RUTCOR
RESEARCH
REPORT

Complexity of Bilevel Coherent Risk Programming

Jonathan Eckstein[a]

RRR 17-2012, April 2012

RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a]Department of Management Science and Information Systems (MSIS) and RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003

Rutcor Research Report
RRR 17-2012, April 2012

Complexity of Bilevel Coherent Risk Programming

Jonathan Eckstein

Abstract. This paper considers a bilevel programming approach to applying coherent risk measures to extended two-stage stochastic programming problems. This formulation technique avoids the time-inconsistency issues plaguing naive models and the incomposability issues which cause time-consistent formulations to have complicated, hard-to-explain objective functions. Unfortunately, the analysis here shows that such bilevel formulations, when using the standard mean-semideviation and average-value-at-risk measures, are $\mathcal{NP}$-hard. While not necessarily indicating that solution of such models is impractical, these results suggest that it may prove difficult and will likely require some kind of implicit enumeration method.

Acknowledgements: This work was supported by a grant from the Air Force Office of Scientific Research, award FA9550-11-1-0164. The author would like to thank Andrzej Ruszczyński for his assistance.

1 Introduction

This paper concerns the computational complexity of risk-averse stochastic optimization in which decisions and the information available evolve together over time. The simplest situation in which the theory may be evolved is an extended two-stage stochastic decision problem, which may be described as follows: first, we have a finite probability space Ω with elements ω, partitioned into a collection of sets $\mathcal{S}$ whose elements we call *scenarios*. We use $\mathrm{P}\{\omega\}$ to denote the probability of sample path ω, and let $\mathrm{P}\{S\} = \sum_{\omega \in S} \mathrm{P}\{\omega\}$ denote the probability of scenario S. For simplicity, we will assume that the problem is linear:

- At the first stage, the decision maker sets the values of the decision variables $x_1 \in \mathbb{R}^{n_1}$, with costs $c_1^\top x_1$ and subject to constraints $A_{11}x_1 \leq b_1$.
- Next, it is revealed which scenario $S \in \mathcal{S}$ has occurred.
- Then, the decision maker sets the recourse decision variables $x_2(S) \in \mathbb{R}^{n_2(S)}$, with costs $c_2(S)^\top x_2(S)$ and subject to constraints $A_{21}(S)x_1 + A_{22}(S)x_2(S) \leq b_2(S)$.
- It is next revealed which sample path $\omega \in S$ has occurred.
- Finally, the decision maker may set the values of the further recourse decision variables $x_3(\omega) \in \mathbb{R}^{n_3(\omega)}$, with costs $c_3(\omega)^\top x_3(\omega)$ and subject to constraints

$$A_{31}(\omega)x_1 + A_{32}(\omega)x_2(S) + A_{33}(\omega)x_3(\omega) \leq b_3(\omega).$$

We let $x_2(\mathcal{S})$ denote the concatenation of all the vectors $x_2(S)$, $S \in \mathcal{S}$, and $x_3(\Omega)$ denote the concatenation of all the vectors $x_3(\omega)$, $\omega \in \Omega$; for any particular $S \in \mathcal{S}$, we also let $x_3(S)$ denote the concatenation of all the vectors $x_2(\omega)$, $\omega \in S$.

In the classical case in which the decision maker seeks to minimize the expected value of the costs incurred, the above situation translates into a standard extensive-form linear program over the variables x_1, $x_2(\mathcal{S})$, and $x_3(\Omega)$. Here, we are instead concerned with a risk-averse decision maker wishing to employ a coherent measure of risk [2, 6, 16] in their objective function. We may think of such a risk measure ρ as a function mapping a random variable to a scalar "certainty equivalent". Such a mapping ρ is called a *coherent risk measure* if it has the following properties:

Monotonicity. If $X_1 \leq X_2$ (that is, the value of X_1 does not exceed the value of X_2 in any possible outcome), then $\rho(X_1) \leq \rho(X_2)$.

Convexity. If $\alpha \in [0, 1]$, then $\rho\big(\alpha X_1 + (1-\alpha)X_2\big) \leq \alpha\rho(X_1) + (1-\alpha)\rho(X_2)$.

Positive homogeneity. If $\alpha \geq 0$, then $\rho(\alpha X) = \alpha\rho(X)$.

Translation invariance. For any $t \in \mathbb{R}$, we have $\rho(X + t) = \rho(X) + t$.

The classical expected-value mapping $\mathbb{E}[\,\cdot\,]$ satisfies these axioms, but so do many other functions. Two popular choices are the mean-semideviation

$$\mathrm{MSD}_\gamma : X \mapsto \mathbb{E}[X] + \gamma\mathbb{E}\big[[X - \mathbb{E}[X]]_+\big] \qquad \text{for } \gamma \in [0,1], \tag{1}$$

and *average value at risk* [1, 7, 14, 15, 22], also called *conditional value at risk*,

$$\mathrm{AVaR}_\alpha : X \mapsto \frac{1}{\alpha}\int_{1-\alpha}^{1} F_X^{-1}(\nu)\,d\nu, \tag{2}$$

where F_X denotes the cumulative distribution function of X and F_X^{-1} denotes its "lower" inverse

$$F_X^{-1}(\nu) = \inf\{x \mid F(x) \geq \nu\} = \inf\{x \mid \mathrm{P}\{X \leq x\} \geq \nu\}. \tag{3}$$

When X has a continuous distribution, an alternative expression is

$$\mathrm{AVaR}_\alpha(X) = \mathbb{E}\big[X \,\big|\, X \geq F_X^{-1}(1-\alpha)\big],$$

that is, the expected value of X given that it lies above its $1-\alpha$ quantile; (2) correctly generalizes this expression to the discrete and general cases.

In discussing the application of such risk measures to dynamic decision problems, it will be helpful to introduce the following "lottery" notation: given two random variables X_1 and X_2, we define

$$X_1 \wedge_p X_2 = BX_1 + (1-B)X_2 = \begin{cases} X_1, & \text{with probability } p, \\ X_2, & \text{with probability } 1-p, \end{cases}$$

where B is a Bernoulli random variable with mean p, independent of X_1 and X_2. Similarly, for some discrete index set $\mathcal{I} = \{i_1, i_2, i_3, \ldots\}$ with a corresponding probability distribution $p(\cdot)$ and indexed collection of random variables X_i, $i \in \mathcal{I}$, we define

$$\bigwedge_{i\in\mathcal{I}}^{p(i)} X_i \;=\; \textstyle\bigwedge_{i\in\mathcal{I}}^{p(i)} X_i \;=\; \begin{cases} X_{i_1}, & \text{with probability } p(i_1), \\ X_{i_2}, & \text{with probability } p(i_2), \\ X_{i_3}, & \text{with probability } p(i_3), \\ \quad\vdots & \qquad\vdots \end{cases},$$

independently of the values of the X_i. Here, the "$\wedge$" symbol is meant to suggest the branching of a tree of outcomes.

In this notation, the total cost incurred in the decision problem described above is the random variable

$$Z\big(x_1, x_2(\mathcal{S}), x_3(\Omega)\big) = c_1^\top x_1 + \bigwedge_{S\in\mathcal{S}}^{\mathrm{P}\{S\}} \left(c_2(S)^\top x_2(S) + \bigwedge_{\omega\in S}^{\frac{\mathrm{P}\{\omega\}}{\mathrm{P}\{S\}}} c_3(\omega)^\top x_3(\omega) \right).$$

In this formula, the quotient $\frac{\mathrm{P}\{\omega\}}{\mathrm{P}\{S\}}$ arises because it is the conditional probability $\mathrm{P}\{\omega \mid S\}$ of sample path ω occurring, given that scenario $S \ni \omega$ has occurred.

At first glance, it might appear that the most natural way to apply a general coherent risk measure ρ to the extended two-stage decision problem described above would be to simply use the objective function $\rho\big(Z\big(x_1, x_2(\mathcal{S}), x_3(\Omega)\big)\big)$, resulting in the formulation

$$\begin{array}{lll} \min & c_1^\top x_1 + \rho\Big(\bigwedge_{S\in\mathcal{S}}^{\mathrm{P}\{S\}} \big(c_2(S)^\top x_2(S) + \bigwedge_{\omega\in S}^{\frac{\mathrm{P}\{\omega\}}{\mathrm{P}\{S\}}} c_3(\omega)^\top x_3(\omega)\big)\Big) & \\ \mathrm{ST} & A_{11}x_1 \le b_1 & \\ & A_{21}(S)x_1 + A_{22}(S)x_2(S) \le b_2(S) & \forall\, S \in \mathcal{S} \\ & A_{31}(\omega)x_1 + A_{32}(\omega)x_2(S) + A_{33}(\omega)x_3(\omega) \le b_3(\omega) & \forall\, S \in \mathcal{S}, \forall\, \omega \in S, \end{array} \tag{4}$$

where the deterministic term $c_1^\top x_1$ may be brought outside the risk measure due to its translation invariance property. Unfortunately, except for very limited, extreme choices of ρ, this model has the undesirable property of *time inconsistency* [17, 21]: specifically, if $\big(x_1^*, x_2^*(\mathcal{S}), x_3^*(\Omega)\big)$ is an optimal solution to (4), then its subvector $\big(x_2^*(S), x_3^*(S)\big)$ may not form an optimal solution to the scenario-S subproblem

$$\begin{array}{ll} \min & \rho\big(c_2(S)^\top x_2(S) + \bigwedge_{\omega\in S}^{\frac{\mathrm{P}\{\omega\}}{\mathrm{P}\{S\}}} c_3(\omega)^\top x_3(\omega)\big) \\ \mathrm{ST} & A_{21}(S)x_1^* + A_{22}(S)x_2(S) \le b_2(S) \\ & A_{31}(\omega)x_1^* + A_{32}(\omega)x_2(S) + A_{33}(\omega)x_3(\omega) \le b_3(\omega) \quad \forall\, \omega \in S. \end{array}$$

Thus, if scenario S occurs and the decision maker still views the risk measure ρ as reflective of their risk-return trade-off preferences, they could in fact choose $x_2(S)$ and $x_3(S)$ differently from the values that helped determine the supposedly optimal first-stage decision vector x_1^*. This property makes a general model like (4) difficult to justify.

In the existing literature, the standard approach to avoiding this kind of difficulty is to use a time-consistent *dynamic* measure of risk [5, 13, 18, 20]. In our current setting and notation, this approach is equivalent to changing the objective function of (4) to

$$c_1^\top x_1 + \rho_1\left(\bigwedge_{S\in\mathcal{S}}^{\mathrm{P}\{S\}}\left(c_2(S)^\top x_2(s) + \rho_2\left(\bigwedge_{\omega\in S}^{\frac{\mathrm{P}\{\omega\}}{\mathrm{P}\{S\}}} c_3(\omega)^\top x_3(\omega)\right)\right)\right), \tag{5}$$

where ρ_1 and ρ_2 are two (possibly identical) coherent risk measures. This approach yields a convex problem while avoiding the difficulties of time inconsistency, but it also has drawbacks. The objective mapping given by (5) no longer expressible as a simple coherent risk mapping applied to the single random variable $Z\big(x_1, x_2(\mathcal{S}), x_3(\Omega)\big)$; instead, it must expressed as a function of a sequence of several random variables over nested probability spaces. Furthermore, the objective function will no longer have a relatively simple form like (1) or (2), because such risk measures are not *composable*, that is, a nested application of MSD risk measures in a form like (5) does not simplify to an MSD risk measure, and similarly for AVaR risk measures.

In fact, in a finite probability space, it was recently shown in [21] that the only coherent risk measures that are composable in this manner are the expected value mapping $\mathbb{E}[\,\cdot\,]$ and the "worst outcome" risk measure

$$\max X = \max\left\{x \in \mathbb{R} \;\middle|\; \mathrm{P}\{X = x\} > 0\right\}$$

(in an infinite probability space, we would substitute essential supremum operation "ess sup" for the simple "max" operation). There is a more general class of "entropic" risk measures that is composable [11], but it is still quite restrictive, and in general violates the positive homogeneity axiom of standard coherent risk measures. Expected value and worst outcome are extreme risk measures, with expectation being completely risk-neutral, and the worst-outcome measure being maximally risk-averse. For these very simple risk measures, furthermore, the objective mapping (5) and the objective of (4) simplify to the same function, and therefore model (4) becomes time-consistent.

Thus, when one uses (5) with, for example, $\rho_1 = \rho_2 = \mathrm{MSD}_\gamma$ or $\rho_1 = \rho_2 = \mathrm{AVaR}_\alpha$, the resulting objective function does not have a simple expression or interpretation. This phenomenon is a serious obstacle to the adoption of multistage decision models involving coherent risk measures, since decision makers may resist using a model whose objective function does not have a simply articulated interpretation as do (1) and (2).

This paper concerns an alternative approach in which the decision maker adopts a particular coherent risk measure ρ as reflective of their risk profile, and we assume that after scenario S is revealed, they will make all further decisions to optimize ρ applied to all remaining costs. At the first stage, we seek to optimize ρ as applied to all costs, under the assumption that we will act optimally in the second stage, with the information that will be available then. Applied to our extended two-stage problem setting, this approach produces a model of the following form:

$$\begin{array}{lll} \min & c_1^\top x_1 + \rho\Big(\bigwedge_{S\in\mathcal{S}}^{\mathrm{P}\{S\}} \big(c_2(S)^\top x_2(s) + \bigwedge_{\omega\in S}^{\frac{\mathrm{P}\{\omega\}}{\mathrm{P}\{S\}}} c_3(\omega)^\top x_3(\omega)\big)\Big) & \\ \mathrm{ST} & A_{11}x_1 \le b_1 & \\ & \big(x_2(S), x_3(S)\big) \in \mathcal{F}_S(x_1) & \forall\, S \in \mathcal{S}, \end{array} \tag{6}$$

where, for all $S \in \mathcal{S}$,

$$\begin{array}{rcll} \mathcal{F}_S(x_1) & = & \mathrm{Arg\ min} & c_2(S)^\top x_2(s) + \rho\big(\bigwedge_{\omega\in S}^{\frac{\mathrm{P}\{\omega\}}{\mathrm{P}\{S\}}} c_3(\omega)^\top x_3(\omega)\big) \\ & & \mathrm{ST} & A_{21}(S)x_1 + A_{22}(S)x_2(S) \le b_2(S) \\ & & & A_{31}(\omega)x_1 + A_{32}(\omega)x_2(S) + A_{33}(\omega)x_3(\omega) \le b_3(\omega) \quad \forall\, \omega \in S. \end{array} \tag{7}$$

This kind of model allows the primary objective funtion to be a simple coherent risk measure of the decision maker's choice, applied to a single random variable reflecting the costs from all stages, without introducing time inconsistency. The price, however, is that it is a bilevel optimization problem, with some constraints of the "leader" problem (6) being defined in terms of the optimal solution set of the parametric "follower" problem (7). In essence, the decision maker is playing a kind Stackelberg game "against himself", or more accurately a collection of possible future versions of himself after having obtained more information. We call problems of the form (6)-(7) *bilevel risk programs* (BLRP's).

Even the simplest form of bilevel programming, bilevel linear programming, has long been known to be $\mathcal{NP}$-hard [3, 4, 9, 10]. However, even for risk measures ρ that can be expressed in a linear programming form, such as MSD_γ and AVaR_α, it is not immediately clear whether the bilevel linear programming problems corresponding to (6)-(7) are completely general; in particular, the objective functions of the leader and follower are very strongly correlated.

This paper shows that for $\rho = \text{MSD}_\gamma$, $\gamma \in (0,1)$, and $\rho = \text{AVaR}_\alpha$, $\alpha \in (0,1)$, solving problems of the form (6)-(7) is $\mathcal{NP}$-hard. While they should not be taken as proving that attempts to solve any particular problem instance of the form (6)-(7) would be fruitless, these results do suggest that one should not attempt to find solution methods with polynomial worst-case running time, and that successful solution or approximate solution methods for these kinds of models will likely involve some form of implicit enumeration, as do most practical approaches to solving instances of other kinds of $\mathcal{NP}$-hard problems.

Formally, we define a parameterized class of problems as follows:

Problem Class $\text{BLRP}(\rho)$ **: Bilevel Risk Programming**

Parameter:	A coherent risk measure ρ.
Input:	All expressed over the rational numbers $\mathbb{Q}$: • A finite probability space Ω, along with a partition $\mathcal{S}$ and probabilities $\text{P}\{\omega\}$ for all $\omega \in \Omega$ • Vectors c_1 and b_1, and a matrix A_{11} • For each $S \in \mathcal{S}$, vectors $c_2(S)$ and $b_2(S)$, and matrices $A_{21}(S)$, $A_{22}(S)$ • For each $\omega \in \Omega$, vectors $c_3(\omega)$ and $b_3(\omega)$, and matrices $A_{31}(\omega)$, $A_{32}(\omega)$, $A_{33}(\omega)$.
Output:	Any optimal solution $\big(x_1, x_2(\mathcal{S}), x_3(\Omega)\big)$ to the problem (6)-(7).

When the coherent risk measure parameter ρ itself has a parameter, as in the case of MSD_γ and AVaR_α, we use a set-valued parameter to denote the version of BLRP in which this risk measure parameter, restricted to the rationals, is encoded as part of the problem input. Thus, $\text{BLRP}(\text{MSD}_{(0,1]})$ denotes the class of all $\text{BLRP}(\text{MSD}_\gamma)$ problems, with $\gamma \in (0,1] \cap \mathbb{Q}$ appended to the problem input, and $\text{BLRP}(\text{AVaR}_{(0,1)})$ denotes the class of all $\text{BLRP}(\text{AVaR}_\alpha)$ problems, with $\alpha \in (0,1) \cap \mathbb{Q}$ appended to the problem input.

The remainder of this paper is organized as follows:

- Section 2 revisits the complexity theory of bilevel linear programming, giving some specialized results used in the remaining analysis

- Drawing on the results of Section 2, Section 3 then shows that the problem classes $\text{BLRP}(\text{MSD}_\gamma)$, for $\gamma \in (0,1] \cap \mathbb{Q}$, and $\text{BLRP}(\text{MSD}_{(0,1]})$ are $\mathcal{NP}$-hard

- Using similar techniques, Section 4 shows that the problem classes BLRP(AVaR_α), for $\alpha \in (0,1) \cap \mathbb{Q}$, and BLRP($\text{AVaR}_{(0,1)}$) are $\mathcal{NP}$-hard.
- Section 5 makes some brief concluding remarks.

2 Bilevel Linear Programming Complexity Revisited

The classical bilevel linear programming problem may be expressed as

$$\begin{array}{lll} \min & f_1^\top y_1 + f_2^\top y_2 & \\ \text{ST} & y_2 \in \ \text{Arg min} & \tilde{f}_2^\top y_2 \\ & \qquad\ \ \text{ST} & B_1 y_1 + B_2 y_2 \le r. \end{array} \tag{8}$$

This problem long been known to be $\mathcal{NP}$-hard [3, 4, 9, 10], with the known proofs relying on reducing various combinatorial problems to (8) with $\tilde{f}_2 = -f_2$, a special case that may be called *oppositional programming*. We begin by focusing on this special case, with the additional restriction that the y_1 must lie in a bounded set.

Problem Class BOLP **: Bounded Oppositional Linear Programming**

Input:	Vectors $f_1 \in \mathbb{Q}^{n_1}$, $f_2 \in \mathbb{Q}^{n_2}$, and $r \in \mathbb{Q}^m$, matrices $B_1 \in \mathbb{Q}^{m \times n_1}$ and $B_2 \in \mathbb{Q}^{m \times n_2}$, and $\zeta \in \mathbb{Q}_+$.
Output:	Any optimal solution (y_1, y_2) of the problem $$\begin{array}{lll} \min & f_1^\top y_1 + f_2^\top y_2 & \\ \text{ST} & y_2 \in \ \text{Arg min} & -f_2^\top y_2 \\ & \qquad\ \ \text{ST} & B_1 y_1 + B_2 y_2 \le r \\ & & \|y_2\|_\infty \le \zeta. \end{array}$$

In principle, the $\mathcal{NP}$-hardness of BOLP may be ascertained by remarking that the existing proofs of the $\mathcal{NP}$-hardness of bilevel linear programming use the special case $\tilde{f}_2 = -f_2$, with all decision variables bounded. For completeness, we give a new proof that is similar in basic spirit to [9], but is simpler and involves reduction from a less complicated decision problem (although at the cost of not demonstrating *strong* $\mathcal{NP}$-hardness as in [9]).

Proposition 1 BOLP *is $\mathcal{NP}$-hard.*

Proof. We prove the result by reduction from the number partition problem (NPP), one of the classical $\mathcal{NP}$-complete decision problems [8]:

Problem Class NPP **: Number Partition**

Input:	$a_1, \ldots, a_n \in \mathbb{Z}$.
Output:	"Yes" if there exists $J \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in J} a_i = \frac{1}{2} \sum_{i=1}^n a_i$, and otherwise "no".

Given an instance $a_1, \ldots, a_n \in \mathbb{Z}$ of NPP, consider the following bilevel program with leader variables $u_0, u_1, \ldots, u_n$ and follower variables $w = (w_1, \ldots, w_n)$:

$$
\begin{array}{ll}
\min & u_0 + \sum_{i=1}^n w_i \\
\text{ST} & u_0 \geq \sum_{i=1}^n a_i u_i - \frac{1}{2}\sum_{i=1}^n a_i \\
& u_0 \geq \frac{1}{2}\sum_{i=1}^n a_i - \sum_{i=1}^n a_i u_i \\
& 0 \leq u_1, \ldots, u_n \leq 1 \\
& w \in \begin{array}[t]{lll} \text{Arg min} & -\sum_{i=1}^n w_i & \\ \text{ST} & w_i \leq u_i & i = 1, \ldots, n \\ & w_i \leq 1 - u_i & i = 1, \ldots, n. \end{array}
\end{array}
\tag{9}
$$

We claim that (9) has an optimal value of zero if and only if the answer to the partition instance is "yes". To prove this claim, we note that the optimality of w for the follower program is equivalent to $w_i = \min\{u_i, 1 - u_i\}$, $i = 1, \ldots, n$, and the optimal value of u_0 is $\left|\sum_{i=1}^n a_i u_i - \frac{1}{2}\sum_{i=1}^n a_i\right|$, so (9) is equivalent to the (nonconvex) single-level optimization problem

$$
\begin{array}{ll}
\min & \left|\sum_{i=1}^n a_i u_i - \frac{1}{2}\sum_{i=1}^n a_i\right| + \sum_{i=1}^n \min\{u_i, 1 - u_i\} \\
\text{ST} & 0 \leq u_1, \ldots, u_n \leq 1.
\end{array}
\tag{10}
$$

Both terms in the objective function of this problem are nonnegative, the first being zero whenever $\sum_{i=1}^n a_i u_i = \frac{1}{2}\sum_{i=1}^n a_i$ and the second being zero whenever $u = (u_1, \ldots, u_n)$ is a binary vector. Thus, if answer to the partition problem is "yes", then there must exist $u \in \{0, 1\}^n$ making the objective of (10) zero, which must be optimal. On the other hand, if the answer to the partition problem is "no", then all choices of $u \in [0, 1]^n$ make the objective of (10) positive, either because $\sum_{i=1}^n a_i u_i \neq \frac{1}{2}\sum_{i=1}^n a_i$ or because at least one of $u_1, \ldots, u_n$ is fractional. Since (10) involves the optimization of a continuous function over a compact set, it must acheive its minimum, and hence the minimum value must be positive. This establishes the claim; it then remains only to observe that problem (9) can be put into the BOLP form by appropriate choices of f_1, f_2, B_1, B_2, and r, with $\zeta = 1$, and that number of bits needed to encode such an instance of BOLP is polynomially bounded in the number of bits needed to encode $a_1, \ldots, a_n$. □

Direct reduction of BOLP or classic bilevel linear programming to problem classes of the form BRLP(ρ) appears to be an intricate task. Thus, we first consider another class of restricted bilevel linear problems and show that it is also $\mathcal{NP}$-hard by reduction from BOLP. In this class of problems, we break the follower variables into two blocks, with the only difference between the leader and follower objectives being that the coefficients for one of the two blocks are scaled by a rational parameter $\beta \neq 1$.

Problem Class BBSBLP(β) **:**
Bounded Block-Scaled Bilevel Linear Programming

Parameter: $\beta \in \mathbb{Q}\backslash\{1\}$.

Input: Vectors $g_1 \in \mathbb{Q}^{n_1}$, $g_2 \in \mathbb{Q}^{n_2}$, $g_3 \in \mathbb{Q}^{n_3}$, and $t \in \mathbb{Q}^m$, matrices $C_1 \in \mathbb{Q}^{m\times n_1}$, $C_2 \in \mathbb{Q}^{m\times n_2}$, and $C_3 \in \mathbb{Q}^{m\times n_3}$ and $\eta_2, \eta_3 \in \mathbb{Q}_+$.

Output: Any optimal solution (y_1, y_2, y_3) of the problem

$$\begin{array}{lll} \min & g_1^\top x_1 + g_2^\top x_2 + g_3^\top x_3 & \\ \text{ST} & (x_2, x_3) \in \ \text{Arg min} & g_2^\top x_2 + \beta g_3^\top x_3 \\ & \qquad\qquad\quad \text{ST} & C_1x_1 + C_2x_2 + C_3x_3 \leq t \\ & & \|x_2\|_\infty \leq \eta_2 \\ & & \|x_3\|_\infty \leq \eta_3. \end{array}$$

We now show that although the leader and follower objectives BBSBLP(β) may appear very similar — if $\beta > 0$, in particular, then the all corresponding leader and follower objective coefficients have the same sign — the possibly small difference between the two objectives is enough to make the problem class $\mathcal{NP}$-hard. The technique used is inspired by the analysis of [12] showing that solutions of bilevel programs need not be Pareto optimal for the leader and follower objectives as long as they are not colinear.

Proposition 2 *For any rational $\beta \neq 1$, the problem class* BBSBLP(β) *is $\mathcal{NP}$-hard.*

Proof. We proceed by reduction from BOLP. Consider any instance $(f_1, f_2, r, B_1, B_2, \zeta)$ of BOLP. We create a corresponding instance of BBSBLP(β) as follows: first, we set

$$g_1 = f_1 \qquad\qquad g_2 = \left(1 - \tfrac{2}{1-\beta}\right) f_2 \qquad\qquad g_3 = f_2. \tag{11}$$

We then set C_1, C_2, C_3, and t to be equivalent to the constraints

$$B_1x_1 + B_2x_2 \leq r \qquad\qquad x_3 = \left(\tfrac{2}{1-\beta}\right)x_2.$$

Specifically, we may accomplish this by setting

$$C_1 = \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} \qquad C_2 = \begin{bmatrix} B_2 \\ -\left(\frac{2}{1-\beta}\right)I \\ \left(\frac{2}{1-\beta}\right)I \end{bmatrix} \qquad C_3 = \begin{bmatrix} 0 \\ I \\ -I \end{bmatrix} \qquad t = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}. \tag{12}$$

Finally, we set $\eta_2 = \zeta$ and $\eta_3 = \left|\frac{2}{1-\beta}\right|\zeta$. Using (11) and that $x_3 = \left(\frac{2}{1-\beta}\right)x_2$ in any feasible solution, the follower problem objective may be rewritten as

$$\left(1 - \tfrac{2}{1-\beta}\right) f_2^\top x_2 + \beta f_2^\top \left(\tfrac{2}{1-\beta}\right) x_2 = \left(\tfrac{-\beta-1}{1-\beta} + \tfrac{2\beta}{1-\beta}\right) f_2^\top x_2 = \left(\tfrac{\beta-1}{1-\beta}\right) f_2^\top x_2 = -f_2^\top x_2.$$

The constraint $\|x_3\|_\infty \le \eta_3$ is equivalent to $\left\|\left(\frac{2}{1-\beta}\right)x_2\right\|_\infty \le \left|\frac{2}{1-\beta}\right|\zeta$, which is exactly the same as the constraint $\|x_2\|_\infty \le \eta_2 = \zeta$, so the follower problem may be written

$$\begin{array}{ll} \min & -f_2^\top x_2 \\ \text{ST} & B_1x_1 + B_2x_2 \le r \\ & \|x_2\|_\infty \le \zeta. \end{array} \tag{13}$$

Next consider the leader objective, which we may rewrite using (11) and $x_3 = \left(\frac{2}{1-\beta}\right)x_2$ as

$$f_1^\top x_1 + \left(1 - \tfrac{2}{1-\beta}\right) f_2^\top x_2 + f_2^\top\left(\tfrac{2}{1-\beta}\right)x_2 = f_1^\top x_1 + \left(\tfrac{-\beta-1}{1-\beta} + \tfrac{2}{1-\beta}\right) f_2^\top x_2 = f_1^\top x_1 + f_2^\top x_2.$$

Combining this observation with the form of the follower problem in (13), it follows that the constructed BBSBLP(β) instance is completely equivalent to the BOLP instance. It remains only to observe that the number of bits needed to encode the BBSBLP(β) instance is clearly bounded by a polynomial function of the number of bits needed to encode the BOLP instance. □

3 Complexity of Bilevel MSD Risk Models

We now consider the complexity of the problem class BLRP(MSD$_\gamma$), for $\gamma \in (0,1) \cap \mathbb{Q}$. Introducing some additional "helper" variables E_1 and $E_2(S)$, $S \in \mathcal{S}$, problem (6)-(7) may in the case of $\rho = \text{MSD}_\gamma$ be expressed as

$$\begin{array}{lll} \min & c_1^\top x_1 + E_1 + \gamma \sum_{S\in\mathcal{S}} \sum_{\omega\in S} \text{P}\{\omega\} \left[c_2(S)^\top x_2(S) + c_3(\omega)^\top x_3(\omega) - E_1\right]_+ & \\ \text{ST} & A_{11}x_1 \le b_1 & \\ & E_1 = \sum_{S\in\mathcal{S}} \text{P}\{S\}\, c_2(S)^\top x_2(S) + \sum_{\omega\in\Omega} \text{P}\{\omega\}\, c_3(\omega)^\top x_3(\omega) & \\ & \big(x_2(S), x_3(S)\big) \in \mathcal{F}_S(x_1) & \forall\, S \in \mathcal{S}, \end{array} \tag{14}$$

where, for each $S \in \mathcal{S}$, $\mathcal{F}_S(x_1)$ denotes the set of $\big(x_2(S), x_3(S)\big)$ portions of all optimal solutions $\big(x_2(S), x_3(S), E_2(S)\big)$ to the scenario-S follower problem

$$\begin{array}{lll} \min & c_2(S)^\top x_2(S) + E_2(S) + \gamma \frac{\text{P}\{\omega\}}{\text{P}\{S\}} \left[c_3(\omega)^\top x_3(\omega) - E_2(S)\right]_+ & \\ \text{ST} & A_{21}(S)x_1 + A_{22}(S)x_2 \le b_2(S) & \\ & A_{31}(\omega)x_1 + A_{32}(\omega)x_2(S) + A_{33}(\omega)x_3(\omega) \le b_3(\omega) & \forall\, \omega \in S \\ & E_2(S) = \sum_{\omega\in S} \left(\frac{\text{P}\{\omega\}}{\text{P}\{S\}}\right) c_3(\omega)^\top x_3(\omega). & \end{array} \tag{15}$$

We now show how to construct a subclass of BLRP(MSD$_\gamma$) problems that is very similar to BBSBLP(β) for an appropriate choice of β. Consider a three-element probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, partitioned into two scenarios $S_1 = \{\omega_1, \omega_2\}$ and $S_2 = \{\omega_3\}$. For two parameters $p_1, p_2 \in (0,1) \cap \mathbb{Q}$, we set up a stochastic decision problem as follows, and as illustrated in Figure 1:

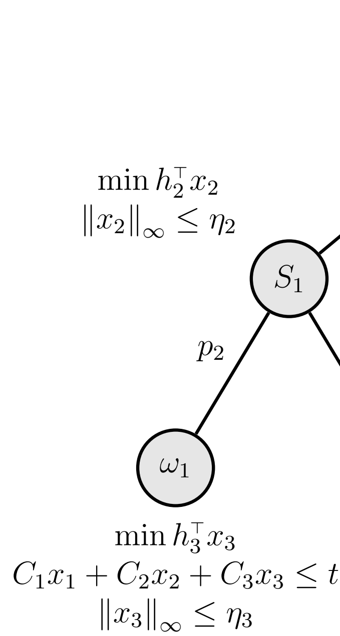

Figure 1: Scenario tree for reduction of BBSBLP(β) instances to BLRP(MSD$_\gamma$) instances.

- The stage-one variables are $x_1 \in \mathbb{R}^{n_1}$, with corresponding cost coefficients $h_1 \in \mathbb{Q}^{n_1}$.
- Scenario S_1 has probability p_1, and hence scenario S_2 has probability $1 - p_1$.
- In scenario S_1, the recourse decision variables are $x_2 \in \mathbb{R}^{n_2}$; here, we omit the "(S_1)" from the original notation $x_2(S_1)$ for brevity, because the only other recourse variables $x_2(S_2)$ will be essentially fixed. The cost coefficient vector is $h_2 \in \mathbb{Q}^{n_2}$, and the constraints are $\|x_2\|_\infty \leq \eta_2$. Given that S_1 occurs, the conditional probability of sample path ω_1 is p_2, and hence the conditional probablity of sample path ω_2 is $1 - p_2$.
  - In sample path ω_1, the final recourse decision variables are $x_3 \in \mathbb{R}^{n_3}$; we omit the "(ω_1)" following x_3 for brevity, because $x_3(\omega_2)$ and $x_3(\omega_3)$ will be essentially fixed. The cost coefficient vector is $h_3 \in \mathbb{Q}^{n_3}$, and the constraints are

$$C_1x_1 + C_2x_2 + C_3x_3 \leq t \qquad\qquad \|x_3\|_\infty \leq \eta_3.$$

  - In sample path ω_2, the final stage incurs a fixed cost of $K_2 = \eta_3\|h_3\|_1$.
- Scenario S_2, from which the only possible final-stage consequence is ω_3, incurs a fixed cost of

$$K_1 = \eta_2\|h_2\|_1 + \left(1 + \tfrac{2p_1p_2}{1-p_1}\right) K_2. \tag{16}$$

Clearly, it is possible configure the input data of a BLRP(MSD$_\gamma$) instance so that the above arrangement is acheived, and the space required is polynomial in the space required to encode $(h_1, h_2, h_3, C_1, C_2, C_3, \eta_2, \eta_3)$. For example, to make sample path ω_2 incur a fixed cost of K_2, we may set $n_3(\omega_2) = 1$, $c_3(\omega_2) = [K_2]$, and

$$A_{31}(\omega_2) = \begin{bmatrix} 0^\top \\ 0^\top \end{bmatrix} \qquad A_{32}(\omega_2) = \begin{bmatrix} 0^\top \\ 0^\top \end{bmatrix} \qquad A_{33}(\omega_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad b_3(\omega_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

effectively yielding the constraint $x_3(\omega_2) = 1$, with cost $K_2 x_3(\omega_2) = K_2$.

The intent of this construction is that K_2 is sufficiently large that the conditional expected value $E_2(S_1)$ of the stage-three costs given that scenario S_1 occurs must always be worse than the outcome along sample path ω_1 for all feasible values of the decision variables. Similarly, although the analysis is more complicated, K_1 is taken sufficiently large that the expected value E_1 at the root of the scenario tree must always be worse than the objective in either of the sample paths ω_1 and ω_2 for all feasible settings of the decision variables. We will now show that these properties mean that the resulting BLRP(MSD$_\gamma$) problem is equivalent to a problem very similar to BBSBLP(β) for an appropriate choice of β.

First, for any rational $\gamma \in (0, 1]$, specializing the formulation (14)-(15) of BLRP(MSD$_\gamma$) to the setting just described yields the following problem, where we abbreviate $E_2(S_1)$ to simply E_2, since $E_2(S_2)$ is a constant:

$$\begin{array}{ll} \min & h_1^\top x_1 + E_1 + \gamma\big(p_1 p_2 \left[h_2^\top x_2 + h_3^\top x_3 - E_1\right]_+ \\ & \qquad\qquad + p_1(1-p_2)\left[h_2^\top x_2 + K_2 - E_1\right]_+ + (1-p_1)\left[K_1 - E_1\right]_+ \big) \\ \text{ST} & E_1 = p_1 p_2 (h_2^\top x_2 + h_3^\top x_3) + p_1(1-p_2)(h_2^\top x_2 + K_2) + (1-p_1)K_1 \\ & (x_2, x_3) \in \begin{array}[t]{ll} \text{Arg min} & h_2^\top x_2 + E_2 + \gamma\left(p_2\left[h_3^\top x_3 - E_2\right]_+ + (1-p_2)\left[K_2 - E_2\right]_+\right) \\ \text{ST} & E_2 = p_2 h_3^\top x_3 + (1-p_2)K_2 \\ & C_1 x_1 + C_2 x_2 + C_3 x_3 \le t \\ & \|x_2\|_\infty \le \eta_2 \\ & \|x_3\|_\infty \le \eta_3. \end{array} \end{array} \tag{17}$$

Now consider the follower problem of (17). By the choice of K_2, we have that for any feasible value of x_3,

$$h_3^\top x_3 \le \|h_3\|_1 \|x_3\|_\infty \le \eta_3 \|h_3\|_1 = K_2,$$

and, since E_2 is a convex combination of $h_3^\top x_3$ and K_2, we therefore always have

$$h_3^\top x_3 \le E_2 \le K_2.$$

Hence, the first $[\,\cdot\,]_+$ term in the follower objective is always zero, and the second $[\,\cdot\,]_+$ term may be written as

$$K_2 - E_2 = K_2 - \big(p_2 h_3^\top x_3 + (1-p_2)K_2\big) = p_2\big(K_2 - h_3^\top x_3\big).$$

Substituting for E_2 and the $[\,\cdot\,]_+$ terms, we obtain the equivalent follower objective function

$$\begin{aligned} & h_2^\top x_2 + p_2 h_3^\top x_3 + (1-p_2)K_2 + \gamma p_2 \cdot 0 + \gamma(1-p_2)p_2\big(K_2 - h_3^\top x_3\big) \\ = \; & h_2^\top x_2 + \big(p_2 - \gamma(1-p_2)p_2\big)h_3^\top x_3 + \big((1-p_2) + \gamma(1-p_2)p_2\big)K_2 \\ = \; & h_2^\top x_2 + p_2(1 - \gamma + \gamma p_2)h_3^\top x_3 + (1-p_2)(1+\gamma p_2)K_2. \end{aligned}$$

Discarding the constant term $(1-p_2)(1+\gamma p_2)K_2$ from the objective, the follower problem thus reduces to

$$\begin{array}{ll} \min & h_2^\top x_2 + p_2(1-\gamma+\gamma p_2)h_3^\top x_3 \\ \text{ST} & C_1x_1 + C_2x_2 + C_3x_3 \le t \\ & \|x_2\|_\infty \le \eta_2 \\ & \|x_3\|_\infty \le \eta_3. \end{array} \tag{18}$$

We now consider the leader problem in (17). We claim that the choice of K_1 in (16) is sufficiently large for all feasible values of (x_2, x_3) that we have

$$E_1 \ge h_2^\top x_2 + K_2 \ge h_2^\top x_2 + h_3^\top x_3. \tag{19}$$

The second inequality in (19) follows immediately from $h_3^\top x_3 \le \|h_3\|_1\|x_3\|_\infty \le \eta_3\|h_3\|_1 = K_2$, so it remains to prove the first inequality. We note that

$$\begin{aligned} E_1 &= p_1\big(h_2^\top x_2 + p_2h_3^\top x_3 + (1-p_2)K_2\big) + (1-p_1)K_1 \\ &= p_1\big(h_2^\top x_2 + p_2h_3^\top x_3 + (1-p_2)K_2\big) + (1-p_1)\left(\eta_2\|h_2\|_1 + \left(1+\tfrac{2p_1p_2}{1-p_1}\right)K_2\right) \\ &= \big[p_1h_2^\top x_2 + (1-p_1)\eta_2\|h_2\|_1\big] + p_1p_2h_3^\top x_3 + p_1(1-p_2)K_2 + (1-p_1+2p_1p_2)K_2. \end{aligned} \tag{20}$$

Since $h_2^\top x_2 \le \|h_2\|_1\,\|x_2\|_\infty \le \eta_2\,\|h_2\|_1$, we have

$$p_1h_2^\top x_2 + (1-p_1)\eta_2\|h_2\|_1 \ge h_2^\top x_2. \tag{21}$$

Next, we observe that

$$h_3^\top x_3 \ge -\|h_3\|_1\,\|x_3\|_\infty \ge -\|h_3\|_1\,\eta_3 = -K_2. \tag{22}$$

Substituting (21) and (22) into (20), we have

$$\begin{aligned} E_1 &\ge h_2^\top x_2 - p_1p_2K_2 + p_1(1-p_2)K_2 + (1-p_1+2p_1p_2)K_2 \\ &= h_2^\top x_2 + \big(-p_1p_2 + p_1 - p_1p_2 + 1 - p_1 + 2p_1p_2\big)K_2 \\ &= h_2^\top x_2 + K_2, \end{aligned}$$

establishing (19). It follows immediately that the first two $[\,\cdot\,]_+$ terms in the leader objective of (17) are always zero. Noting that

$$K_1 > \eta_2\,\|h_2\|_1 + K_2 \ge h_2^\top x_2 + K_2 \ge h_2^\top x_2 + h_3^\top x_3$$

for any feasible (x_2, x_3), it follows from E_1 being a convex combination of $h_2^\top x_2 + h_3^\top x_3$, $h_2^\top x_2 + K_2$, and K_1 that $K_1 \ge E_1$. Therefore, $[K_1 - E_1]_+ = K_1 - E_1$, and the leader objective of (17) may be written, where "$\simeq$" denotes equivalence up to a constant among functions of (x_1, x_2, x_3), as

$$\begin{aligned} &h_1^\top x_1 + E_1 + \gamma(1-p_1)(K_1 - E_1) \\ \simeq\ &h_1^\top x_1 + (1-\gamma+\gamma p_1)E_1 \\ =\ &h_1^\top x_1 + (1-\gamma+\gamma p_1)\big(p_1h_2^\top x_2 + p_1p_2h_3^\top x_3 + p_1(1-p_2)K_2 + (1-p_1)K_1\big) \\ \simeq\ &h_1^\top x_1 + (1-\gamma+\gamma p_1)\big(p_1h_2^\top x_2 + p_1p_2h_3^\top x_3\big) \\ =\ &h_1^\top x_1 + p_1(1-\gamma+\gamma p_1)\big(h_2^\top x_2 + p_2h_3^\top x_3\big). \end{aligned}$$

Combining this form of the leader objective with (18), we may express the entire problem as

$$\begin{array}{lll} \min & h_1^\top x_1 + p_1(1-\gamma+\gamma p_1)h_2^\top x_2 + p_1 p_2(1-\gamma+\gamma p_1)h_3^\top x_3 & \\ \text{ST} & (x_2,x_3) \in \;\; \operatorname{Arg\,min} \;\; h_2^\top x_2 + p_2(1-\gamma+\gamma p_2)h_3^\top x_3 & \\ & \qquad\qquad\;\; \text{ST} \qquad C_1x_1 + C_2x_2 + C_3x_3 \le t & \\ & \qquad\qquad\qquad\qquad \|x_2\|_\infty \le \eta_2 & \\ & \qquad\qquad\qquad\qquad \|x_3\|_\infty \le \eta_3. & \end{array} \tag{23}$$

This problem form has exactly the same constraint structure as BBSBLP(β). We can now exploit the differing relative scaling of the $h_2^\top x_2$ and $h_3^\top x_3$ terms in the two objective functions of (23) to reduce BBSBLP(β) to BLRP(MSD$_\gamma$) for an appropriate choice of β, thus proving that BLRP(MSD$_\gamma$) is $\mathcal{NP}$-hard.

Proposition 3 *For any $\gamma \in (0,1] \cap \mathbb{Q}$, the problem class* BLRP(MSD$_\gamma$) *is $\mathcal{NP}$-hard.*

Proof. The proof is by reduction from BBSBLP($1-\gamma/2$); note that since $\gamma > 0$, it follows that $1-\gamma/2 \neq 1$, and thus that BBSBLP($1-\gamma/2$) is $\mathcal{NP}$-hard by Proposition 2. Also, since $\gamma \le 1$, we have $1-\gamma/2 > 0$. Now consider any instance $(g_1, g_2, g_3, C_1, C_2, C_3, t, \eta_2, \eta_3)$ of BBSBLP($1-\gamma/2$), fix $p_1 = p_2 = 1/2$, and set

$$\begin{aligned} h_1 &= g_1 \\ h_2 &= \left(\frac{1}{p_1(1-\gamma+\gamma p_1)}\right) g_2 = \left(\frac{2}{1-\gamma/2}\right) g_2 \\ h_3 &= \left(\frac{1}{p_1 p_2(1-\gamma+\gamma p_1)}\right) g_3 = \left(\frac{4}{1-\gamma/2}\right) g_3. \end{aligned}$$

We now use $(h_1, h_2, h_3, C_1, C_2, C_3, t, \eta_2, \eta_3)$ to construct a BLRP(MSD$_\gamma$) problem instance of the form (17); the space required to encode (h_1, h_2, h_3) is polynomial in the space required to encode (g_1, g_2, g_3), so the size of the resulting BLRP(MSD$_\gamma$) instance is polynomial in the encoding sized of the BBSBLP($1-\gamma/2$) instance $(g_1, g_2, g_3, C_1, C_2, C_3, t, \eta_2, \eta_3)$. From the analysis above, the resulting BLRP(MSD$_\gamma$) instance is equivalent to (23). Substituting the above choices of h_1, h_2, and h_3 into the leader objective of (23), along with $p_1 = p_2 = 1/2$, we obtain the leader objective $g_1^\top x_1 + g_2^\top x_2 + g_3^\top x_3$, exactly as in BBSBLP($1-\gamma/2$). Making the same substitutions into the follower objective of (23), we obtain

$$\begin{aligned} h_2^\top x_2 + p_2(1-\gamma+\gamma p_2)h_3^\top x_3 &= \left(\frac{2}{1-\gamma/2}\right) g_2^\top x_2 + \left(\frac{(1/2)(1-\gamma/2)\cdot 4}{1-\gamma/2}\right) g_3^\top x_3 \\ &= \left(\frac{2}{1-\gamma/2}\right) g_2^\top x_2 + 2 g_3^\top x_3. \end{aligned}$$

Applying the positive scaling factor $(1-\gamma/2)/2$ to both terms in its objective does not make any difference to the solution set of follower problem, so we can equivalently use the follower objective $g_2^\top x_2+(1-\gamma/2)g_3^\top x_3$. In summary, the BLRP(MSD$_\gamma$) instance we have constructed is equivalent to the problem

$$\begin{array}{lllll} \min & g_1^\top x_1 + g_2^\top x_2 + g_3^\top x_3 & & & \\ \text{ST} & (x_2, x_3) \in & \text{Arg min} & g_2^\top x_2 + (1-\gamma/2) g_3^\top x_3 \\ & & \text{ST} & C_1 x_1 + C_2 x_2 + C_3 x_3 \le t \\ & & & \|x_2\|_\infty \le \eta_2 \\ & & & \|x_3\|_\infty \le \eta_3, \end{array}$$

precisely the BBSBLP$(1-\gamma/2)$ instance encoded by $(g_1, g_2, g_3, C_1, C_2, C_3, t, \eta_2, \eta_3)$. Since the BLRP(MSD$_\gamma$) instance encoding size is polynomial in the size of the BBSBLP$(1-\gamma/2)$ instance, existence of a polynomial-time solution algorithm for BLRP(MSD$_\gamma$) would imply polynomial-time algorithm for the $\mathcal{NP}$-hard problem class BBSBLP$(1-\gamma/2)$. □

Corollary 4 *The problem class* BLRP(MSD$_{(0,1]}$)*, with γ encoded as part of the problem input, is also $\mathcal{NP}$-hard.*

Proof. Consider any instance of the problem class BLRP(MSD$_{1/2}$), which is $\mathcal{NP}$-hard by Proposition 3. Appending $\gamma = 1/2$ to the encoding of this instance only increases the problem size by a constant, so a polyomial-time algorithm for BLRP(MSD$_{(0,1]}$) would imply a polynomial-time algorithm for BLRP(MSD$_{1/2}$). □

4 Complexity of Bilevel AVaR Risk Models

We now consider the complexity of bilevel models using the AVaR$_\alpha$ risk measure instead of the MSD$_\gamma$ risk measure; the overall technique of the analysis is similar to the MSD$_\gamma$ case, but involves a reduction from BBSBLP(0), regardless of the value of α.

To set up the analysis, we construct a simple scenario tree similar to that of Section 3, but with different probabilities, all based on the parameter α, and sample path ω_3 representing a highly desirable outcome rather than an highly undesirable one:

- The stage-one variables are $x_1 \in \mathbb{R}^{n_1}$, with corresponding cost coefficients $h_1 \in \mathbb{Q}^{n_1}$.
- Scenario S_1 has probability α, and hence scenario S_2 has probability $1-\alpha$.
- Scenario S_1 has the recourse decision variables $x_2 \in \mathbb{R}^{n_2}$, with corresponding cost coefficient vector $h_2 \in \mathbb{Q}^{n_2}$ and subject to the constraint $\|x_2\|_\infty \le \eta_2$. Given that S_1 occurs, the conditional probability of sample path ω_1 is $1-\alpha$, and hence the conditional probablity of sample path ω_2 is α.
  - In sample path ω_1, the final recourse decision variables are $x_3 \in \mathbb{R}^{n_3}$, with corresponding cost coefficients $h_3 \in \mathbb{Q}^{n_3}$ and subject to the constraints

$$C_1 x_1 + C_2 x_2 + C_3 x_3 \le t \qquad\qquad \|x_3\|_\infty \le \eta_3.$$

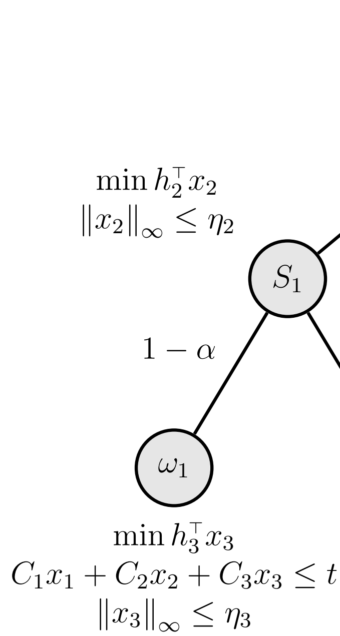

Figure 2: Scenario tree for reduction of BBSBLP(0) instances to BLRP(AVaR_α) instances.

  - In sample path ω_2, the final stage incurs a fixed cost of $K_2 = \eta_3\|h_3\|_1 + 1$.

- Scenario S_2, from which the only possible final-stage consequence is ω_3, incurs a fixed cost of $-K_1$ (that is, a benefit), where $K_1 = \eta_2\|h_2\|_1 + K_2 = \eta_2\|h_2\|_1 + \eta_3\|h_3\|_1 + 1$.

This slightly modified scenario structure is shown in Figure 2. As long as $\alpha \in \mathbb{Q}$, the space required to express the resulting BLRP(AVaR_α) problem instance is polynomially bounded in the space required to express $(h_1, h_2, h_3, C_1, C_2, C_2, t, \eta_2, \eta_3)$. This BLRP($\text{AVaR}_\alpha$) instance has the form

$$\begin{array}{lll} \min & h_1^\top x_1 + \text{AVaR}_\alpha\Big(\big(h_2^\top x_2 + (h_3^\top x_3 \wedge_{1-\alpha} K_2)\big) \wedge_\alpha -K_1\Big) & \\ \text{ST} & (x_2, x_3) \in \begin{array}{ll} \text{Arg min} & h_2^\top x_2 + \text{AVaR}_\alpha(h_3^\top x_3 \wedge_{1-\alpha} K_2) \\ \text{ST} & C_1x_1 + C_2x_2 + C_3x_3 \le t \\ & \|x_2\|_\infty \le \eta_2 \\ & \|x_3\|_\infty \le \eta_3, \end{array} \end{array} \tag{24}$$

As before, we write x_2 instead of $x_2(S_1)$, since the value of $x_2(S)$ is only material for $S = S_1$, and similarly we write x_3 instead of $x_3(\omega_1)$.

The second term in the follower objective of (24) may be written $\text{AVaR}_\alpha\big(Z_3(x_3)\big)$, where $Z_3(\,\cdot\,)$ denotes the random-variable-valued function given by

$$Z_3(x_3) = h_3^\top x_3 \wedge_{1-\alpha} K_2 = \begin{cases} h_3^\top x_3, & \text{with probability } 1-\alpha \\ K_2, & \text{with probability } \alpha. \end{cases}$$

To provide some intuition, the problem instance has been constructed so that all quantiles of $Z_3(x_3)$ above the $1-\alpha$ quantile are simply K_2 for any feasible value of x_3, and the $\mathrm{AVaR}_\alpha(\,\cdot\,)$ term in the follower objective of (24) is equivalent to a constant. We now verify this claim using the formal definition (2) of the AVaR_α risk measure. Noting that for any feasible value of x_3, we have

$$h_3^\top x_3 \le \|h_3\|_1 \|x_3\|_\infty \le \eta_3 \|h_3\|_1 < \eta_3 \|h_3\|_1 + 1 = K_2, \tag{25}$$

the lower inverse cumulative function $F^{-1}_{Z_3(x_3)}(\,\cdot\,)$ as defined through (3) takes the form

$$F^{-1}_{Z_3(x_3)}(\nu) = \begin{cases} h_3^\top x_3, & \text{if } \nu \in (0, 1-\alpha] \\ K_2, & \text{if } \nu \in (1-\alpha, 1], \end{cases}$$

hence we have

$$\mathrm{AVaR}_\alpha\big(Z_3(x_3)\big) = \frac{1}{\alpha}\int_{1-\alpha}^{1} F^{-1}_{Z_3(x_3)}(\nu)\, d\nu = \frac{1}{\alpha}\int_{1-\alpha}^{1} K_2\, d\nu = \tfrac{1}{\alpha}(\alpha K_2) = K_2,$$

since the $F^{-1}_{Z_3(x_3)}(\nu) = K_2$ throughout $[1-\alpha, 1]$ except for the singleton $\{1-\alpha\}$, which has measure zero. Thus, the follower objective of (24) may be replaced, without any change to the follower problem, by $h_2^\top x_2 + K_2$, or dropping the constant, equivalently simply $h_2^\top x_2$.

Next, consider now the leader objective in (24). The second term in this objective may be written as $\mathrm{AVaR}_\alpha\big(Z_2(x_2, x_3)\big)$, where $Z_2(\,\cdot\,,\,\cdot\,)$ denotes the random-variable-valued function given by

$$Z_2(x_2, x_3) = \big(h_2^\top x_2 + (h_3^\top x_3 \wedge_{1-\alpha} K_2)\big) \wedge_\alpha - K_1 = \begin{cases} -K_1, & \text{with probability } 1-\alpha \\ h_2^\top x_2 + h_3^\top x_3, & \text{with probability } \alpha(1-\alpha) \\ h_2^\top x_2 + K_2, & \text{with probability } \alpha^2. \end{cases}$$

We note that

$$-K_1 = -\eta_2 \left\|h_2\right\|_1 - \eta_3 \left\|h_3\right\|_1 - 1 < h_2^\top x_2 + h_3^\top x_3$$

for any feasible values of x_2 and x_3. Further, since we have already established in (25) that $h_3^\top x_3 < K_2$ for all feasible values of K_3, we have that

$$-K_1 < h_2^\top x_2 + h_3^\top x_3 < h_2^\top x_2 + K_2$$

for all feasible values of x_2 and x_3. Thus, the lower inverse cumulative function $F^{-1}_{Z_2(x_2,x_3)}(\,\cdot\,)$ takes the form

$$F^{-1}_{Z_2(x_2,x_3)}(\nu) = \begin{cases} -K_1 & \text{if } \nu \in (0, 1-\alpha] \\ h_2^\top x_2 + h_3^\top x_3 & \text{if } \nu \in (1-\alpha, 1-\alpha^2] \\ h_2^\top x_2 + K_2 & \text{if } \nu \in (1-\alpha^2, 1]. \end{cases}$$

Applying the definition of the AVaR_α risk measure, we have

$$\begin{aligned}
\mathrm{AVaR}_\alpha\big(Z_2(x_2,x_3)\big) &= \frac{1}{\alpha}\int_{1-\alpha}^{1} F^{-1}_{Z_2(x_2,x_3)}(\nu)\,d\nu \\
&= \frac{1}{\alpha}\left(\int_{1-\alpha}^{1-\alpha^2}(h_2^\top x_2 + h_3^\top x_3)\,d\nu + \int_{1-\alpha^2}^{1}(h_2^\top x_2 + K_2)\,d\nu\right) \\
&= \frac{1}{\alpha}\Big(\big(1-\alpha^2-(1-\alpha)\big)(h_2^\top x_2 + h_3^\top x_3) + \alpha^2(h_2^\top x_2 + K_2)\Big) \\
&= \tfrac{1}{\alpha}\big(\alpha h_2^\top x_2 + (\alpha-\alpha^2)h_3^\top x_3 + \alpha^2 K_2\big) \\
&= h_2^\top x_2 + (1-\alpha)h_3^\top x_3 + \alpha K_2.
\end{aligned}$$

Substituting this equation into the leader objective of (24) and discarding the constant αK_2, and also substituting the already established equivalent follower objective $h_2^\top x_2$, we arrive at the problem instance

$$\begin{array}{lll l}
\min & h_1^\top x_1 + h_2^\top x_2 + (1-\alpha)h_3^\top x_3 & & \\
\mathrm{ST} & (x_2,x_3) \in & \operatorname{Arg\,min} & h_2^\top x_2 \\
 & & \mathrm{ST} & C_1x_1 + C_2x_2 + C_3x_3 \le t \\
 & & & \|x_2\|_\infty \le \eta_2 \\
 & & & \|x_3\|_\infty \le \eta_3.
\end{array} \tag{26}$$

This problem is essentially identical in form to BBSBLP(0), a property that we now exploit.

Proposition 5 *For any $\alpha \in (0,1)\cap\mathbb{Q}$, the problem class* $\mathrm{BLRP}(\mathrm{AVaR}_\alpha)$ *is $\mathcal{NP}$-hard.*

Proof. Consider any instance $(g_1, g_2, g_3, C_1, C_2, C_3, t, \eta_2, \eta_3)$ of BBSBLP(0), and set

$$h_1 = g_1 \qquad\qquad h_2 = g_2 \qquad\qquad h_3 = \left(\tfrac{1}{1-\alpha}\right) g_3.$$

Now construct a $\mathrm{BLRP}(\mathrm{AVaR}_\alpha)$ instance of the form (24), which by the immediately preceding analysis is equivalent to (26). Substituting the definitions of h_1, h_2, h_3 above into (26), we obtain the equivalent problem

$$\begin{array}{lll l}
\min & g_1^\top x_1 + g_2^\top x_2 + g_3^\top x_3 & & \\
\mathrm{ST} & (x_2,x_3) \in & \operatorname{Arg\,min} & g_2^\top x_2 \\
 & & \mathrm{ST} & C_1x_1 + C_2x_2 + C_3x_3 \le t \\
 & & & \|x_2\|_\infty \le \eta_2 \\
 & & & \|x_3\|_\infty \le \eta_3,
\end{array}$$

which is precisely the original BBSBLP(0) problem instance. The space needed to encode the $\mathrm{BLRP}(\mathrm{AVaR}_\alpha)$ instance is polynomially bounded in the space to encode the BBSBLP(0) instance, so existence of a polynomial-time algorithm for $\mathrm{BLRP}(\mathrm{AVaR}_\alpha)$ would imply the existence of a polynomial-time algorithm for BBSBLP(0), which is $\mathcal{NP}$-hard. □

Corollary 6 *The problem class* $\mathrm{BLRP}(\mathrm{AVaR}_{(0,1)})$*, with the quantile parameter α encoded as part of the problem input, is also $\mathcal{NP}$-hard.*

Proof. Similar to the proof of Corollary 4. □

5 Brief Concluding Remarks

Finally, we make some straightforward concluding observations. To begin with, we note that the parameter ranges $\gamma \in (0, 1]$ for the MSD_γ risk measure and $\gamma \in (0, 1)$ for the AVaR_α risk measure cover all the cases of serious interest. In particular, when $\rho = \mathbb{E}[\,\cdot\,]$, problem (6)-(7) simplifies to a linear program, and thus any instance of $\text{BLRP}\big(\mathbb{E}[\,\cdot\,]\big)$ reduces to a linear programming problem of size polynomial in its input data. Since the MSD risk measure reduces to expectation when $\gamma = 0$ — that is, $\text{MSD}_0 = \mathbb{E}[\,\cdot\,]$ — we thus have $\text{BLRP}(\text{MSD}_0) = \text{BLRP}\big(\mathbb{E}[\,\cdot\,]\big) \in \mathcal{P}$. Furthermore, MSD_γ is not a coherent measure of risk for $\gamma > 1$, so the analysis of $\gamma \in (0, 1]$ covers all the cases of interest for the MSD risk measure. In the case of AVaR_α, the limiting cases $\alpha \to 0$ and $\alpha \to 1$ correspond to the expection and worst-outcome risk measures $\mathbb{E}[\,\cdot\,]$ and "max" respectively. In both of these cases, problem (6)-(7) simplifies to a linear program of size polynomial in the input data of the $\text{BLRP}(\,\cdot\,)$ instance and the resulting problem class is thus polynomial-time solvable. Thus, $\alpha \in (0, 1)$ comprises all cases of interest for the AVaR_α risk measure.

Again, as noted in Section 1, the results above should not be interpreted as indicating that any attempt to obtain optimal or provably near-optimal solutions to problem instances of the form $\text{BLRP}(\rho)$ should be abandoned. However, they do suggest that research into solution methods should not focus on algorithms with polynomial worst-case run-time guarantees (except perhaps for heuristics), and that some aspect of implicit enumeration is likely to be needed.

Although these $\mathcal{NP}$-hardness results cover only two specific families of risk measures, it seems reasonable to conjecture that they will extend to any family of risk measures which has a polyhedral dual form [2, 6, 19]; this subject is a matter for further research.

Finally, we note that reducibility of the "oppositional" problem form BOLP respectively through $\text{BBSBLP}(1-\gamma/2)$ or BBSBLP(0) to $\text{BLRP}(\text{MSD}_\gamma)$ or $\text{BLRP}(\text{AVaR}_\alpha)$ indicates that it is possible to contrive extended two-stage stochastic programming problems instances in which the use of the MSD_γ or AVaR_α risk measure breaks time consistency in a particularly dramatic way. Specifically, $\text{BLRP}(\,\cdot\,)$ instances constructed using such two-step reductions from BOLP possess a second-stage scenario which, if it is revealed to have occured, effectively reverses second-stage portion of the objective as compared to the perspective of the first stage. This phenonenon underscores that when using non-composable measures of risk, the revelation of partial information can dramatically change a decision maker's preferenences among the remaining courses of action.

References

[1] C. Acerbi. Spectral measures of risk: a coherent representation of subjective risk aversion. *J. Banking Finance*, 26, 2002.

[2] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Math. Finance*, 9:203–228, 1999.

[3] J. F. Bard. Some properties of the bilevel programming problem. *J. Optim. Theory Appl.*, 68(2):371–378, 1991.

[4] O. Ben-Ayed and C. E. Blair. Computational difficulties of bilevel linear programming. *Oper. Res.*, 38(3):556–560, 1990.

[5] P. Cheridito, F. Delbaen, and M. Kupper. Dynamic monetary risk measures for bounded discrete-time processes. *Electron. J. Prob.*, 11:57–106, 2006.

[6] F. Delbaen. Coherent risk measures on general probability spaces. In *Advances in finance and stochastics*, pages 1–37. Springer, Berlin, 2002.

[7] H. Föllmer and A. Schied. *Stochastic finance: an introduction in discrete time, second edition*. DeGruyter, Berlin, 2004.

[8] M. R. Garey and D. S. Johnson. *Computers and intractability: a guide to the theory of NP-completeness*. W. H. Freeman, San Francisco, 1979.

[9] P. Hansen, B. Jaumard, and G. Savard. New branch-and-bound rules for linear bilevel programming. *SIAM J. Sci. Statist. Comput.*, 13(5):1194–1217, 1992.

[10] R. G. Jeroslow. The polynomial hierarchy and a simple model for competitive analysis. *Math. Programming*, 32(2):146–164, 1985.

[11] M. Kupper and W. Schachermayer. Representation results for law invariant time consistent functions. *Math. Financ. Econ.*, 2(3):189–210, 2009.

[12] P. Marcotte and G. Savard. A note on the Pareto optimality of solutions to the linear bilevel programming problem. *Comput. Oper. Res.*, 18(4):355–359, 1991.

[13] F. Riedel. Dynamic coherent risk measures. *Stoch. Processes and Their Appl.*, 112:185–200, 2004.

[14] R. T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *J. Risk*, 2:21–42, 2000.

[15] R. T. Rockafellar and S. Uryasev. Conditional value-at-risk for general loss distributions. *J. Banking Finance*, 26:1442–1471, 2002.

[16] R. T. Rockafellar, S. Uryasev, and M. Zabarankin. Generalized deviations in risk analysis. *Finance Stoch.*, 10(1):51–74, 2006.

[17] A. Ruszczyński. Risk-averse dynamic programming for Markov decision processes. *Math. Programming*, 125(2):235–261, 2010.

[18] A. Ruszczyński and A. Shapiro. Conditional risk mappings. *Math. Oper. Res.*, 31(3):544–561, 2006.

[19] A. Ruszczyński and A. Shapiro. Optimization of convex risk functions. *Math. Oper. Res.*, 31(3):433–452, 2006.

[20] G. Scandolo. *Risk Measures in a Dynamic Setting*. PhD thesis, Università degli Studi di Milano, Milan, Italy, 2003.

[21] A. Shapiro. Time consistency of dynamic risk measures. Optimization Online preprint, Georgia Institute of Technology, 2012.

[22] S. Uryasev and R. T. Rockafellar. Conditional value-at-risk: optimization approach. In *Stochastic optimization: algorithms and applications (Gainesville, FL, 2000)*, volume 54 of *Appl. Optim.*, pages 411–435. Kluwer, Dordrecht, 2001.