RUTCOR
RESEARCH
REPORT

Triangle inequality for resistances

Vladimir Gurvich [a]

RRR 20-2012, July, 2012

RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003; email: gurvich@rutcor.rutgers.edu

RUTCOR RESEARCH REPORT
RRR 20-2012, JULY, 2012

TRIANGLE INEQUALITY FOR RESISTANCES

Vladimir Gurvich

Abstract. Given an electrical circuit each edge e of which is an isotropic conductor with a monomial conductivity function $y_e^* = y_e^r / \mu_e^s$. In this formula, y_e is the potential difference and y_e^* current in e, while μ_e is the resistance of e, while r and s are two strictly positive real parameters common for all edges.
In 1987, Gvishiani and Gurvich [4] proved that, for every two nodes a, b of the circuit, the effective resistance $\mu_{a,b}$ is well-defined and for every three ordered nodes a, b, c the inequality $\mu_{a,b}^{s/r} \leq \mu_{a,c}^{s/r} + \mu_{c,b}^{s/r}$ holds. It obviously implies the standard triangle inequality $\mu_{a,b} \leq \mu_{a,c} + \mu_{c,b}$ when $s \geq r$. In 1992, the same authors showed [5] that the equality holds if and only if c belongs to every path between a and b.
Recently, Pavel Chebotarev has found several earlier works of 1967 by Gerald Subak-Sharpe [16, 17, 18] in which the inequality was shown for the case $s = r = 1$. Furthermore, it was rediscovered in 1993 by Douglas J. Klein and Milan Randić.
In this report we provide the story of the considered inequality with more details.

Key words: distance, metric, ultrametric; potential, voltage, current, Ohm law, Joule-Lenz heat, Maxwell's minimum energy dissipation principle, pressure, flow, maximum flow, minimum cut, shortest path, bottleneck path.

In the Moscow Institute of Physics and Technology they teach five semesters of general physics. The third one is Electricity. So, in December of 1970, I had to take the exam. Traditionally, it begins with a topic chosen by the student. I chose to compute the resistance of a n-node network that contains $\binom{n}{2}$ passive resistances, only. From high-school, I was sure that there should be a formula. In 1970, I was able derive it, having the concept of a determinant in my possession.

It was not too difficult. Take the $n \times n$ matrix L whose entry $\lambda_{i,j}$ is the conductance of the edge $e_{i,j}$ between nodes i and j (if there is none, $\lambda_{i,j} = 0$) and $\lambda_{i,i}$ is the sum of all conductances in the row (or column) i. The rest is simple. Eliminate row i and column j (respectively, two rows and two columns i and j) from L and denote the obtained matrix by $L'_{i,j}$ (respectively, by $L''_{i,j}$). Then the effective conductance $\Lambda_{i,j}$ between the poles i and j is just the ratio of the corresponding two determinants, more precisely,

$$\Lambda_{i,j} = |\det L'_{i,j} / \det L''_{i,j}|. \tag{1}$$

I still do not know who was the first to derive this formula; perhaps, Kirchhoff knew it, already. (Actually, L is called the so-called *Laplace matrix.*) Yet, I felt that did more than they do during the 8th year of the high-school computing "infinitely" many resistances of series-parallel networks. But will it be enough to impress the examiner? I was not completely sure of it and decided to check the triangle inequality for the corresponding resistances:

$$\Lambda_{i,k}^{-1} + \Lambda_{k,j}^{-1} \geq \Lambda_{i,j}^{-1} \ \ \forall\, i, j, k. \tag{2}$$

To my pleasant surprise, (2) indeed followed from (1), but not that easily. It took several hours and pages to derive it.

Next morning I reported, with pride, all this to an examiner. But, to my unpleasant surprise, he was not happy. Not at all.

- What devices are in your networks?

- Only passive resistances.

- And this is all?! No capacitors, launching coils, or generators?!

- No, nothing like that...

- How dare you to come with this?! We taught you Maxwell equations... And here... Resistances!

I shrugged this question off. He was pretty loud, so another examiner came to us and saved me from almost certain failure. I vaguely recall that his name was Kaloshnikov and he was teaching PhysLabs.

- What is all this noise about?

My examiner explained.

- So what?! I find all this curious. Let me finish the exam with this student...

In fact, they demonstrated two possible reactions to resistance distances that are still typical for chemists, physicists, and electrical engineers.

I should confess that I did not give much importance to this result either and did not publish it until 1987. Thenadays, I was working for the Schmidt Institute of Physics of the Earth of the Soviet Academy of Sciences, in the Laboratory of Mathematical Methods of Earthquakes Prediction directed by Alexei Gvishiani. He was graduated in 1975 from Mech.-Mat. MSU, where studied functional analysis advised by Israel Gelfand and Alexander Kirillov. (Also, he is a grandson of the Soviet prime minister, Alexei Kosygin, 1904–1980.)

Although network resistances are pretty far from earthquakes, yet, not farer than the functional analysis is. So Gvishiani and I decided to generalize and publish the result.

We replaced the classical Ohm law $y_e^* = y_e/\mu_e$ by a monomial conductivity law

$$y_e^* = y_e^r/\mu_e^s, \tag{3}$$

where e is a conductor of the network, y_e is the tension (or the potential drop) on e and y_e^* is the current through e, furthermore, μ_e is the resistance of e, while r and s are real strictly positive parameters that do not depend on e.

For example, the quadratic law, $r = 1/2$, is typical for hydraulics or gas dynamics, where y_e is the pressure drop and y_e^* is the flow through the pipe e.

It appears that the effective resistance $\mu_{i,j}$ is well defined for any two nodes i, j, and for every three ordered nodes i, j, k the following metric inequality holds:

$$\mu_{i,k}^{s/r} + \mu_{k,j}^{s/r} \geq \mu_{i,j}^{s/r} \ \ \forall\, i, j, k. \tag{4}$$

Interestingly, there are several proofs for the linear conductivity law, $r = 1$. Yet, I know only one general proof, but it is certainly the simplest one.

Let us fix potentials x_i and x_j in the nodes i and j. Without any loss of generality, assume that $x_i \geq x_j$. Then, some current $y_{i,j}^*$ will flow from i to j and some potential x_k in the node k will appear such that $x_i \geq x_k \geq x_j$.

Now, let us similarly fix x_i and x_k and determine the flow $y_{i,k}^*$. All we have to prove is the inequality $y_{i,j}^* \geq y_{i,k}^*$. And this is not difficult. Then, by symmetry, we have $y_{i,j}^* \geq y_{j,k}^*$. Finally, any good high-school student could derive the triangle inequality from the the above two inequalities as follows.

$$y_{i,j}^* = \frac{(x_i - x_j)^r}{\mu_{i,j}^s} \geq \frac{(x_i - x_k)^r}{\mu_{i,k}^s} = y_{i,k}^*; \quad y_{i,j}^* = \frac{(x_i - x_j)^r}{\mu_{i,j}^s} \geq \frac{(x_k - x_j)^r}{\mu_{k,j}^s} = y_{k,j}^*, \tag{5}$$

which can be obviously rewritten as

$$\left(\frac{\mu_{i,k}}{\mu_{i,j}}\right)^{s/r} \geq \frac{x_i - x_k}{x_i - x_j}; \quad \left(\frac{\mu_{k,j}}{\mu_{i,j}}\right)^{s/r} \geq \frac{x_k - x_j}{x_i - x_j}. \tag{6}$$

Summing up these two inequalities one gets exactly (4).

It is also easy to show that (4) holds with equality iff $y^*_{i,j} = y^*_{i,k} = y^*_{k,j}$, which happens iff node k belongs to every path between the nodes i and j. The metric spaces that have the latter properties are called *geodetic* [2]. However, we noticed the geodetic property of resistance distances only later, when the paper was already published. So, this observation appeared only in 1992, in our book [5] published in Russian. For the linear case, $r = s = 1$ it was rediscovered “in the next millenium”; see for example, [2].

To submit our result, we chose the journal of “Functional Analysis and its Applications”, founded and led by Israel Gelfand. I still believe that the metric inequality for resistances does belong to the functional analysis. Moreover, as far as applications are concerned, our paper was just fine, unlike most of the papers in this journal.

However, Gelfand was not too happy. I can only guess about his reasons. Perhaps, in contrast to my physics-examiner, who decided that there are no serious “devices” and (2) is too ”mathematical”, Gelfand decided that there are too many “devices” and (4) is not enough “functionally analytic”. Anyway, he suggested to withdraw the paper from ”Functional Analysis” and resubmit it to the “Communications of the Moscow Math. Soc. in Russian Math. Surveys.” This was a good journal and a reasonable compromise.

It would be natural for Gelfand to call Gvishiani, his former student, and tell him the verdict. Yet, he chose a more sophisticated way. I got a call from Andrei Zelevinsky, my former classmate and another Gelfand’s student (perhaps, the best one; now he is a famous mathematician, the inventor of Cluster Algebras). I asked Andrei what is wrong with submitting to the “Functional Analysis” and got a pretty curious answer. He said that it is one of a few journals that have a peculiar style and submitting our paper there is like coming in a ski suit (although in a very good one) to a banquet, where all other wear fracs.

Anyway, the offer was decent and it was accepted. The only disadvantage (but quite a serious one) was that I had to squeeze the eight pages long paper to just two pages, because the “Comm. of the Moscow Math. Soc.” were (and still are) strictly limited to this size.

In any case, the paper [4] was never cited before I republished a more detailed (10 pages) version of it in the “Discrete Applied Mathematics”, in 2010 [7]. Also, I presented the metric inequality for resistances and some related results as a series of the “high-school problems” in the Russian journal Mathematical Enlightenment [6].

It was recently (in the Fall of 2011) discovered by Pavel Chebotarev that the metric inequality, for the linear case, $r = s = 1$, was published much earlier, in 1967, by Gerald E. Subak-Sharpe [16, 17, 18], a Professor of Electrical Engineering from New York City. Till 2012 his works were not cited either. Professor Subak-Sharpe passed away [19] on September 11th, 2011, just shortly before his breakthrough of 1967 was finally recognized by colleagues.

The name ”resistance distance” was coined later by Douglas J. Klein and Milan Randić. They also rediscovered the metric inequality, for the linear case, and published it in 1993 in the J. of Math. Chemistry [9]. Their paper got 111 citations.

References

[1] D. Babić, D.J. Klein, I. Lukovits, S. Nikolić, and N. Trinajstić, Resistance-Distance Matrix: A Computational Algorithm and Its Applications, Int. J. Quant. Chem. **90** (2002) 166-176.

[2] P. Chebotarev, A class of graph-geodetic distances generalizing the shortest-path and the resistance distances, Discrete Applied Mathematics **159:5** (2011) 295-302.

[3] M. Deza and E. Deza, Encyclopedia of Distances, Springer-Verlag, 2009.

[4] A.D. Gvishiani and V.A. Gurvich, Metric and ultrametric spaces of resistances, Communications of the Moscow Mathematical Society, in Russ. Math. Surveys, **42:6** (258) (1987) 235–236.

[5] A.D. Gvishiani and V.A. Gurvich, Dynamical classification problems and convex programming in applications, Moscow, Nauka (Science), 1992 (in Russian).

[6] V.A. Gurvich, Metric and ultrametric spaces of resistances, Math. Enlightenment **13** (2009) 134–141 (in Russian).

[7] V.A. Gurvich, Metric and ultrametric spaces of resistances, Discrete Applied Math. **158** (2010) 1496–1505.

[8] D.J. Klein, J.L.L. Palacios, M. Randić, and N. Trinajstić, Random Walks and Chemical Graph Theory, J. Chem. Inf. Comput. Sci. **44:5** (2004) 15211525.

[9] D.J. Klein and M.J. Randić, Resistance distance, J. Math. Chem., **12** (1993) 81-95.

[10] D.J. Klein, Resistance-Distance Sum Rules, Croat. Chem. Acta **75** (2002) 633-649.

[11] O.V. Lyashko, Why resistance does not decrease? Kvant **1** (1985) 10-15 (in Russian). English translation in Quant. Selecta, Algebra and Analysis II, S. Tabachnikov ed., AMS, Math. World **15** (1999) 63-72.

[12] L. Lukovits, S. Nikolić, and N. Trinajstić, Resistance Distance in Regular Graphs, Int. J. Quant. Chem. **71** (1999) 217-225.

[13] L. Lukovits, S. Nikolić, and N. Trinajstić, Note on the Resistance Distances in the Dodecahedron, Croat. Chem. Acta **73** (2000) 957-967.

[14] Minty G. J. *Monotone networks* // Proceedings of the Royal Society of London, Ser. **A** **257** (1960) 194–212.

[15] J.L.L. Palacios, Closed-Form Formulas for Kirchhoff Index, Int. J. Quant. Chem. **81** (2001) 135-140.

[16] G.E. Sharpe, Solution of the (m+1)-terminal resistive network problem by means of metric geometry, in: Proceedings of the First Asilomar Conference on Circuits and Systems, Pacific Grove, CA (November 1967) 319–328.

[17] G.E. Sharpe, Theorem on resistive networks, Electron. Lett. **3** (1967) 444–445.

[18] G.E. Sharpe, Violation of the 2-triple property by resistive networks, Electron. Lett. **3** (1967) 543544.

[19] Gerald Subak-Sharpe died on Sep. 26th 2011, Obituary at http://www.columbiapaper.com/index.php/obituaries/2243-by-staff .

[20] W. Xiao and I. Gutman, Resistance Distance and Laplacian Spectrum, Theor. Chem. Acc. **110** (2003) 284-289.

[21] Wikipedia, Resistance Distance, at http://en.wikipedia.org/wiki/Resistance_distance .

[22] WolframMathWorld, Resistance Distance, at http://mathworld.wolfram.com/ResistanceDistance .