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On tame, pet, miserable, and strongly miserable impartial games

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Abstract. We consider tame impartial games and develop the Sprague-Grundy theory for misère playing the sum of such games that looks simpler than the classical theory suggested by Conway in 1976, which is based on the concept of genus.
An impartial game is called *pet* if the sets of P-positions of its normal and misère versions are disjoint. We provide several equivalent characterizations and show that the pet games form a proper subfamily of the tame games.
For example, NIM, Wythoff's NIM, and game Euclid are tame but not pet, while all subtraction games, the Fraenkel extension NIM(a) of Wythoff's NIM(1), as well as its further extension NIM(a, b) recently suggested by the author are pet games whenever $a > 1$. Thus, very many important impartial games are tame or pet.

Keywords: combinatorial, impartial, tame, pet, miserable, and strongly miserable games; normal and misère play, Sprague-Grundy function, NIM, Wythoff's NIM, Fraenkel's NIM, game Euclid, game Mark, subtraction games; swap, tame, and critical positions.

1 Introduction: the Sprague-Grundy theory

The first results on impartial games are due to Bouton (1901) [7] and Wythoff (1907) [42].

The general theory was developed independently by Sprague (1935–1937) [40, 41] and Grundy (1939) [22]. For more detailed presentation of the subject we refer the readers to Conway (1976, in particular, Chapter 12) and Berlecamp et al (2001–2004, especially, Chapters 12, 13) [3].

1.1 Minimum excludant functions mex and mex_b

The *minimum excludant* function $mex(S)$ is defined for any subset $S \subset \mathbb{Z}_{\geq 0}$ of the non-negative integers as the minimum $z \in \mathbb{Z}_{\geq 0}$ such that $z \notin S$; in particular, $mex(\emptyset) = 0$.

We will also need the following generalization suggested recently by Gurvich (2010) [26].

Given an integer $b \geq 1$ and a finite set $S \subset \mathbb{Z}_{\geq 0}$ of m pairwise distinct non-negative integers, $0 = s_0 < s_1 < \cdots < s_{m-1}$, let us define $mex_b(S) = s_i + b$, where s_i is the smallest number in S such that $s_{i+1} - s_i > b$. By convention, we assume that $s_m = +\infty$ and that $mex_b(\emptyset) = 0$. It is easily seen that mex_b coincide with mex when $b = 1$, that is, $mex_1 = mex$.

1.2 Impartial games, normal and misère versions

An *impartial game* is a two-person game in which both players have perfect information, there are no moves of chance, and the allowed moves are the same for both players. Such a game is modeled by a directed acyclic graph $G = (V, E)$, each vertex $v \in V$ of which is a position and each arc $e = (v, v') \in E$ is a move of the corresponding game. Let us mentioned that G might be infinite but it must be *potentially finite*, that is, the set of all (not only immediate) successors of any position $v_0 \in V$ must be finite; in particular, the set V_T of the terminal positions is not empty. We assume that a token is initially placed in a vertex $v_0 \in V$, and two players take turns moving the token from a current vertex v to one of its immediate successors v' (that is, $(v, v') \in E$). The game ends when the token reaches V_T.

The player who has then to move (but cannot) is claimed the winner in the misère version of the game G^M and looser in the normal version G^N. The goal of this paper is the joint analysis of these two versions.

1.3 The Sprague-Grundy function

Given a digraph $G = (V, E)$, the Sprague-Grundy (SG) function $g^N : V \to \mathbb{Z}_{\geq 0}$ is defined recursively by the formula $g^N(v) = mex\{g(w) | (v, w) \in E\}$, that is, $g^N(v)$ takes the minimum value not taken by any immediate successor of v. This recursion starts in the terminal positions, all of which receive the value 0, that is, $g^N(v) = 0$ for all $v \in V_T$.

The misère SG function g^M is defined by the same formula as g^N but the initialization is different: by convention $g^M(v) = 1$ (while $g^N(v) = 0$) for all terminal positions $v \in V_T$. Let us modify the original digraph $G = G^N$ adding one new vertex v^* and the edge (v, v^*) for every $v \in V_T$ and denote the obtained digraph G^M; see Figure 1 as an example.

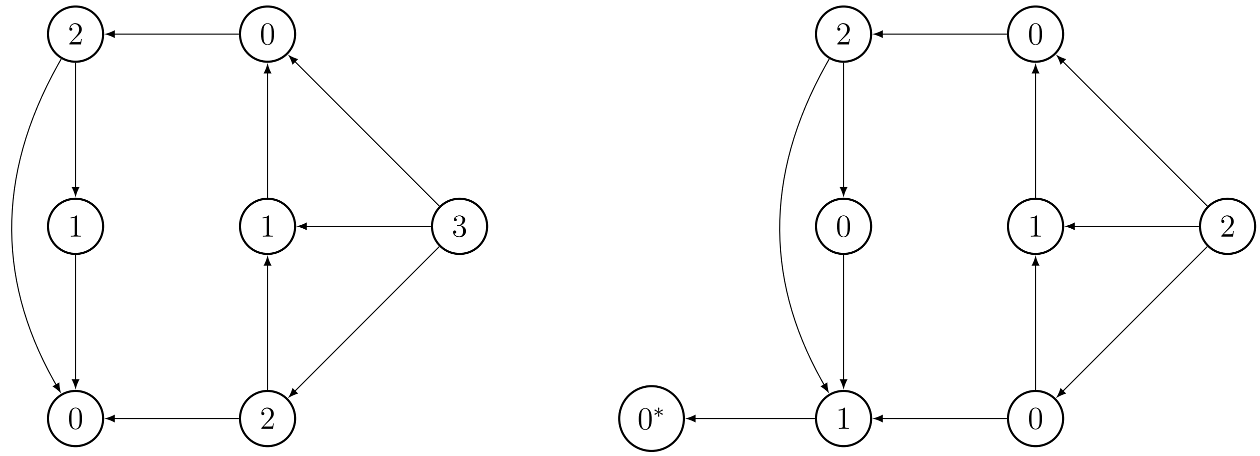

Figure 1: This game is not a minion (and not tame), since some $g(v)$ take values $(2,0)$ and $(3,2)$.

Then, obviously, the misère SG function of G^N is the normal SG function of G^M, that is, $g^M_{G^N} = g^N_{G^M}$. In other words, the normal-misère swap is an involution: $g^{MM} = g^N$.

Given $k, \ell \in \mathbb{Z}_{\geq}$, let V_k^N and V_ℓ^M denote the subsets of V in which functions g^N and g^M take values k and ℓ, respectively; in other words,

$$V_k^N = \{v \in V \mid g^N(v) = k\} \text{ and } V_\ell^M = \{v \in V \mid g^M(v) = \ell\}.$$

Furthermore, let us set $V_{k,\ell} = V_k^N \cap V_\ell^M$. A position $v \in V_{k,\ell}$ (for which $g^N(v) = k$ and $g^M(v) = \ell$, or equivalently, $g(v) = (g^N(v), g^M(v)) = (k, \ell)$) will be called a (k,ℓ)-*position*.

In particular, $v \in V_{0,1} \cup V_{1,0}$ will be called a *swap* position and $v \notin V_{0,1} \cup V_{1,0}$ a *tame* position. Let us note that, by definition, each terminal is a $(0,1)$-position, $V_T \subseteq V_{0,1}$.

Given two subsets of positions $V', V'' \subseteq V$, we say that V'' is *reachable* from V' if for every $v' \in V'$ there is a move $e = (v', v'') \in E$ such that $v'' \in V''$.

Lemma 1 *(Sprague (1935) and Grundy (1939) [40, 22]). Set V_i^N is reachable from V_j^N whenever $i < j$. Furthermore, $g^N(v) \neq g^N(v')$ for any move $(v, v') \in E$.*

Proof: It follows immediately from the definition of the minimum excludant. □

Of course, we get the similar statement for the misère version just replacing every superscript N by M and replacing the digraph G^N by G^M as explained above.

1.4 Kernels, P- and N-positions

The zeros of the SG function g^N are called the *P-positions*.

Lemma 2 *The subset $P \subseteq V$ of all P-positions of a digraph $G = (V, E)$ is*

- *(p)* independent*: $(v, v') \in E$ for no $v, v' \in P$, and*

- *(pp)* absorbing*: for any $v \notin P$ there is a move $(v, v') \in E$ such that $v' \in P$.*

Proof: It follows immediately from Lemma 1. □

Properties (p) and (pp) show that a player cannot stay in P and, being out of P, can always enter it. Hence, a player who enters P (the *Previous* one) wins the normal version of the game. Indeed, (s)he can always re-enter P, while the opponent must always leave it. The play will terminate in V_T in a finite number of moves, since we assume that each vertex has a finite number of successors. Furthermore, $V_T \subseteq P$, by (pp).

In graph theory, an independent and absorbing vertex-subset $P \subseteq V$ of a digraph $G = (V, E)$ is called a *kernel*. This concept was introduced by von Neumann and Morgenstern (1944) [32] in their seminal book, where it was shown that every acyclic digraph has a unique kernel, which can be found by the following recursive algorithm.

Set $K_0^1 = V_T$ (obviously $K_0^1 \subseteq P$). Let K_1^1 denote the set of all vertices from which K_0^1 is reachable (obviously, $K_1^1 \subseteq V \setminus P$). Delete $K_0^1 \cup K_1^1$ from V and repeat, etc., thus, getting sets K_0^i and K_1^i in every step $i = 1, 2, \dots$ Then, $P = \cup_{i=1}^{\infty} K_0^i$, while $N = V \setminus P = \cup_{i=1}^{\infty} K_1^i$. The latter set is usually referred to as the set of *N-positions*, in which the *Next* player wins.

1.5 Sums of games

1.5.1 The Sprague-Grundy function of the sum

Given n digraphs $G_i = (V_i, E_i),\ i \in [n] = \{1, \dots, n\}$, their sum $G = G_1 \oplus \dots \oplus G_n = (V, E)$ is defined by the vertex-set $V = V_1 \times \dots \times V_n = \{(v_1, \dots, v_n) \mid v_i \in V_i, i \in [n]\}$ and the set E of directed edges that contains all pairs (v, v') such that $v = (v_1, \dots, v_n)$ and $v' = (v'_1, \dots, v'_n)$ differ by exactly one coordinate $i \in [n]$ and $(v_i, v'_i) \in E_i$. In other words, there are n impartial games and a token in each. In a current position $v = (v_1, \dots, v_n)$, a player chooses an arbitrary game $i \in [n]$ and move $(v_i, v'_i) \in E_i$ in it. Two player take turns. The play begins in a given initial position $v^0 = (v_1^0, \dots, v_n^0) \in V$ and (in a finite number of moves) ends in a terminal position $v^T = (v_1^T, \dots, v_n^T) \in V$, where v_i^T is a terminal position of V_i for all $i \in [n]$. The normal and misère versions are defined standardly.

The fundamental property of the SG function is given by the equality

$$g^N(v) = g^N(v_1) \oplus \dots \oplus g^N(v_n), \text{ for all } v = (v_1, \dots, v_n), \ v_i \in V_i, ; i \in [n], \tag{1}$$

where $\oplus$ is the so-called *bitwise XOR (or NIM) sum*: we take the binary representations of $g^N(v_i)$ for all $i \in [n]$ and take their bitwise binary sum. For example,

$3 \oplus 5 = 011_2 + 101_2 = 110_2 = 6$; similarly, $5 \oplus 6 = 3$, $6 \oplus 3 = 5$, and $3 \oplus 5 \oplus 6 = 0$.

Thus, one can get $g^N(v_1, \dots, v_n)$ just computing $g^N(v_i)$ for all $i \in [n]$. Recall that the zeros of g^N are exactly the P-positions of the sum.

Remark 1 *Let us note that the sets P_i of the P-positions for all $i \in [n]$ are not sufficient for computing the P-positions of the sum, yet, since both options $v \in P$ and $v \notin P$ are possible for a position $v = (v_1, v_2)$ such that $v_1 \notin P_1$ and $v_2 \notin P_2$.*

1.5.2 Normal playing sums

The SG theory allows to play the normal version of a sum $G = G_1 \oplus \ldots \oplus G_n$.

We assume that the n summands form the input. Although the size of G is already exponential in n, nevertheless, we can efficiently solve the sum, since, somewhat surprisingly, to do so we need not to construct G explicitly. Given a position $v = (v_1, \ldots, v_n)$, let us compute the SG function $g^N(v_i)$ for each $i \in [n]$, as well as the NIM-sum $g^N(v)$ of the obtained n numbers. If $g^N(v) = 0$ then v is a P-position of G^N and there is no winning move from v. Yet, if $g^N(v) > 0$ then such moves exist and are easy to find. To do so, let us consider the leading 1-bit in the binary representation (1) of $g^N(v)$. By definition of the NIM-sum, there is an odd number of $i \in [n]$ such that $g_i^N(v_i)$ also have a 1 in this bit. For each such i there exists a unique positive integer $y_i \in \mathbb{Z}_>$ such that $g^N(v)$ will be reduced to 0 by subtracting y_i from $g_i^N(v_i)$. Furthermore, by Lemma 1, such moves exist in G_i. Moreover, they form the set of winning moves in $G(v)$ that reduce the value of the SG function $g^N(v)$.

Remark 2 *Yet, there might be other winning moves, which enlarge the value of $g^N(v)$.*

1.5.3 Misère playing sums

For each $i \in [n]$ both the normal and misère SG functions (g_i^N and g_i^M) can be computed in linear time. Yet, the equality (1) will fail if we replace all superscripts N by M. In other words, the whole SG theory fails for the misère play. This principal difficulty was outlined by Conway (1976, Chapter 12) [12]. However, on the positive side, he introduced the concept of the *genus* and made use of it to show how to solve the misère version of the sum in some special cases. In Section 2.1, we suggest a simpler approach to one of these cases.

Misère impartial games were considered in many papers: [1, 2, 3, 12, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 29, 35, 36, 37, 39, 43] beginning with the pioneer Bouton (1901) article [7].

Let us separately mention the comparative analysis of the misère versus normal play by Plambeck (1992, 2003, 2005) [35, 36, 37].

1.5.4 NIM as the simplest sum

Although we postpone examples in Section 3, yet, one of them we will consider now, since it illustrates perfectly many basic points of the SG theory for the normal and misère play.

The ancient Chinese game of NIM is played as follows. There are n piles of $x_1, \ldots, x_n$ stones (or matches). Two players alternate turns. By one move, a player chooses a pile $i \in [n] = \{1, \ldots, n\}$ and takes y_i stones from it such that $0 < y_i \leq x_i$. The game is over as soon as there are no stones left. Standardly, the player who took the last stone wins in the normal version and loses in the misère one. Both versions were solved by Bouton (1901) [7].

Obviously, the n-pile NIM is the sum of n one-pile NIMs. The SG function of the one-pile NIM is trivial: $g^N(x_i) = x_i$. Hence, $g^N(x_1, \ldots, x_n) = x_1 \oplus \ldots \oplus x_n$ is the NIM-sum.

The main point of the whole SG theory is that playing (the normal version of) the sum of any n impartial games is just a little bit more difficult that playing NIM with n piles. To analyze position $v = (v_1, \ldots, v_n)$ of the sum $G = G_1 \oplus \ldots \oplus G_n$ one has to compute the SG function $g^N(v_i)$ for all summands $i \in [n]$ and play G like NIM, replacing $x_i = g^N(x_i)$ by $g^N(v_i)$. If an optimal move for NIM is to reduce x_i by y_i for some $i \in [n]$ then any move reducing $g^N(v_i)$ by y_i in G_i will be optimal in G. And any such move can be realized, by Lemma 1. (Let us also remark that the number of such optimal $i \in [n]$ is always odd.)

All above is related to the normal version. What about misère play? As we already mentioned, in general, it is difficult, since the SG theory fails in this case. Yet, the misère version of NIM is still easy. For the one-pile NIM, the misère SG function coincide with the normal one whenever $x_i > 1$, that is, $g^M(x_i) = g^N(x_i) = x_i$, and the values swap when $x_i \leq 1$, that is, $g^M(1) = g^N(0) = 0$ and $g^M(0) = g^N(1) = 1$. A position $x = (x_1, \ldots, x_n)$ will be called *swap* if $x_i \leq 1$ for all $i \in [n]$, otherwise x will be called *tame*. Furthermore, x will be called *critical* if it is tame but a swap position is reachable from it (by one move); in other words, if there is a unique $i \in [n]$ such that $x_i > 1$. Obviously, exactly two swap positions x' and x'' can be reached from x, since x_i can be reduced to 1 or to 0.

Lemma 3 *The normal and misère SG function of the n-pile NIM coincide for the tame positions, $g^M(x) = g^N(x)$, while for the swap positions $g^M(x) = 1 - g^N(x) = \frac{1}{2}(1 - (-1)^m)$, where $m = m(x)$ is the number of the non-zero coordinates (that is, non-empty piles) in $x = (x_1, \ldots, x_n)$. In other words, for the swap positions, $g^M(x) + g^N(x) \equiv 1$ and $g^M(x) = 1 - g^N(x)$ is 0 or 1 when the number of non-empty piles is odd or even, respectively.*

Proof: It follows immediately from the definitions of g^M, g^N, and NIM-sum. □

It is easy to see that in a swap position x the result of NIM depends only on parity but not on players' skills. Indeed, the play beginning from x is essentially unique (if we assume that all one-stone piles are equivalent). The number of moves in it is the number $m(x)$ of non-empty piles of x. If this number is even then x is a P-position in the normal version of the game and an N-position in the misère one; if $m(x)$ is odd then vice versa. (Recall that the Previous player wins in a P-position and the Next player wins in an N-position.)

Furthermore, each critical position is an N-position for both normal and misère versions of the game. Yet, the Next player must be careful and reduce the unique $x_i > 1$ either to 1 or to 0 depending on the version played. For each, normal or misère, version one of these two moves is winning, while the other one is loosing.

As for the remaining (that is, non-critical) tame positions the Next player may not care of which version is played. The winning moves lead to a P-position, for which the SG function is 0, and by Lemma 3, $g^N(x) = g^M(x)$, unless x is a swap position. Thus, only in a critical position the Next player should inquire which version is actually played.

These arguments appeared already in the seminal paper by Bouton (1901) [7].

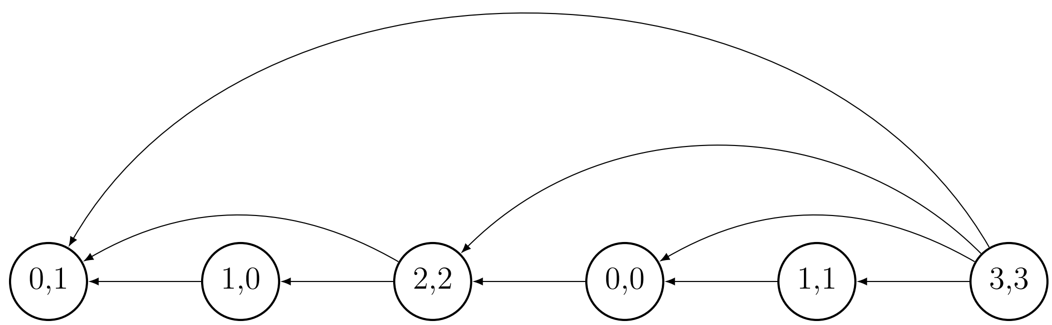

Figure 2: A minion that is not tame (and not miserable).

Remark 3 *Interestingly, the misère (rather than normal) version of NIM was played by the students in 19th century and in ancient China as well. Thus, Bouton treated the normal version of NIM as the misr̀e version of the standard game.*

Gurvich (2007) [25] noticed that the same arguments also work for the game Euclid, in which the Fibonacci (rather than $(0, 1)$) positions become the swap positions; see Section 3.5 for more details. In this paper, we extend further the class of games whose misère version is almost as simple as the normal one.

2 Summary: main concepts and results

Let us consider game G in Figure 1 together the corresponding SG functions g_G^M and g_G^N. Their values "differ a lot". Here we will study games in which they "differ just slightly".

2.1 Tame or miserable games

2.1.1 Minions

A game will be called a *minion* if it contains only $(0, 1)$- $(1, 0)$- and (k, k)-positions; furthermore, it will be called a 0-*minion* if $k \geq 0$ and a 2-*minion* if $k \geq 2$. Obviously, the latter class is a subclass of the former one and minions and 0-minions are the same. For example, the game in Figure 2 is a 0-minion, while the game in Figure 1 is not.

Lemma 4 *In a minion, from a* $(1, 0)$*-position (respectively, from a non-terminal* $(0, 1)$*-position) there is a move to a* $(0, 1)$*-position (respectively, to a* $(1, 0)$*-position); in other words,* $V_{0,1}$ *is reachable from* $V_{1,0}$*, while* $V_{1,0}$ *is reachable from* $V_{0,1} \setminus V_T$*.*

Proof: Indeed, by Lemma 1, in every game, from each $(1,0)$-position (respectively, from a non-terminal $(0,1)$-position) there is move to a $(0,k)$-position (respectively, to a $(k,0)$-position). Furthermore, $k=0$ cannot hold, again by Lemma 1, while $k \leq 1$ must hold, since the considered game is a minion. Hence, $k=1$ and the statement follows. □

2.1.2 Miserable games

A game will be called *miserable* if from every non-swap position $v \in V \setminus (V_{0,1} \cup V_{1,0})$ either both swap subclasses $V_{0,1}$ and $V_{1,0}$ are reachable from v, or none of them; in other words, if from a position v, set $V_{0,1}$ is reachable, while $V_{1,0}$ is not (respectively, vice versa) then $v \in V_{1,0}$ (respectively, $v \in V_{0,1}$).

Proposition 1 *Every miserable game is a minion but not vice versa.*

Proof: Let us derive the claim by backward induction. By definition, $V^0 = V_T \subseteq V_{0,1}$. Let $V^1 \subseteq V \setminus V^0$ denote the set of positions all moves from which lead to V^0. Since the digraph G is acyclic, $V^1 = \emptyset$ if and only if $V = V^0$. By Lemma 1, $V^1 \subseteq V_{1,0}$. Then, let $V^2 \subseteq V \setminus (V^0 \cup V^1)$ denote the set of positions all moves from which lead to $V^0 \cup V^1$. In particular, V^1 is reachable from V^2. Since G is acyclic, $V^2 = \emptyset$ if and only if $V = V^0 \cup V^1$. Furthermore, $V^2 \subseteq V_{0,1} \cup V_{1,0} \cup V_{2,2}$, because G is miserable. Then, let $V^3 \subseteq V \setminus (V^0 \cup V^1 \cup V^2)$ denote the set of positions all moves from which lead to $V^0 \cup V^1 \cup V^2$. In particular, V^2 is reachable from V^3. Since G is acyclic, $V^3 = \emptyset$ if and only if $V = V^0 \cup V^1 \cup V^2$. Furthermore, $V^3 \subseteq V_{0,1} \cup V_{1,0} \cup (\cup_{k=0}^{3} V_{k,k})$, because G is miserable; etc.

In general, by induction for any integer $\ell \in \mathbb{Z}_{\geq}$ we derive that $V^\ell \subseteq V_{0,1} \cup V_{1,0} \cup (\cup_{k=0}^{\ell} V_{k,k})$; in other words G can contain only $(0,1)$-, $(1,0)$-, and (k,k)-positions, where $k \geq 0$.

The obtained containment is strict. For example, the game in Figure 2 is a minion that is not miserable. □

Remark 4 *Miserable games were recently introduced by Gurvich (2011) [27], where it was additionally required that $V_{0,1}$ is reachable from $V_{1,0}$ and $V_{1,0}$ is reachable from $V_{0,1} \setminus V_T$. Yet, by Lemma 4, these extra requirements are superfluous. A similar, but slightly narrower, class was introduced earlier by Gurvich (2007) [25], where "even and odd forced positions" were considered instead of $(0,1)$- and $(1,0)$-positions. It was shown that the obtained class contains games NIM and Euclid; see Section 3.1 and 3.5 below.*

2.1.3 Tame and wild games

Actually, it is not difficult to understand that the above definition of the miserable games is just a reformulation of the classical concept of *tame* games introduced by Conway (1976) [12]; see also Berlecamp et al (2001–2004)) [3].

Our labels $(0,1)$, $(1,0)$, and (k,ℓ) correspond respectively to 0, 1-, and k^ℓ in the cited books, where the family of tame games is defined inductively as follows:

A game $G(v)$ is tame if $G(v')$ is tame for every immediate successor v' of v and the next additional condition holds: Let $L(v)$ be the set of labels of the direct followers of v; if $L(v)$ contains exactly one of the labels 0 and 1 then it must not contain 0^0 nor 1^1.

Proposition 2 *A game is miserable if and only if it is tame.*

Proof: "If part". Let us suppose that $V_{0,1}$ is reachable from a position v, while $V_{1,0}$ is not (respectively, vice versa). Then, by tameness, $V_{0,0} \cup V_{1,1}$ is not reachable from v either. Hence, v is a $(1,0)$-position (respectively, a $(0,1)$-position).

"Only if part". Indeed, $V_{0,1}$ and $V_{1,0}$ are reachable from v, or not, only simultaneously, unless v is a swap position. In the latter case, if $v \in V_{0,1}$ (respectively, $v \in V_{1,0}$) then $V_{1,0}$, but not $V_{0,1}$ (respectively, $V_{0,1}$, but not $V_{1,0}$) is reachable from v, by Lemmas 1 and 4. $\Box$

The non-tame games are called *wild*. For example, the game in Figure 1 is wild; moreover, it is not a minion, since it contains $(2,0)$- and $(3,2)$-positions. The game in Figure 2 is a minion but still wild, since from the $(3,3)$-position, $V_{0,1}$ is reachable, while $V_{1,0}$ is not.

2.1.4 Tameness respects sums

The following property of tame games plays very important role.

Theorem 1 *(Convay (1976), Berlecamp et al (2001–2004)) The sum of tame games is tame.*

We will need also the following slightly more detailed statement.

Theorem 2 *(Gurvich (2011)) Let $G(v) = G_1(v_1) \oplus \ldots \oplus G_n(v_n)$ be the sum of n games, then the sum G is tame whenever all n summands are tame. Furthermore, $v = (v_1, \ldots, v_n)$ is a swap position of G if and only if v_i is a swap position of G_i for all $i \in [n] \in \{1, \ldots, n\}$. Moreover, in this case, v is a $(1,0)$-position of G if and only if v_i is a $(1,0)$-position of G_i for an odd number of indices $i \in [n]$.*

Proof: Let v_i be a swap position position of G_i for all $i \in [n]$. Then, by Lemma 4, the previous player can maintain the number $m = m(v)$ of $(1,0)$-position among them, while the next player can reduce it by 1 and then maintain this reduced number. Hence, the results of both the normal and misère versions depend only on parity $m(v)$: if it is odd then v is a P-position of G^M and an N-position of G^N and vice versa when $m(v)$ is even. This exactly means that v is a swap position of G.

Then, applying backward induction, as in the proof of Lemma 1, we conclude that all remaining positions are tame, that is, each of them is a (k,k)-position with some $k \geq 0$. $\Box$

Remark 5 *Again, the n-pile NIM, which is the sum of n one-pile NIMs, can serve as a model example. According to Section 1.5.4, this game contains 2^n swap positions, with at most one (that is, 0 or 1) stone in each pile. There are 2^{n-1} $(0,1)$-positions, for which the number of non-empty piles is even, and 2^{n-1} $(1,0)$-positions, for which it is odd.*

Let us note, however, for any swap position v of NIM there is a unique play that begins in v (up to a natural isomorphism: all piles with one stone can be viewed as equivalent). Furthermore, in each such play the $(0,1)$- and $(1,0)$-positions alternate. Hence, the result of the game does not depend on the players' skills at all, it only depend on parity. In contrast, in a general tame sum, there may be several distinct moves from a swap position. Moreover, some of these moves may lead to tame positions. For example, $(0,1)$-, $(3,3)$-, and $(0,0)$-, or $(1,0)$-, $(2,2)$-, and $(1,1)$-positions might successively appear in an optimal play.

2.2 Playing the misère sum of tame games

Conway (1976) [12] introduced the concept of *genus* and made use of it to show that the misère version of the sum of tame games can be efficiently solved. Here, we will obtain the same result simpler by showing that misère playing the sum of miserable games is just a little bit more difficult than misère playing NIM and already Bouton (1901) demonstrated that the latter is very simple; see Section 1.5.4.

In a swap position $v \in V_{0,1} \cup V_{1,0}$ the Next player swaps, that is, (s)he makes a move (v, v') to another swap position v' such that $v' \in V_{1,0}$ if $v \in V_{0,1}$ and $v' \in V_{0,1}$ when $v \in V_{1,0}$. The existence of such moves is guaranteed by Lemma 4.

The Next player must be "especially" careful in a critical position, that is, in a position v, which is tame but has a move (v, v') to a swap position, say, $v' \in V_{0,1}$. By Theorem 2 and definition of tameness, there is an alternative move (v, v'') to a position $v'' \in V_{1,0}$. Exactly one of these two moves is winning in the misère play and it must be chosen. (Of course, $v'' \in V_{0,1}$ when $v' \in V_{1,0}$.)

As for the other, non-critical, tame positions the Next player may not care at all of which version is played. The winning moves lead to a P-position, for which the SG function is 0, and by Lemma 3, $g^N(v) = g^M(v)$, unless v is a swap position. Thus, only in a critical position the Next player should inquire which version, normal or misère, is actually played.

2.3 Strongly miserable or pet games

A game will be called *strongly miserable* (or *pet*) if $V_{0,1} = V_0^N = V_1^M$ and $V_{1,0} = V_1^N = V_0^M$, or in other words, when the normal and misère kernels are disjoint, $V_0^N \cap V_0^M = \emptyset$.

One can also say that a strongly miserable game is a tame game in which $V_{0,0} = V_{1,1} = 0$.

In this case the definition can be simplified as follows. By Lemma 1, V_i^N is reachable from V_j^N when $i < j$. In particular, both V_0^N and V_1^N are reachable from any V_j^N with $j > 1$ and V_0^N is reachable from V_1^N. Thus, it remains to check that V_1^N is reachable from $V_0^N \setminus V_T$.

We can summarize all above observations as follows.

Theorem 3 *The next seven properties of a game G are equivalent:*
(i) G is strongly miserable; (i') G is a 2-minion; (i") G is miserable and $V_{0,0} = V_{1,1} = \emptyset$;
(i"') $g(v) = (g^N(v), g^M(v))$ take only values $(0,1), (1,0)$, and (k,k) for $k \geq 2$;
(ii) $V_{0,0} = \emptyset$; (iii) V_1^N is reachable from $V_0^N \setminus V_T$; (iii') V_1^M is reachable from V_0^M.

Property (iii) was introduced by Ferguson (1974) [15], where he proved that it holds for all subtraction games; see Section 3.7 for more details. Thus, to demonstrate that a game is not strongly miserable it is sufficient to show a non-terminal position of the SG value 0 from which there is no move to a position of the SG value 1. An alternative way is to show

$$v \in (V_0^N \setminus V_{0,1}) \cup (V_1^N \setminus V_{1,0}) = (V_0^N \cup V_1^N) \setminus (V_{0,1} \cup V_{1,0}).$$

Proof of Theorem 3. Obviously, (i',i",i"') are just reformulations of (i) and imply (ii).

It is also easy to show that (i) implies (iii). Indeed, let us assume indirectly that there is a position $v \in V_0^N \setminus V_T$ from which V_1^N is not reachable, that is, $g^N(v) = 0$ and $g^N(v') = 1$ for no move $(v, v') \in E$. Then, by (i), $g^M(v') = 0$ for no move $(v, v') \in E$. Hence, by Lemma 1, $g^M(v) = 0$ and $g(v) = (g^N(v), g^M(v)) = (0, 0)$ in contradiction with (i).

By similar arguments, we prove that (i) implies (iii').

It remains to show that, conversely, (i') results from (ii) or (iii) or (iii').

We will prove this indirectly, by induction. First, let us recall that $g(v) = (0, 1)$ for any $v \in V_T$, by definition of the SG function, and hence, (i') holds in this case. Second, let us assume indirectly that (i') fails for a position $v \in V$ but holds for all its (immediate) successors and show that (iii), (iii') and (ii) also fail.

Case 1: $g(v) = (0, 0)$. Then (ii) fails for v, by definition. If (iii) holds for v then there is a move (v, v') such that $g^N(v') = 1$ and, by (i'), $g^M(v') = 0$. Yet, $g^M(v) = 0$ too and, by Lemma 1, $g^M(v') \neq 0$, which is a contradiction.

Similarly, if (iii') holds for v then there is a move (v, v') such that $g^M(v') = 1$ and, by (i'), $g^N(v') = 0$. Yet, $g^N(v) = 0$ too and, by Lemma 1, $g^N(v') \neq 0$, which is a contradiction.

Case 2: $g(v) = (k, 0)$ (respectively, $g(v) = (k, 1)$) , where $k > 1$. By Lemma 1, there is a move (v, v') such that $g^N(v') = 1$ (respectively, $g^N(v') = 0$). Then, by (i'), $g(v') = (1, 0)$ (respectively, $g(v') = (0, 1)$) and $g^M(v) = g^M(v')$, in contradiction with Lemma 1.

Case 3: $g(v) = (k, \ell)$, where $k > \ell > 1$. By Lemma 1, there is a move (v, v') such that $g^N(v') = \ell$. Then, by (i'), $g^M(v') = g^N(v') = \ell$. Hence, $g^M(v) = g^M(v') = \ell$, in contradiction with Lemma 1.

Similar arguments work for the symmetric cases: $k = g^M(v) > g^N(v) \in \{0, 1, \ell\}$.

Let us also remark that cases 2 and 3 are impossible just by the induction hypothesis, while the assumptions (iii), (iii'), and (ii) are irrelevant. $\square$

3 Examples

3.1 NIM

As we already mentioned, the n-pile NIM is the sum of n one-pile NIMs. The SG function of the one-pile NIM is trivial: $g^N(x_i) = x_i$. Hence, $g^N(x_1, \ldots, x_n) = x_1 \oplus \ldots \oplus x_n$. For example,

$$g^N(0, 2) = g^N(2, 0) = 0 \oplus 2 = 2, g^N(1, 2) = g^N(2, 1) = 1 \oplus 2 = 3, g^N(2, 2) = 2 \oplus 2 = 0.$$

These simple computations imply the following claim.

Lemma 5 *Already the two-pile NIM is not strongly miserable.*

Proof: Only positions $(0,2),(1,2)$ and $(2,0),(2,1)$ are reachable from $(2,2)$. It is easy to verify that $g^M(2,2) = g^N(2,2) = 0$, while $g^N(2,1) = 3$ and $g^N(2,0) = 2$; hence, conditions (ii) and (iii) of Theorem 3 fail. $\square$

However, Bouton (1901) proved that miserability holds even for n piles; see Section 1.5.4.

Lemma 6 *(Bouton (1901) [7]). Game NIM is miserable.*

Proof: Bouton's original arguments were reproduced in Section 1.5.4. Also, the statement immediately follows from Theorem 1. Indeed, the one-pile NIM is (strongly) miserable and the n-pile NIM is the sum of n one-pile NIMs. $\square$

The above two lemmas are summarized by the next statement.

Proposition 3 *Game NIM is miserable but not strongly misearable.* $\square$

Thus, the sum of strongly miserable games is miserable but not necessarily strongly.

3.2 Wythoff's NIM

Wythoff (1907) [42] introduced the following interesting modification of the two-pile NIM.

Two piles contain x and y stones. Two players move alternately. By one move, a player can take either any number of stones from one pile (and nothing from the other), or equal numbers of stones from both. It is not allowed to pass one's turn.

In other words, $V = \{(x,y) \in \mathbb{Z}^2_{\geq 0}\}$ and from a position (x,y) there are moves to

(j) $\{(x',y) \mid 0 \leq x' < x\}$, (jj) $\{(x,y') \mid 0 \leq y' < y\}$, and

(jjj) $\{(x',y') \mid 0 < x - x' = y - y' \leq \min(x,y)\}$.

Let us notice that positions (x,y) and (y,x) are equivalent, due to obvious symmetry. For this reason, we assume that $x \leq y$, unless it is explicitly said otherwise, and we will keep this assumption through subsections 3.2 – 3.5.

Wythoff proved that the P-position (x_n, y_n) are given by the following recursion

$$x_n = mex\{x_i, y_i \mid 0 \leq i < n\}, \quad y_n = x_n + n; \quad n \in \mathbb{Z}_{\geq 0}, \tag{2}$$

which can be solved explicitly as follows;

$$x_n = \lfloor \phi n \rfloor, \quad y_n = x_n + n = \lfloor (\phi+1)n \rfloor; \quad n \in \mathbb{Z}_{\geq 0}, \tag{3}$$

where $\phi = \frac{1}{2}(1+\sqrt{5})$ is the golden section; see also Coxeter (1953) [13].

Somewhat surprisingly, no explicit formula is known for the SG function of the Wythoff game; see Berlecamp et all (2001-2004), Blass and Fraenkel (1990), and Nivasch (2009) [3, 5, 34] for partial results in this direction.

However, the zeros of the SG function are well (and simply) defined by (2) or (3).

Lemma 7 *(See, for example, Berlecamp et al [3]). The normal and misère SG functions of Wythoff's NIM differ only in six positions given below and coincide in all other positions:*

$g^N(0,0) = g^N(1,2) = g^N(2,1) = g^M(0,1) = g^M(1,0) = g^M(2,2) = 0,$
$g^M(0,0) = g^M(1,2) = g^M(2,1) = g^N(0,1) = g^N(1,0) = g^N(2,2) = 1,$
$g^N(x,y) = g^M(x,y)$ *for all* $(x,y) \notin \{(0,0),(1,2),(2,1);(0,1),(1,0),(2,2)\} = V'$.
In other words, the Wythoff game is miserable with the sets of 0*- and* 1*-swap positions* $V_{0,1} = \{(0,0),(1,2),(2,1)\}$ *and* $V_{1,0} = \{(0,1),(1,0),(2,2)\}$*, respectively.*

Proof: It is not difficult to chack miserability. Indeed, $V_{0,1}$ is reachable from $V_{1,0}$ and $V_{1,0}$ is reachable from $V_{0,1} \setminus \{(0,0\} = \{(1,2),(2,1)\}$, according to the rules of the game.

Furthermore, each "column" $x = 0, x = 1, x = 2$, each "row" $y = 0, y = 1, y = 2$, and each "diagonal" $y = x \pm 1$ and $y = x$ contain exactly two positions of $V' = V_{0,1} \cup V_{1,0}$, one from $V_{0,1}$ and one from $V_{1,0}$; while all other rows, columns, and diagonals contain none.

Let us also notice that both $V_{0,1}$ and $V_{1,0}$ are reachable from $(1,1)$. Thus, from every non-swap position $(x,y) \notin V'$, either each set $V_{0,1}$ and $V_{1,0}$, or none of them is reachable. $\square$

Lemma 8 *Wythoff's game is not strongly miserable.*

Proof: The zeros of the SG function g^N, given by (2) or (3), form an infinite set. By Lemma 7 this set and the set of zeros of the misère SG function "almost" coincide; more precisely, their symmetric difference is V'. Thus $g(x,y) = (g^N(x,y), g^M(x,y)) = (0,0)$ for infinitely many positions (x,y) defined by (2) or (3), e.g., for $(3,5),(4,7),(6,10),\ldots$, which contradicts condition (ii) of Theorem 3.

3.3 Fraenkel's NIM(a)

In 1982, Fraenkel generalized the Wythoff NIM keeping (j, jj) and replacing (jjj) by

(jjj') $\{(x',y') \mid 0 \le x' \le x,\ 0 \le y' \le y,\ x'+y' < x+y, \text{ and } |(x-x')-(y-y')| < a\}$,

where a is a strictly positive integer parameter; see Fraenkel (1982) [17] and also Fraenkel (1984) [18].

In other words, in Fraenkel's game a player can take either any strictly positive number of stones from one pile, and nothing from the other, or "approximately" equal numbers of stones from both piles; more precisely, the difference must be at most $a - 1$.

Obviously, Fraenkel's NIM(a) turns into Wythoff's NIM when $a = 1$. Fraenkel showed that the recursive solution (2) of NIM(1) should by just slightly modified to solve NIM(a):

$$x_n = mex\{x_i, y_i \mid 0 \le i < n\}, \quad y_n = x_n + an; \quad n \in \mathbb{Z}_{\ge 0}. \tag{4}$$

Moreover, he solved this recursion and got the following explicit formula for (x_n, y_n).

Let $\alpha = \alpha(a) = \frac{1}{2}(2-a+\sqrt{a^2+4})$ be the (unique) positive root of the quadratic equation $z^2 + (a-2)z - a = 0$, or equivalently $\frac{1}{z} + \frac{1}{z+a} = 1$. In particular, $\alpha(1) = \frac{1}{2}(1+\sqrt{5})$ is the golden section and $\alpha(2) = \sqrt{2}$. The explicit solution is given by the following formula

$$x_n = \lfloor \alpha n \rfloor, \;\; y_n = x_n + an \equiv \lfloor n(\alpha + a) \rfloor; \;\; n \in \mathbb{Z}_{\geq 0}. \tag{5}$$

Fraenkel refers to the following "folk-lemma" going back at least to Beatty (1926) [4].

Lemma 9 *(Beatty (1926) [4]) Let α and β be positive irrationals satisfying $\alpha^{-1} + \beta^{-1} = 1$ then sets $A = \{\lfloor n\alpha \rfloor \mid n \in \mathbb{Z}_{>0}\}$ and $B = \{\lfloor n\beta \rfloor \mid n \in \mathbb{Z}_{>0}\}$ partition $\mathbb{Z}_{>0}$.* □

A very elegant and short proof given by Fraenkel (1982) [17] he attributes to Ostrovsky.

Fraenkel (1982) [17] mentioned that the explicit formula (5) solves the game $NIM(a)$ in linear time, in contrast to recursion (4) providing only an exponential algorithm. A recursive solution of the misère version of NIM(a) can be found in page 69 of Fraenkel (1984) [18].

$$x_n = mex\{x_i, y_i \mid 0 \leq i < n\}, \;\; y_n = x_n + an + 1; \;\; n \in \mathbb{Z}_{\geq 0}; \;\; a \in \mathbb{Z}_{\geq 2} = \mathbb{Z}_{\geq 0} \setminus \{0, 1\}. \tag{6}$$

Let us recall that NIM(1) is the Wythoff game, which is (not strongly) miserable.

Lemma 10 *Game $NIM(a)$ is strongly miserable when $a > 1$.*

Proof: One can easily verify condition (ii) of Theorem 3 just comparing (4) and (6). Indeed, $y_n = x_n + an$ for the normal version and $y_m = x_m + am + 1$ for the misère one. If $(x_n, y_n) = (x_m, y_m)$ then $y_n - y_m = x_n - x_m + a(n - m) - 1 = a(n - m) - 1 = 0$. Hence, $a(n - m) = 1$, which is impossible for $a > 1$. □

3.4 Game NIM(a, b)

Further generalization was suggested by Gurvich (2010) [26]. Given two positive integers a and b, we keep (jjj') and replace two sets defined by (j) and (jj) by one set defined as follows:

(j–jj) $\{(x', y') \mid 0 \leq x' \leq x, \; 0 \leq y' \leq y, \;\; x' + y' < x + y, \;$ and $\; [x - x' < b \;$ or $\; y - y' < b]\}$.

In other words, in NIM(a, b) a player can take either (j–jj) any positive number of stones from one pile and at most $b - 1$ from the other, or (jjj') any "approximately equal" numbers of stones from both piles; more precisely, these two numbers may differ by at most $a - 1$.

We still assume that two player move alternately, it is not allowed to pass one's turn, the player who has no move loses in the normal version and wins in the misère one.

Obviously, NIM($a, 1$) is exactly the Fraenkel NIM(a), since $0 \leq z - z' < 1$ iff $z = z'$.

The recursive solutions of NIM(a, b), for the normal and misère versions, were recently obtained by Gurvich (2010) [26]. They are similar to the corresponding Fraenkel recursions (4) and (6) for NIM(a), but mex is replaced by mex_b. For the normal version the P-positions are given by

$$x_n = mex_b\{x_i, y_i \mid 0 \leq i < n\}, \; y_n = x_n + an; \;\; n \in \mathbb{Z}_{\geq 0}, \; a, b \in \mathbb{Z}_{>0}. \tag{7}$$

while for the misère version and $a \geq 2$ we have

$$x_n = mex_b\{x_i, y_i \mid 0 \leq i < n\},\ y_n = x_n + an + 1;\ n \in \mathbb{Z}_{\geq 0};\ b \in \mathbb{Z}_{>0},\ a \in \mathbb{Z}_{\geq 2}. \quad (8)$$

Case $a = 1$ for the misère version appears to be special for all b, not only for $b = 1$, and it is similar to NIM$(1,1)$ = NIM(1), that is, to the classical Wythoff NIM.

Lemma 11 *(Gurvich (2010) [26]). The normal and misère SG functions of NIM$(1,b)$ differ only in six positions given below and coincide in all other positions:*

$g^N(0,0) = g^N(b,b+1) = g^N(b+1,b) = g^M(0,1) = g^M(1,0) = g^M(b+1,b+1) = 0,$

$g^M(0,0) = g^M(b,b+1) = g^M(b+1,b) = g^N(0,1) = g^N(1,0) = g^N(b+1,b+1) = 1,$

$g^N(x,y) = g^M(x,y)$ *for all* $(x,y) \notin \{(0,0),(b,b+1),(b+1,b);(0,1),(1,0),(b+1,b+1)\} = V'$.

In other words, game NIM$(1,b)$ is miserable with the 0- and 1-swap positions $V_{0,1} = \{(0,0),(b,b+1),(b+1,b)\}$ and $V_{1,0} = \{(0,1),(1,0),(b+1,b+1)\}$.

Proof: It is easy to see that Lemma 11 turns into Lemma 7 when $b = 1$; so we will mimic the proof of the latter. and verify miserability. Indeed, $V_{0,1}$ is reachable from $V_{1,0}$ and $V_{1,0}$ is reachable from $V_{0,1} \setminus \{(0,0)\} = \{(b,b+1),(b+1,b)\}$, in accordance with the rules of the game. Then, let us introduce the set of positions

$$V'' = \{(x,y) \mid 0 \leq x \leq 2b, \text{ or } 0 \leq y \leq 2b, \text{ or } |x-y| \leq 1\} \subseteq V$$

consisting of $2b+1$ rows $0 \leq y \leq 2b$, columns $0 \leq x \leq 2b$, and $2a+1 = 3$ diagonals $|x-y| \leq 1$. It is easy to see that both

$$V_{0,1} = \{(0,0),(b,b+1),(b+1,b)\} \text{ and } V_{1,0} = \{(0,1),(1,0),(b+1,b+1)\}$$

are reachable from $V'' \setminus V'$ but none of these two sets is reachable from $V \setminus V''$. □

Similarly, we extend Lemma 8 from the Wythoff NIM$(1,1)$ to NIM$(1,b)$ for all $b \in \mathbb{Z}_{>0}$.

Lemma 12 *Game NIM$(1,b)$ is not strongly miserable for all $b \in \mathbb{Z}_{>0}$.*

Proof: The zeros of the SG function g^N, given by (7) or (8), form an infinite set. By Lemma 11 this set and the set of zeros of the misère SG function "almost" coincide; more precisely, their symmetric difference is V'. Thus $g(x,y) = (g^N(x,y), g^M(x,y)) = (0,0)$ for infinitely many positions (x,y) defined by (7) or (8), which contradicts condition (ii) of Theorem 3. □

Thus, game NIM$(1,b)$ is miserable but not strongly miserable.

Lemma 13 *For any $b \in \mathbb{Z}_{\geq 0}$, game NIM$(a,b)$ is strongly miserable when $a > 1$.*

Proof: We can just copy the proof of Lemma 10 for the Fraenkel game NIM(a). □

The case $a = 0$ is also of interest. In this case the moves of type (jjj') become impossible, since $a - 1 = -1 < 0$. Hence, the obtained game NIM$(0, b)$ is a simple generalization of the standard two-pile NIM. It is easy to verify, see Gurvich (2010) [26], that

$x_n = y_n = bn$ for all $n \geq 0$ in the normal version, while

$x_0 = y_0 = (0, 1)$, $x'_0 = y'_0 = (1, 0)$, and $x_n = y_n = bn + 1$ for all $n \geq 1$ in the misère version.

These recursions imply the next claim.

Lemma 14 *The game NIM$(0, b)$ is strongly miserable when $b > 1$; in this case the sets P_N and P_M are the zeros and ones of the SG function; in particular, $P_N \cap P_M = \emptyset$.*

In contrast, the game NIM$(0, 1)$ is miserable but not strongly miserable; in this case $P_N \setminus P_M = \{(0, 0), (1, 1)\}$ and $P_M \setminus P_N = \{(0, 1), (1, 0)\}$.

Finally, the last four lemmas are summarized by the following statement.

Proposition 4 *(Gurvich (2010) [26]). For any $a, b \in \mathbb{Z}_{\geq 0}$, the game NIM$(a, b)$ is strongly miserable unless $a = 1, b \geq 1$ or $b = 1, a \leq 1$, in which cases it is only miserable.* □

Let us note that no explicit formula is known not only for the SG function of NIM(a, b) but even for its P-positions. It is shown by Gurvich (2010) [26] that $b \leq x_{n+1} - x_n \leq 2b$. Thus, for $b = 1$ only values 1 and 2 are taken, which enables us to apply Beatty's lemma; see Fraenkel (1982, 1984) [17, 18]. Yet, in general, differences $x_{n+1} - x_n$ demonstrate some "pseudo-random" behavior and it is hardly possible to give an explicit formula for x_n.

Gurvich (2010) [26] conjectured that x_n can be computed in polynomial time and that limits $\ell(a, b) = \lim_{n\to\infty} x_n(a, b)/n$ exist and are irrational algebraic numbers. for all $a, b \in \mathbb{Z}_{>0}$. Both these conjectures were recently proven Boros et al (2011) [6]. A linear time algorithm was obtained, as well as the formula $\ell(a, b) = \frac{a}{r-1}$, where $r > 1$ is the unique positive real root (so called Perron root) of the polynomial

$$P(z) = z^{b+1} - z - 1 - \sum_{i=1}^{a-1} z^{\lceil ib/a \rceil},$$

which is the characteristic polynomial of a non-negative $(b + 1) \times (b + 1)$ integer matrix depending only on parameters a and b, and where $|r'| < r$ for any other root r' of $P(z)$.

Let us remark that the above formulas hold only for relatively prime a and b, while $x_n(ka, kb) = kx_n(a, b)$ and hence, $y_n(ka, kb) = ky_n(a, b)$ and $\ell(ka, kb) = k\ell(a, b)$, as it was shown by Gurvich (2010) [26].

Finally, let us remark that the proofs are based on the Perron-Frobenius theory and, in particular, on the Collatz-Wielandt formula; see Chapter 8 of the textbook by Meyer (2000) [31]. It can be also shown that $P(z)$ satisfies all conditions of the Cauchy-Ostrovsky theorem; see theorems 1.1.3, 1.1.4 in the textbook by Prasolov (2010) [38].

3.5 Game Euclid

In Cole and Davie (1969) [9] introduced a game inspired by the Euclidean algorithm. The positions of this game are all pairs (x, y) of positive integers. Two players move alternately. By one move a player is allowed to subtract any positive multiple of the smaller number from the larger one, provided the difference is still positive. The game ends when no more moves are possible. It is easily seen that the game has a unique terminal position (z, z), where $z = gcd(x, y)$, the greatest common divisor of x and y. Again, positions (x, y) and (y, x) are equivalent. Also, positions $(\ell x, \ell y)$ are equivalent for all positive integer ℓ.

Cole and Davie (1969) [9] proved that (x, y) is a P-position if and only if $x < \phi y$ and $y < \phi x$, where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the golden section.

Moreover, Nivasch (2004) [33] got a very nice formula for the SG function:

$$g^N(x, y) = \lfloor |x/y - y/x| \rfloor \quad \forall\, x, y \in \mathbb{Z}_{>0}. \tag{9}$$

Gurvich (2007) [25] showed that the game is miserable with the swap positions defined as follows. Let F_j be the jth Fibonacci number, $F_j = 1, 1, 2, 3, 5, 8, ...$ for $j = 0, 1, 2, 3, 4, 5, ...$ Let us call (x, y) a *Fibonacci position* of rank i if (x, y) or (y, x) equals $(\ell F_i, \ell F_{i+1})$, where $i \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$. It is easy to see that in a Fibonacci position there is a unique move when $i > 0$ and no move when $i = 0$. Moreover, this unique move leads to the Fibonacci position of rank $i - 1$, since $(\ell F_i, \ell F_{i+1} - \ell F_i) = (\ell F_i, \ell F_{i-1})$. Then the next move is again unique (if $i > 1$) and it leads to the Fibonacci position of rank $i - 2$, etc. until the play terminates in the Fibonacci position (ℓ, ℓ) of rank 0.

Lemma 15 *(Lengyel (2003) [30]). The following two claims are equivalent:*

- *(t) (x, y) is a Fibonacci position,*
- *(tt) beginning from (x, y), each further move is forced.*

Proof: Obviously, each of these two claims is equivalent to

(ttt) y/x expands into a continued fraction whose every incomplete quotient is 1. □

Lengyel (2003) [30] showed that the winning strategy in position (x, y) can be defined in terms of the incomplete quotient of y/x for game Euclid, as well as for several similar and more general games. The following obvious reformulations of Lemma 15 will be useful.

Lemma 16 *A Fibonacci position (x, y) of rank i is a P-position if and only if i is even.*

This position is terminal if $i = 0$ and if $i > 0$ then there is unique move from it, which leads to a Fibonacci position of rank $i - 1$. □

Lemma 17 *(Gurvich (2007) [25]). From a non-Fibonacci position, either no move enters a Fibonacci one, or two moves enter two Fibonacci positions whose ranks differ by 1.*

For example, from (22,4) there are moves to (18,4),(14,4)(10,4),(6,4), and (2,4), the last two of which are Fibonacci positions of rank 2 and 1 respectively, while from (16,3) there are moves to (13,3),(10,3),(7,3),(4,3),and (1,3), none of which is a Fibonacci position.

Proof of Lemma 17. We will show that if a move from a position v leads to a Fibonacci position $v_i = (\ell F_i, \ell F_{i+1})$, of rank i, then there is another move from v that leads either to the Fibonacci position $v_{i-1} = (\ell F_i, \ell F_{i-1})$, of rank $i-1$, or to the Fibonacci position $v_{i+1} = (\ell F_{i+2}, \ell F_{i+1})$, of rank $i+1$.

Case 1: $v = (\ell F_i, \ell F_{i+1} + \ell' F_i)$. Then subtraction of $\ell' F_i$ from the second coordinate results in v_i. Yet, subtraction of $(\ell' + \ell)F_i$ results in v_{i-1}.

Case 2: $v = (\ell F_i + \ell' F_{i+1}, \ell F_{i+1})$. Then subtraction of $\ell' F_{i+1}$ from the first coordinate results in v_i. Yet, subtraction of $(\ell' - \ell)F_{i+1}$ results in v_{i+1}.

It remains to notice that $\ell, \ell' \in \mathbb{Z}_{>0}$ and that v_i is not reachable from other positions. □

Lemma 18 *Game Euclid is not strongly miserable.*

Proof: From position $(3,4)$ there is only one move, which leads to $(3,1)$. However, $g^N(3,4)g^M(3,4) = 0$, while $g^N(3,1) = 2$. Hence, conditions (ii) and (iii) of Theorem 3 fail. Equivalently, both containments, $V_{0,1} \subset V_0^N$ and $V_{1,0} \subset V_1^N$, are strict. □

Four previous lemmas, result in the following claim.

Proposition 5 *Game Euclid is not strongly miserable but it is miserable with the* 0- *and* 1-*swap positions being the Fibonacci positions of even and odd rank, respectively.* □

3.6 Game 3-Euclid and its versions

Collins (2005) and Collins and Lengyel (2008) [10, 11] presented an extension of the above game to three dimensions that they called 3-Euclid. In this game, a position is a triplet of positive integers. Each move is to subtract from one of the integers a positive integer multiple of one of the others as long as the result remains positive. Generally, from a position (x_1, x_2, x_3) where $x_1 \le x_2 \le x_3$, there are three types of moves in 3-Euclid: $i-j$ moves: subtracting a multiple of x_i from x_j, provided $x_i < x_j$, where $i, j \in \{1,2,3\}$ and $i < j$. Recently, Ho (2011) [28] considered two modifications of 3-Euclid, in which only $1-3$ and, respectively, $1-2$ and $1-3$ moves are allowed. He proved that the kernels of these two games coincide. Moreover, his results imply (see Corollary 3 of Ho (2011) [28]) that both games are strongly miserable; see also Cairns et al [8].

It is an open question whether the game 3-Euclid itself is (strongly) miserable.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$g^N(n)$	0	1	0	2	1	2	0	1	0	2	0	1	2	1	0	2	1	2	0
$g^M(n)$	1	0	1	2	0	2	0	1	2	1	0	1	2	1	0	2	0	1	0

Table 1: Game Mark is not miserable: some $g(v)$ take values $(0, 2)$ and $(2, 1)$.

3.7 Subtraction games

Given a (finite or infinite) non-empty subset of positive integers $S \subseteq \mathbb{Z}_{>0}$, a *subtraction game* G_S is defined by the set of positions $V = \mathbb{Z}_{\geq 0}$ and possible moves $(x, x - s)$, where $x \in V$ and $s \in S$. All subtraction games are strongly miserable, as the next statement shows.

Proposition 6 *(Ferguson (1974) [15]). Every subtraction game G_S satisfies the Ferguson property (iii).*

Since the proof by Ferguson (1974) [15]) is very short and elegant, we copy it here for readers' convenience. It is based on the following lemma that is of independent interest.

Lemma 19 *(Ferguson (1974) [15]). Let k be the smallest element of S. Then $g^N_S(x) = 0$ implies $g^N_S(x + k) = 1$. Conversely, $g^N_S(x) = 1$ implies $g^N_S(x - k) = 0$.*

Proof: Since $k \in S$, $g^N_S(x) = 0$ implies $g^N_S(x + k) \neq 0$. Assume the conclusion is false and find the smallest x such that $g^N_S(x) = 0$ and $g^N_S(x + k) > 1$. By the latter, there is an $s \in S$ such that $g^N_S(x+k-s) = 1$ and $x-s \geq 0$, since k is the smallest element of S. Furthermore, $g^N_S(x) = 0$ implies $g^N_S(x - s) > 0$. Thus, there exists an $s' \in S$ such that $g^N_S(x - s - s') = 0$. This, together with $g^N_S(x+k-s) = 1$ entails $g^N_S(x-s-s'+k) > 1$. Thus, $y = x-s-s' < x$ also satisfies $g^N_S(y) = 0$ and $g^N_S(y + k) > 1$ contradicting the choice of x as the smallest one.

Conversely, if $g^N_S(x) = 1$ and $g^N_S(x-k) \neq 0$, there is an $s \in S$ such that $g^N_S(x-k-s) = 0$. From the first part of the theorem, this implies $g^N_S(x - s) = 1$. It contradicts $g^N_S(x) = 1$. □

Proof of Proposition 6. Given any nonterminal x such that $g^N_S(x) = 0$, one has $g^N_S(x - k) \neq 0$, where k is the smallest element of S. This implies that there is an $s \in S$ such that $g^N_S(x - k - s) = 0$. From the lemma, $g^N_S(x - s) = 1$. □

3.8 Not miserable impartial games

Of course, "most" of the impartial games are not miserable. For example, let us consider game Mark recently introduced by Fraenkel (2011) [20]. The set of positions of Mark is $\mathbb{Z}_{\geq 0}$ and for each $n \in \mathbb{Z}_{\geq 0}$ only the moves to $n - 1$ and $\lfloor n/2 \rfloor$ are allowed. The first SG values of the normal and misère versions of this game are given in Table 1. This table shows that Mark is not miserable. Indeed, for some n, pairs $g(n) = (g^N(n), g^M(n))$ take values $(0, 0), (0, 2)$ and $(2, 1)$.

Let us remark that Mark is not a subtraction game. Moreover, it has very different properties. For example, all subtraction games are strongly miserable, while Mark is not even miserable. It is also shown Fraenkel (2011) [20] that the SG function of Mark is not periodic, unlike the SG functions of all subtraction games.

Another example of a non-misrable game is given by the digraphs G^N and G^M in Figure 1. For this game $g(v) = (g^N(v), g^M(v))$ takes values $(0,0), (2,0)$ and $(3,2)$.

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