w_j}(m_i) &= r_{w_j}(m_i) + 1 \quad \text{for all} \quad i = 1,...,n-1, \text{ and } j = 1,...,n-1, \\ r_{w_n}(m_n) &= \ell_n, \\ r_{w_n}(m_{n-1}) &= n, \\ r_{m_i}(w_n) &= n \quad \text{for all} \quad i = 1,...,n-1, \\ r_{m_n}(w_{n-1}) &= 2, \end{aligned} \tag{20}$$

and where we keep the preferences of the first $n-1$ men over the first $n-1$ women, as in σ, and where we fill in the preferences of man m_n and woman w_n, not defined in (20), arbitrarily. We claim that the women-oriented algorithm for this enlarged σ' terminates with $((1,...,1),(\ell_1+1,...,\ell_{n-1}+1,\ell_n))$. Since this is clearly the men-optimal rank-profile for σ', the lemma follows from this claim.

To see this claim, observe first that in the women-optimal matching all women must get at least as good as in the men-optimal one, and hence the rank of w_j's partner cannot be more than $\ell_j + 1$, for $j = 1,...,n-1$. Observe next that the first $n-1$ women all make an offer first to man m_n, and the first $n-2$ are rejected by him, according to the preferences given in (20). Due to these two observations, and to (19), the first $n-2$ women must have in the women-optimal matching a partner from among the first $n-2$ men. Thus, the partner of woman w_n in the women-optimal matching can be either man m_{n-1} or man m_n. However, man m_{n-1} is her last choice, and in the men-optimal matching she gets her ℓ_nth choice, where $\ell_n < n$ by our assumption, therefore she must have man m_n as her partner in the women-optimal matching. This implies that woman w_{n-1} must also have a partner from among the first $n-1$ men. Thus all women w_j, $j = 1, ..., n-1$ end up with men m_j, $j = 1, ..., n-1$. Since in σ' these men and women have the same relative preference orders as in σ, they must end up with the same optimal matching, as in σ. This proves our claim, and completes the proof of the lemma. □