RUTCOR RESEARCH REPORT

Closed Loop Layout

Sadegh Niroomand[a] Béla Vizvári[b]

RRR 21-2012, September 2012

RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a]Dept. of Industrial Engineering, Eastern Mediterranean University, sadegh.niroomand@cc.emu.edu.tr
[b]Dept. of Industrial Engineering, Eastern Mediterranean University, bela.vizvari@emu.edu.tr

Rutcor Research Report
RRR 21-2012, September 2012

Closed Loop Layout

Sadegh Niroomand   Béla Vizvári

Abstract. In layout problem of manufacturing cells, rectangular cells to be positioned without overlapping. The objective is to minimize the total transportation cost. The types of layouts are categorized according to the shape of the transportation system's track. In the case of a closed loop layout, the track has a rectangular shape. A common difficulty of all layout problems is the manner in which distances are measured. A frequently used approximation is the Manhattan distance. However, it is significantly shorter than the real distance in many cases. Both meta-heuristics and exact models suggested by earlier studies use the Manhattan distance.

In this paper, a new mathematical model is suggested for the closed loop layout with exact distances. Many feasible solutions are generated for benchmark problems that are competitive with the solutions provided by meta-heuristics.

1 Introduction

One of the most important costs in manufacturing systems is the material handling cost. Material handling and layout costs constitute approximately 20-50 percent of the total operating costs in manufacturing [Tompkins *et al.* 1996]. In the last two decades, several mathematical models were created to decrease the material handling cost in manufacturing systems, which is called a facility layout problem.

A facility layout design problem can be classified as either (1) a general Facility Layout Problem (FLP), which only considers each department's area and the determination of the shapes of the cells is the part of the problem, or (2) a Machine Layout Problem (MLP), which considers each department's, or machine's, specific shape [Chae-Peters 2006]. This study addresses problem (2), i.e., the MLP in a flexible manufacturing system environment.

There are four commonly used FMS layout design shapes: the spine, circular (closed loop), ladder and open field layouts [Luggen 1991]. The spine layout is a configuration in which cells are located on a single, direct line, which is the material handling path between cells. This configuration may be on one side of a line [Samarghandi *et al.* 2010] or the cells may be located on both sides of the line [Ting, J.-H., Tanchoco 2001] without overlapping. All pick-up/drop-off points are also placed on the line. In a closed loop layout, the material handling path is a rectangle in which cells are either located inside or outside the rectangle, but all pick-up/drop-off points are on the edges of the rectangle [Chae-Peters 2006, Tavakkoli-Moghaddam and Panahi 2007]. In this type of configuration, there may be shortcuts available to connect two opposite sides [Lasardo and Nazzal 2011]. The ladder layout includes several vertical and horizontal direct lines that serve as material handling paths; one or more cells are placed in each rectangle formed by those lines. In the open field layout, there is no restriction on the layout pattern [Rajasekharan *et al.* 1998]. This means that there is no limitation in the material handling path. In this paper, the focus will be on the closed loop configuration.

[Das 1993] introduced a mixed integer linear programming (MILP) model for the open field layout type and proposed a heuristic method that included four steps; the method was named the 'four-step open field layout'. In the first step, using the spine method, an upper bound to the FLP objective was obtained. A determination of the orientation and spatial sequencing of the cells was performed in the second step. In the third step, taking into account all of the other decisions made in step 2, the interference relationships of the cells were determined. In the last step, all decisions made in the second and third steps were considered as fixed variables and the locations of the pick-up/drop-off points and the spatial coordinates of the cells were determined. The spine method solution from step 1 was also taken into account in this final step.

[Rajasekharan *et al.* 1998], applied a genetic algorithm to Das's MILP model as an alternative solution procedure. This algorithm improved the quality and the computational time of the solution. The solution methodology consists of two steps. The first step considers the open field floor area for each cell's location, and the second step applies the genetic algorithm to find a good solution in the restricted open field floor.

[Chae-Peters 2006] continued Rajasekharan's and Das's studies. They restricted the material handling between cells to be located on a rectangular closed loop, and all pick-up/drop-off points had to be placed on this path. Although the problems are originally open field layout problems, the authors obtained acceptable solutions in some problems and ever better solutions in a few problems. They followed Das's MILP model and designed an algorithm to locate cells on a large-enough loop. In the next steps, they applied a simulated annealing method to improve the arrangement of cells in that fixed loop size. Then, the loop size becomes smaller, and the best arrangement in this situation will again be obtained by simulated annealing. The procedure is applied in an organized way for several loop shapes.

In all previous studies, the Manhattan distances between pick-up/drop-off points are considered to evaluate the material handling cost of the layout shape. Although this distance is correct in some arrangements, in other cases, it is not the real distance between two pick-up/drop-off points. For example, in the closed loop that was applied in Chae's paper, when two cells are on opposite sides of the rectangle and material is forced to move on that rectangular closed loop, the real distance between these two cells is greater than the Manhattan distance between them. The same problem may also occur in an open field layout when there is one cell between a pair of cells. In this case, the real distance between that pair of cells is again greater than the Manhattan distance between them.

In this paper, a new MILP model is introduced for the closed loop layout to eliminate such cases. The paper is organized as follows. In Section 2, the basic model of Das is introduced. The new model is discussed in Section 3. Section 4 contains some remarks about the model. The computational experiments are described in the next section. The final section of the paper contains the conclusions.

2 The Basic Model of Das

In all models, it is assumed that the rectangles of the cells have only horizontal and vertical edges and that the cells are not rotated in any other way.

An exact model of the layout of the rectangular cells must satisfy the following constraints:

- the cells must not overlap,
- the cells can be rotated by 90, 180 or 270 degrees,
- the method used to measure distances must be defined.

The model of [Das 1993] gives a perfect solution for avoiding overlap and applying rotation but contains only an approximation for distances.

Notations

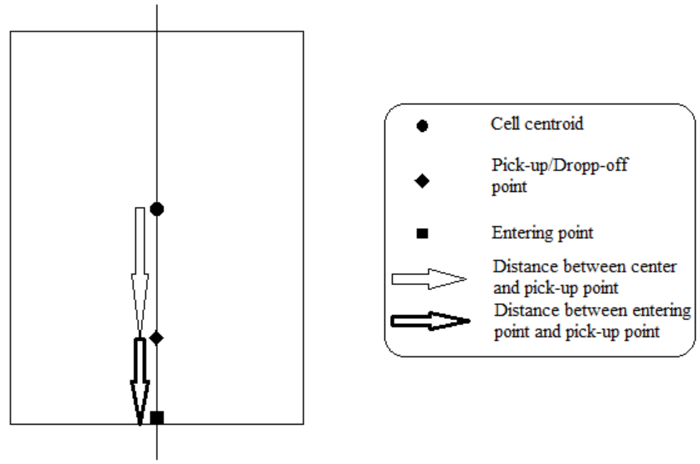
Table 1. Constants and variables used in the model of Das

n	the number of cells	parameter
i, j	indices of cells	index
x_i (y_i)	horizontal (vertical) coordinate of the center of cell i	variable
z_i	binary variable; it is 1 (0) if cell i is in a vertical (horizontal) position	variable
s_i	the length of the shorter edge of cell i	parameter
t_i	the length of the longer edge of cell i	parameter
ω_i	the distance of the pick-up point of cell i from the center of the cell	parameter
a_i (b_i)	horizontal (vertical) coordinate of the pick-up point of cell i	variable
e_{ij}	$\max\{0, x_i - x_j\}$	variable
f_{ij}	$\max\{0, x_j - x_i\}$	variable
g_{ij}	$\max\{0, y_i - y_j\}$	variable
h_{ij}	$\max\{0, y_j - y_i\}$	variable
p_{ij}	$\max\{0, a_i - a_j\}$	variable
q_{ij}	$\max\{0, a_j - a_i\}$	variable
π_{ij}	$\max\{0, b_i - b_j\}$	variable
ρ_{ij}	$\max\{0, b_j - b_i\}$	variable
φ_{ij}	the flow value between cells i and j	parameter
α_{ij}	a binary variable; it is 1 if $x_i \geq x_j$	variable
β_{ij}	a binary variable; it is 1 if $y_i \geq y_j$	variable
δ_{ij}	a binary variable; if it is 1, then cells i and j are not overlapping vertically, and if it is 0, then cells i and j are not overlapping horizontally	variable
M	a large positive number	
λ_{1i}, λ_{2i}, λ_{3i}, λ_{4i},	binary variables describing the position of the pick-up point of cell i according to its rotation	variable

In the discussion that follows, the cells are represented by their center. The position of a cell is vertical (horizontal) if the position of its longer edge is vertical (horizontal). In the case of a square, the tie can be broken arbitrarily.

Two cells are overlapping if and only if their centers are too close to each other. The minimal required horizontal (vertical) distance such that two cells are not overlapping is half the sum of the length of their edges in the horizontal (vertical) position. The sum depends on the rotation of the cells. Notice that $e_{ij} + f_{ij}$ ($g_{ij} + h_{ij}$) is the horizontal(vertical) distance of the centers of the cells i and j. If there is no horizontal (vertical) overlap, then the distance must be at least as long as the sum of the two horizontal (vertical) half edges.

Figure 1: Cell with entering points and pick-up points.



This requirement is described by the following inequalities:

$$\forall i,j,i\neq j:\ e_{ij}+f_{ij}-\frac{1-z_i}{2}t_i-\frac{z_i}{2}s_i-\frac{1-z_j}{2}t_j-\frac{z_j}{2}s_j\geq -M\delta_{ij} \tag{1}$$

and

$$\forall i,j,i\neq j:\ g_{ij}+h_{ij}-\frac{1-z_i}{2}s_i-\frac{z_i}{2}t_i-\frac{1-z_j}{2}s_j-\frac{z_j}{2}t_j\geq -M(1-\delta_{ij}). \tag{2}$$

It is difficult to use the formulae of e_{ij}, f_{ij}, g_{ij}, and h_{ij} explicitly in an optimization problem; therefore, they are described implicitly by the following constraints:

$$\begin{aligned}
\forall i,j,i\neq j:\ x_i-x_j &= e_{ij}-f_{ij} &(3)\\
\forall i,j,i\neq j:\ y_i-y_j &= g_{ij}-h_{ij} &(4)\\
\forall i,j,i\neq j:\ e_{ij} &\leq M\alpha_{ij} &(5)\\
\forall i,j,i\neq j:\ f_{ij} &\leq M(1-\alpha_{ij}) &(6)\\
\forall i,j,i\neq j:\ g_{ij} &\leq M\beta_{ij} &(7)\\
\forall i,j,i\neq j:\ h_{ij} &\leq M(1-\beta)_{ij}. &(8)
\end{aligned}$$

Notice that the inequalities (1) and (2) handle both overlapping and rotation. Constraints (5) and (6) with nonnegativity (see constraint (17) below) ensure that either e_{ij} or f_{ij} is equal to zero.

The next main step is the formulation of the objective function. It is the minimization of the sum of the flow between cells weighted by the distance of the pick-up points of the cells. The first step is to determine the coordinates of the pick-up points. The pick-up point is on one of the middle lines of the cell at distance ω_i (see Fig. 1). If the pick-up point is on the shorter edge, then $\omega_i = t_i/2$, and if the pick-up point is on the longer edge, then

$\omega_i = s_i/2$. However, the point can also be inside the cell. This means that if $\omega_i = 0$ then the two coordinates of the pick-up point are the same as those of the center point of the cell; otherwise, one coordinate is different, and the other one is equal to the same coordinate as the center point. In the latter case, the coordinates depend (i) on the definition of the position of the point, i.e., the point is on the middle line connecting the two shorter/longer edges, and (ii) on the rotation of the cell in the layout. The rotation of cell i is described by four binary variables, λ_{1i}, λ_{2i}, λ_{3i}, and λ_{4i} defined as follows:

$$\lambda_{1i} = \begin{cases} 1 & \text{if the pick-up point is on the right side of the center} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{2i} = \begin{cases} 1 & \text{if the pick-up point is below the center} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{3i} = \begin{cases} 1 & \text{if the pick-up point is on the left side of the center} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{4i} = \begin{cases} 1 & \text{if the pick-up point is above the center} \\ 0 & \text{otherwise} \end{cases}$$

The λ and z variables are not independent. Two equations must hold between them. If the pick-up point is on the middle line connecting the two shorter edges, then the cell is in a vertical position if the pick-up point is below or above the center. Thus,

$$z_i \;=\; \lambda_{2i} + \lambda_{4i} \tag{9}$$

implying that

$$1 - z_i \;=\; \lambda_{1i} + \lambda_{3i}. \tag{10}$$

If the pick-up point is on the middle line connecting the two longer edges, then the form of the equations is as follows (using the same equation numbering):

$$z_i \;=\; \lambda_{1i} + \lambda_{3i} \tag{9}$$

and

$$1 - z_i \;=\; \lambda_{2i} + \lambda_{4i}. \tag{10}$$

Finally, the two co-ordinates of the pick-up point of cell i are

$$\forall i: \quad a_i \;=\; x_i \,+\, \omega_i(\lambda_{1i} - \lambda_{3i}) \tag{11}$$

and

$$\forall i: \quad b_i \;=\; y_i \,+\, \omega_i(\lambda_{4i} - \lambda_{2i}). \tag{12}$$

The Manhattan distance of the cells i and j can be described by nonnegative variables p_{ij}, q_{ij}, π_{ij}, and ρ_{ij} as follows.

$$p_{ij} - q_{ij} = a_i - a_j \tag{13}$$

$$\pi_{ij} - \rho_{ij} = b_i - b_j. \tag{14}$$

Then, the Manhattan distance of the two cells is

$$p_{ij} + q_{ij} + \pi_{ij} + \rho_{ij}.$$

In the model of [Das 1993], the total Manhattan distance weighted with the flow among the cells is minimized, i.e., the objective function is

$$\min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \varphi_{ij}(p_{ij} + q_{ij} + \pi_{ij} + \rho_{ij}) \tag{15}$$

To complete the model, the technical constraints defining the type of the variables must be mentioned. Without loss of generality, we may assume that the cells are in the nonnegative quarter of the plane:

$$\forall\, i: \; x_i,\, y_i,\, a_i,\, b_i \;\geq\; 0. \tag{16}$$

The distance variables are also nonnegative:

$$\forall\, i,j: \; e_{ij},\, e_{ij},\, f_{ij},\, g_{ij},\, h_{ij},\, p_{ij},\, q_{ij},\, \pi_{ij},\, \rho_{ij},\; \geq\; 0. \tag{17}$$

All other variables are binary:

$$\forall\; i,j: \; z_i,\, \alpha_{ij},\, \beta_{ij},\, \delta_{ij},\, \lambda_{1i},\, \lambda_{2i},\, \lambda_{3i},\, \lambda_{4i}, = \; 0 \; \text{ or } \; 1. \tag{18}$$

The model (1)-(18) was developed by [Das 1993]. In the next section, the modification of the model for a closed loop layout with exact distances is elaborated.

3 Closed Loop Layout with Exact Distances

The vehicle transporting something to a cell enters the cell and moves to the pick-up point. The entering point is the middle point of the edge just in front of the pick-up point. The transportation performance within the cells is constant and is determined by the flow matrix and the position of the pick-up points within the cells. This quantity is introduced by [Das 1993] and is denoted in that paper by $TVLP_{LB}$. *The model developed below does not contain the quantity* $TVLP_{LB}$. *The total transportation among the entering points of the cells is minimized.* Hence, a_i and b_i denotes the coordinates of the entering point.

If the distance of the cells is measured as the Manhattan distance between well-defined points of the cells, then this distance can be shorter than what the vehicle must

Figure 2: The real distance and the Manhattan distance.

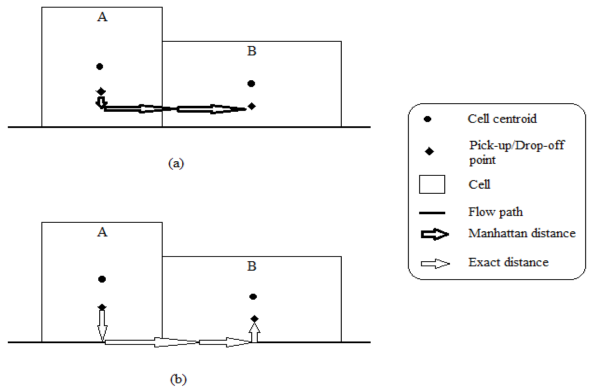
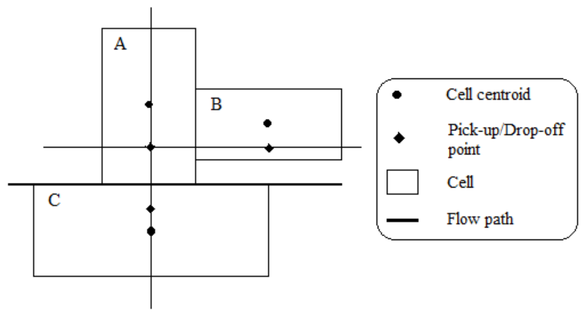


Figure 3: A solution that is optimal for the Manhattan distance, but is not optimal for the real distance.



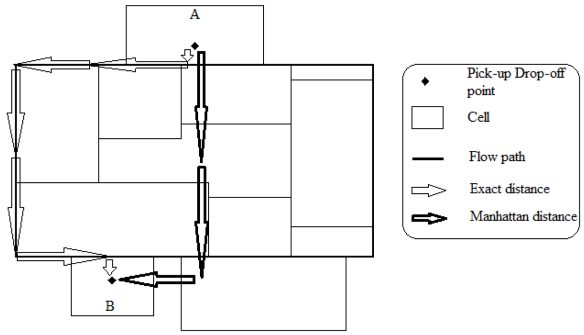
pass. Fig. 2 compares the Manhattan distance and the exact distance of two neighboring cells lying on a line. The Manhattan distance is shorter than the exact distance because the vehicle does not follow the whole path within the cell that it is required to follow. A 3-cell example is shown in Fig. 3; as an objective function, the Manhattan distance gives an optimal solution that can be improved in an obvious way for the real distances by shifting cell B down. Further on, there are configurations for which the real path cannot use the logic of the Manhattan distance, e.g., see Fig. 4.

In the case of closed loop layout, the shape of the track of the vehicle is a rectangle. *The entering points of the cells are on one edge of the rectangle.* The vehicle may use both directions. Between two entering points, the vehicle uses the direction that yields the shorter path.

Notations # 2

Table 2. Further notations related to distances

Figure 4: The route that is optimal for the Manhattan distance cannot be used.



k, l	indices of the edges of the track	index
w_{klij}	binary variable; it is 1 if cell i is on edge k and cell j is on edge l	variable
v_1, v_2	the two vertical coordinates of the track ($v_2 \geq v_1$)	variable
h_1, h_2	the two horizontal coordinates of the track ($h_2 \geq h_1$)	variable
d_{ij}^k	the distance of cells i and j if both are on edge k; 0 otherwise	variable
d_{ij}^{kl}	the distance of cells i and j if cell i is on edge k and cell j is on edge l ($k \neq l$); 0 otherwise	variable
ψ_{ki}	a binary variable; it is 1 if cell i is on edge k	variable
ϑ_i	a binary variable; it is 1 if cell i is outside the track	variable
a_i (b_i)	the x (y) coordinate of the entering point of cell i	variable
u_{13ij}, u_{31ij} u_{24ij}, u_{42ij}	binary variables; they select the minimal path for the vehicle if cells i and j are on opposite edges	variable

The edges of the track have the following indices: the upper horizontal edge is 1, the right vertical edge is 2, the lower horizontal edge is 3, and the left vertical edge is 4.

Notice that

$$w_{klij} = \psi_{ki}\psi_{lj}.$$

This relation can be described equivalently by two linear inequalities using an old integer programming technique:

$$2w_{klij} \leq \psi_{ki} + \psi_{lj}$$
$$\psi_{ki} + \psi_{lj} - 1 \leq w_{klij}.$$

The equivalence is based on the fact that all three variables are binary. Therefore, the first set of constraints of the model is

$$2w_{klij} \ \leq \ \psi_{ki} + \psi_{lj} \quad 1 \leq i < j \leq n, \ k,l = 1,2,3,4 \tag{19}$$

$$\psi_{ki} + \psi_{lj} - 1 \leq w_{klij} \quad 1 \leq i < j \leq n, \ k,l = 1,2,3,4. \tag{20}$$

The distances are restricted in the model only from below. In the optimal solution, the optimality condition forces them to be equal to their maximal lower bound.

There are several cases according to the (potential) position of the two cells.

Case 1: cells i and j are both on edge k. One of the coordinates of the two entering points is the same. If they are on edge 1 or 3, then the common coordinate is the y coordinate; otherwise, it is the x coordinate. Let d_{ij}^k be the distance of the entering points of the two cells. In any other case, d_{ij}^k is 0. Then, the distance must satisfy the following inequalities.

$$d_{ij}^k + M(1 - w_{kkij}) \ \geq \ a_i - a_j \quad 1 \leq i < j \leq n, \ k = 1,3 \tag{21}$$

$$d_{ij}^k + M(1 - w_{kkij}) \ \geq \ a_j - a_i \quad 1 \leq i < j \leq n, \ k = 1,3 \tag{22}$$

$$d_{ij}^k + M(1 - w_{kkij}) \ \geq \ b_i - b_j \quad 1 \leq i < j \leq n, \ k = 2,4 \tag{23}$$

$$d_{ij}^k + M(1 - w_{kkij}) \ \geq \ b_j - b_i \quad 1 \leq i < j \leq n, \ k = 2,4 \tag{24}$$

Notice that the constraints (21)-(24) are not restrictive if both of cells i and j are not on edge k; $w_{kkij} = 0$, and the constraint is satisfied automatically.

Case 2: cells i and j are on two adjacent edges. The vehicle must pass the intersection point of the two edges of the track. For example, if cell i is on edge 1 and cell j is on edge 2, then the vehicle must go through the upper left corner of the track. The coordinates of this point are (v_2, h_2). The pick-up point of cell i is to the left of this point, and the pick-up point of cell j is under this point. Hence, the distance d_{ij}^{12} must satisfy an inequality similar to the ones in (21)-(24):

$$d_{ij}^{12} + M(1 - w_{12ij}) \geq v_2 - a_i \ + \ h_2 - b_j. \tag{25}$$

Similarly, the distances of Case 2 must satisfy the following inequalities.

$$d_{ij}^{21} + M(1 - w_{21ij}) \geq v_2 - a_j \ + \ h_2 - b_i \tag{26}$$

$$d_{ij}^{23} + M(1 - w_{23ij}) \geq v_2 - a_j \ + \ b_i - h_1 \tag{27}$$

$$d_{ij}^{32} + M(1 - w_{32ij}) \geq v_2 - a_i \ + \ b_j - h_1 \tag{28}$$

$$d_{ij}^{34} + M(1 - w_{34ij}) \geq a_i - v_1 \ + \ b_j - h_1 \tag{29}$$

$$d_{ij}^{43} + M(1 - w_{43ij}) \geq a_j - v_1 \ + \ b_i - h_1 \tag{30}$$

$$d_{ij}^{14} + M(1 - w_{14ij}) \geq a_i - v_1 \ + \ h_2 - b_j \tag{31}$$

$$d_{ij}^{41} + M(1 - w_{41ij}) \geq a_j - v_1 \ + \ h_2 - b_i \tag{32}$$

Case 3: cells i and j are on two parallel edges. Assume that cell i is on edge 1 and that cell j is on edge 3. Any path between them must reach one of the vertical edges of the track first on a horizontal edge. After that, the path must pass the vertical distance $h_2 - h_1$. Finally, the path must reach the target cell on the other horizontal edge. If the vehicle starts to move to the right, then the two distances on the horizontal edges are $v_2 - a_i$ and $v_2 - a_j$. If the vehicle moves in the opposite direction, then the two distances are $a_i - v_1$ and $a_j - v_1$. The vehicle must choose the shorter of the two paths. Thus, in that case,

$$d_{ij}^{13} \;=\; \min\{h_2 - h_1 + 2v_2 - a_i - a_j, h_2 - h_1 - 2v_1 + a_i + a_j\}$$

It is not easy to use the minimum function in a model. Therefore, a new binary variable, u_{13ij}, is introduced, which will select the minimum from the two above-mentioned distances. Thus, d_{ij}^{13} must satisfy the following two inequalities:

$$d_{ij}^{13} + M(1 - w_{13ij}) + Mu_{13ij} \;\geq\; h_2 - h_1 + 2v_2 - a_i - a_j \tag{33}$$

and

$$d_{ij}^{13} + M(1 - w_{13ij}) + M(1 - u_{13ij}) \;\geq\; h_2 - h_1 - 2v_1 + a_i + a_j. \tag{34}$$

Notice that if $w_{13ij} = 0$, then neither (33) nor (34) is binding. In that case, d_{ij}^{13} can be on the lower bound, which is 0, as will be described later. As was mentioned above, the objective function will determine the value of u_{13ij} in such a way that d_{ij}^{13} is as small as possible. Formally, there are feasible solutions satisfying both (33) and (34) with a strict inequality, but they are not optimal. Based on a similar analysis, the following inequalities must be satisfied:

$$d_{ij}^{31} + M(1 - w_{31ij}) + Mu_{31ij} \;\geq\; h_2 - h_1 + 2v_2 - a_i - a_j \tag{35}$$

$$d_{ij}^{31} + M(1 - w_{31ij}) + M(1 - u_{31ij}) \;\geq\; h_2 - h_1 - 2v_1 + a_i + a_j \tag{36}$$

$$d_{ij}^{24} + M(1 - w_{24ij}) + Mu_{24ij} \;\geq\; v_2 - v_1 + 2h_2 - b_i - b_j \tag{37}$$

$$d_{ij}^{24} + M(1 - w_{24ij}) + M(1 - u_{24ij}) \;\geq\; v_2 - v_1 - 2h_1 + b_i + b_j \tag{38}$$

$$d_{ij}^{42} + M(1 - w_{42ij}) + Mu_{42ij} \;\geq\; v_2 - v_1 + 2h_2 - b_i - b_j \tag{39}$$

$$d_{ij}^{42} + M(1 - w_{42ij}) + M(1 - u_{42ij}) \;\geq\; v_2 - v_1 - 2h_1 + b_i + b_j. \tag{40}$$

The next set of constraints determines the positions of the cells from the closed loop point of view. Each cell must be either completely inside or completely outside the track. Furthermore, the edge of the cell where the vehicle may enter the cell must lie on one of the edges of the track.

The coordinates of the four corner points of cell i depend on the rotation of the cell described by the binary variable z_j. They are as follows:

$$(x_i - \frac{1-z_i}{2}t_i - \frac{z_i}{2}s_i, y_i - \frac{1-z_i}{2}s_i - \frac{z_i}{2}t_i), \;\; (x_i - \frac{1-z_i}{2}t_i - \frac{z_i}{2}s_i, y_i + \frac{1-z_i}{2}s_i + \frac{z_i}{2}t_i),$$

$$(x_i + \frac{1-z_i}{2}t_i + \frac{z_i}{2}s_i, y_i + \frac{1-z_i}{2}s_i + \frac{z_i}{2}t_i), \;\; (x_i + \frac{1-z_i}{2}t_i + \frac{z_i}{2}s_i, y_i - \frac{1-z_i}{2}s_i - \frac{z_i}{2}t_i).$$

A cell is inside the track if

$$h_1 \ \le \ x_i - \frac{1-z_i}{2}t_i - \frac{z_i}{2}s_i, \quad h_2 \ \ge \ x_i + \frac{1-z_i}{2}t_i + \frac{z_i}{2}s_i$$

and

$$v_1 \ \le \ y_i - \frac{1-z_i}{2}s_i - \frac{z_i}{2}t_i, \quad v_2 \ \ge \ y_i + \frac{1-z_i}{2}s_i + \frac{z_i}{2}t_i.$$

Furthermore one of these inequalities must be satisfied.

A binary variable ϑ_i is introduced to describe whether cell i is outside or inside the track. $\vartheta_i = 1$ if cell i is outside.

The cell must satisfy different conditions if it is inside the track than if it is outside.

Inside constraints. A pair of inequalities must be satisfied for each of the four edges of the track. The first inequality claims that the cell is inside the track, and the second one claims that its entering point is on the edge. Obviously, the first constraint must not be claimed if the cell is outside, and the cell can be on only one of the edges. This means that the constraints must be satisfied automatically in certain cases, and this situation is ensured with the binary variables ψ_{ki}'s and ϑ_i's.

$$\text{Edge 1: } y_i + \frac{1-z_i}{2}s_i + \frac{z_i}{2}t_i - M\vartheta_i \le h_2,\ y_i + \frac{1-z_i}{2}s_i + \frac{z_i}{2}t_i + M\vartheta_i + M(1-\psi_{1i}) \ge h_2 \quad (41)$$

$$\text{Edge 2: } x_i + \frac{1-z_i}{2}t_i + \frac{z_i}{2}s_i - M\vartheta_i \le v_2,\ x_i + \frac{1-z_i}{2}t_i + \frac{z_i}{2}s_i + M\vartheta_i + M(1-\psi_{2i}) \ge v_2 \quad (42)$$

$$\text{Edge 3: } y_i - \frac{1-z_i}{2}s_i - \frac{z_i}{2}t_i + M\vartheta_i \ge h_1,\ y_i - \frac{1-z_i}{2}s_i - \frac{z_i}{2}t_i - M\vartheta_i - M(1-\psi_{3i}) \le h_1 \quad (43)$$

$$\text{Edge 4: } x_i - \frac{1-z_i}{2}t_i - \frac{z_i}{2}s_i + M\vartheta_i \ge v_1,\ x_i - \frac{1-z_i}{2}t_i - \frac{z_i}{2}s_i - M\vartheta_i - M(1-\psi_{4i}) \le v_1 \quad (44)$$

$$i = 1, 2, \ldots, n$$

The fact that the entering point of cell i must be on exactly one edge is expressed by the equation

$$\psi_{1i} + \psi_{2i} + \psi_{3i} + \psi_{4i} = 1 \quad i = 1, 2, \ldots, n. \qquad (45)$$

Notice that the first constraints are automatically satisfied if cell i is outside as $\vartheta = 1$; thus, a "large M" helps to make this possible. The two constraints of an edge together satisfy the equation if and only if $\vartheta_i = 0$, i.e., the cell is inside, and $\psi_{ki} = 1$, i.e., the indicator variable claims that the cell is on edge k.

Outside constraints. The lower edge of a cell cannot be higher than the upper edge of the track; otherwise, the cell is not on the track. Similarly, its left (upper, right) edge cannot be right (under, left) of the right (lower, left) edge of the track. However, the two edges must be on the same line if the indicator variable ψ_{ki} claims it. Moreover, if cell i is on a horizontal (vertical) edge of the track, then the horizontal (vertical) coordinate of its center point must be in the horizontal (vertical) range of the track. Hence, two pairs of inequalities must be satisfied for each edge of the track.

Edge 1:

$$\begin{aligned} y_i - \frac{1-z_i}{2}s_i - \frac{z_i}{2}t_i &\leq h_2, \\ y_i - \frac{1-z_i}{2}s_i - \frac{z_i}{2}t_i + M(1-\vartheta_i) + M(1-\psi_{1i}) &\geq h_2, \\ v_2 + M(1-\psi_{1i}) \geq x_i &\geq v_1 - M(1-\psi_{1i}) \end{aligned} \tag{46}$$

Edge 2:

$$\begin{aligned} x_i - \frac{1-z_i}{2}t_i - \frac{z_i}{2}s_i &\leq v_2, \\ x_i - \frac{1-z_i}{2}t_i - \frac{z_i}{2}s_i + M(1-\vartheta_i) + M(1-\psi_{2i}) &\geq v_2 \\ h_2 + M(1-\psi_{2i}) \geq y_i &\geq h_1 - M(1-\psi_{2i}) \end{aligned} \tag{47}$$

Edge 3:

$$\begin{aligned} y_i + \frac{1-z_i}{2}s_i + \frac{z_i}{2}t_i &\geq h_1, \\ y_i + \frac{1-z_i}{2}s_i + \frac{z_i}{2}t_i - M(1-\vartheta_i) - M(1-\psi_{3i}) &\leq h_1 \\ v_2 + M(1-\psi_{3i}) \geq x_i &\geq v_1 - M(1-\psi_{3i}) \end{aligned} \tag{48}$$

Edge 4:

$$\begin{aligned} x_i + \frac{1-z_i}{2}t_i + \frac{z_i}{2}s_i &\geq v_1, \\ x_i + \frac{1-z_i}{2}t_i + \frac{z_i}{2}s_i - M(1-\vartheta_i) - M(1-\psi_{4i}) &\leq v_1 \\ h_2 + M(1-\psi_{4i}) \geq y_i &\geq h_1 - M(1-\psi_{4i}) \\ i = 1, 2, \ldots, n \end{aligned} \tag{49}$$

Notice that the constraints claiming that a cell must lay on a certain edge of the track are satisfied automatically again in all other cases.

For the sake of completeness, the nature of the new variables is claimed again:

$$\forall\, i, j, k, l: \; w_{klij}, \psi_{ki}, \vartheta_i \; = \; 0 \text{ or } 1 \tag{50}$$

and

$$\forall\, i, j, k, l: \; d_{ij}^k, d_{ij}^{kl} \geq 0. \tag{51}$$

The objective function is the minimization of the total distance weighted by the flow values, i.e., it is

$$\min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \varphi_{ij} \sum_{k=1}^{4} \left(d_{ij}^k + \sum_{l \neq k} d_{ij}^{kl} \right). \tag{52}$$

The model of the closed loop layout with exact distances is the optimization of (52) under the constraints (1)-(14) and (16)-(51).

4 Degenerated Solutions and the Multiplicity of the Solutions

The spine layout is a degenerated version of the closed loop layout. In the case of the spine solution, all cells lie on the same line. If the line is horizontal (vertical), then $h_1 = h_2$ ($v_1 = v_2$). This type of solution can also occur for the case in which one cell closes the track at the end of the track, i.e., the cell is rotated 90 degrees toward the track and its center line is the line of the track.

The problem has a high degree of symmetry. In the case of the non-degenerated solutions, the layout can be rotated by 90, 180, and 270 degrees such that the solution remains geometrically congruent to the original solution. Moreover, the solution can be mirrored to the horizontal and vertical middle lines of the closed loop. Obviously, the application of any sequence of these transformations results in a congruent layout. The transformations generate the well-known dihedral group of the square. It consists of 8 elements: identity, the four reflections (to the horizontal and vertical axes and the two diagonals) and the three rotations (90, 180, and 270 degrees) (see Figure 5). It is a non-Abelian group, i.e., the operation is not commutative.

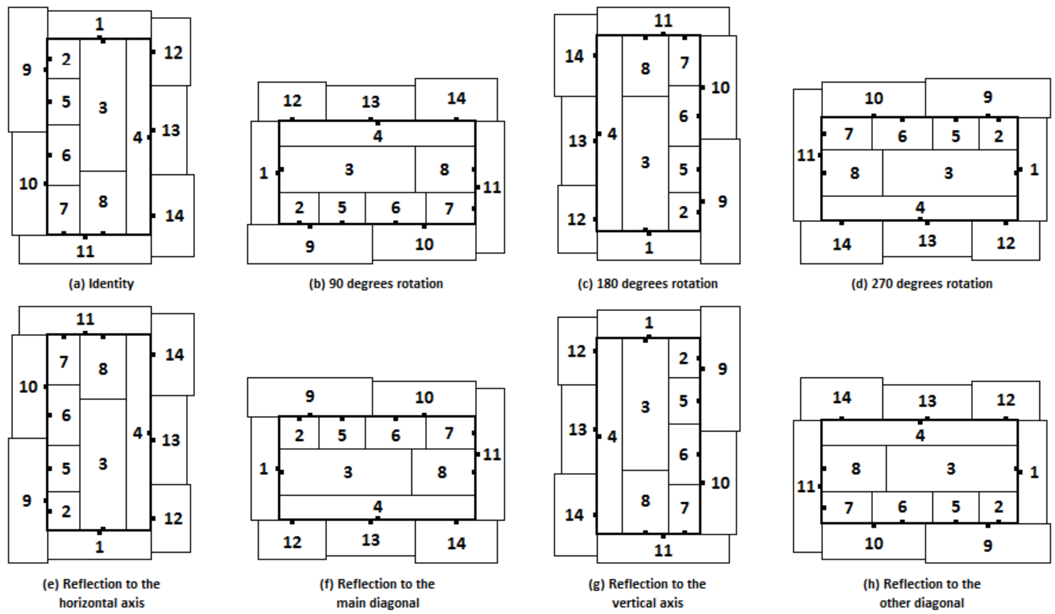
5 Computational Experiments

The high symmetry of the problem causes computational problems. This is true particularly in a branch and bound frame because there are eight equivalently good branches. The symmetry was broken by constraints similar to those used in [Sherali *et al.* 2003]. Any solution can be shifted on the plane without changing the transportation cost. It is equivalent to fixing the values h_1, and v_1. Both of them were fixed to 40. In this way there is enough space for all cells to be in the nonnegative quarter of the plane. Furthermore, the whole configuration can be put in a bounded area. For example, if the sums of the lengths, and widths of the cells are L, and W then all cells can be fitted into an area of size $L \times W$. The cell with the largest transportation flow was claimed to be in the lower left part of configuration. Finally, it was claimed that the layout has standing position, i.e., $v_2 \leq h_2$.

The computational experiments are carried out on the sequence of problems used by several authors. The sequence contains problems for $n = 4, 6, 8, 12, 14, 16, 18$. The first four problems were proposed by [Das 1993] and the last three were added to the sequence by [Rajasekharan *et al.* 1998]. The same problems are used in [Chae-Peters 2006].

An optimal solution is found, and its optimality is proven only for $n = 4$ and 6. This solution is shown on Figure 6. In all other cases, only feasible solutions are obtained.

Figure 5: 8 equivalent solutions according to the 8 elements of the dihedral group.



The objective function values of the best-known feasible solutions are listed in Tables 3, and 4. It is important to emphasize that the previous values listed in the second column of Table 4 concern to the Manhattan distances; however, the values obtained from the model given in (1)-(14), (16)-(51), and (52) which are listed in Table 3, are exact. Thus, a higher exact value may represent a smaller real transportation performance than the transportation performance of a layout determined by using the Manhattan distance. *Another difficulty of the comparison is that the previously generated layouts are unknown*, with the exception that [Das 1993] published three figures for the 6-cell problem. In all three cases, it is obvious that the Manhattan value is strictly less than the real objective function value. Moreover, the solutions called TAA-X and 4-STEP can be improved for real distances. The reconstructions of the 4-STEP solution are not unique because cells 3 and 4 can be shifted horizontally. The TAA-X solution is reconstructed from [Das 1993] and is shown in Figure 7. Its real objective function value is 4225.8, which is greater than the value of the best closed loop layout solution, which is 3255.8.

Table 3. The objective function values of the best-known feasible solutions for closed loop layout. The distances are exact. Solutions for 4, and 6 cells are optimal.

n	closed loop	inter-cell transportation cost	total transportation cost
4	547.5	1003.4	1550.9
6	1601.5	1654.3	3255.8
8	6634.5	4381.4	11015.9
10	14635.0	6627.4	21262.4
12	39765.0	14331.6	54096.6
14	65335.0	12980.0	78315.0
16	73942.0	15551.0	89493.0
18	96529.0	18525.0	115044.0

Table 4. The objective function values of the best-known feasible solutions for open field layout obtained from the literature and by optimizer. The distances are non-exact. Solutions for 4, and 6 cells are optimal.

	total costs	
n	literature	optimizer
4	1393.6	1393.6
6	2556.0	2556.0
8	8905.7	8793.3
10	15629.3	16245.1
12	36676.5	39940.6
14	41691.3	47661.5
16	55064.1	63506.4
18	66489.2	80090.4

The calculations were carried out by an Xpress-IVE system. The CPU times are long, i.e., greater than 10,000 seconds. Each problem was tried with different parameters of Xpress. The following parameters have importance. The total number of physically existing and logical processors is XPRS_THREADS. The B&B tree is different for different values of XPRS_THREADS; thus, different sets of feasible solutions are generated. Xpress uses several heuristics, even during the B&B procedure. They can be applied in smaller or larger environments and with different frequencies. The parameter XPRS_SEARCHEFFORT controls the number of calculations made by heuristics. The default value of the parameter is 1. The parameter is a multiplier, e.g., if XPRS_SEARCHEFFORT=1, then the heuristics work double compared with the default case. The frequency of the application of heuristics is controlled by XPRS_HEURFREQ. Heuristics are applied only at nodes such that their index is a positive integer multiple of XPRS_HEURFREQ. The types of heuristics applied in the root and in the tree are selected by XPRS_HEURSEARCHROOTSELECT and XPRS_HEURSEARCHTREESELECT, respectively.

Figure 6: The optimal closed loop layout solution of the 4-cell, and 6-cell problem.

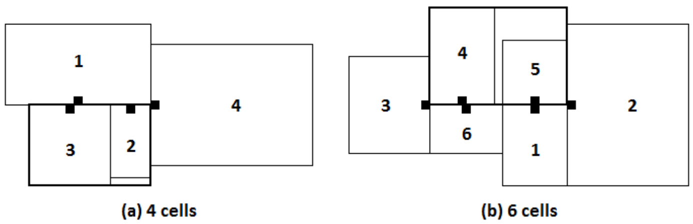
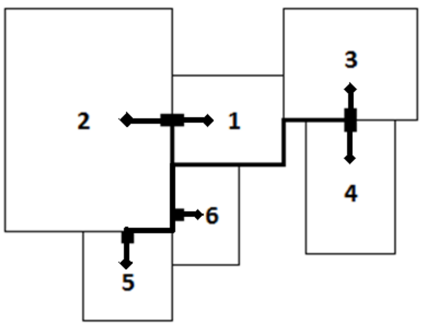


Figure 7: The TAA-X layout of the 6-cell problem. The layout is reconstructed from [Das 1993]. Notice that because the pick-up points are in the interiors of the cells, only the (1,2), (1,5), (2,6), (3,4), and (5,6) pairs have a Manhattan distance.



The experiments show that the generation of feasible solutions is sensitive for the values of the parameters. Differences can exist even among similar problems. From the point of view of feasible solutions, the 14-cell problem was more difficult than any other problem. In the case of the closed loop layout problems, a good set of parameter values is as follows:

XPRS_HEURSEARCHEFFORT=50, XPRS_HEURFREQ=20,
XPRS_HEURSEARCHROOTSELECT=7, and XPRS_HEURSEARCHTREESELECT=7.

Table 5. Best-known closed loop solutions of problems with 4, 6, and 8 cells of [Das 1993].

Problem	C4				C6				C8			
Cell	x_i	y_i	a_i	b_i	x_i	y_i	a_i	b_i	x_i	y_i	a_i	b_i
1	105	19	100	19	18	0	18	5	10	0	10	7
2	95.5	12.5	100	12.5	29.5	5	22	5	20	3.5	20	7
3	95	20	100	20	0	5	5	5	15	22	10	12
4	100	0	100	10	9	11	9	5	0	14.5	10	14.5
5					18	9	18	5	10	43.5	10	37
6					9.5	2	9.5	5	14	21	10	21
7									13	29.5	10	29.5
8									4.5	29.5	10	29.5
	h_1	h_2	v_1	v_2	h_1	h_2	v_1	v_2	h_1	h_2	v_1	v_2
	10	25	90	100	5	17	5	22	7	37	10	20

Table 6. Best-known closed loop solutions of problems with 10, 12, and 14 cells.

Problem	C10				C12				C14			
Cell	x_i	y_i	a_i	b_i	x_i	y_i	a_i	b_i	x_i	y_i	a_i	b_i
1	7.5	27	12.5	27	24.5	19.5	28	19.5	16	4	16	7.5
2	18.5	25	12.5	25	33	10	28	10	19	24.5	19	19.5
3	32	44.5	24.5	44.5	12	26	6	26	39.5	25.5	39.5	19.5
4	29	14.5	24.5	14.5	22	28	22	32	7.5	3.5	7.5	7.5
5	15.5	33.5	12.5	33.5	0	26	6	26	23.5	13.5	23.5	19.5
6	0	43.5	12.5	43.5	16	36	16	32	29.5	23.5	29.5	19.5
7	20.5	0	20.5	10	6	0	6	10	51.5	27	51.5	19.5
8	15	42	12.5	42	13.5	15.5	6	15.5	0	14	7.5	14
9	18	15.5	18	10	27	36.5	27	32	7.5	24	7.5	19.5
10	6	10	12.5	10	6	37	6	32	12.5	12.5	12.5	7.5
11					18.5	4.5	18.5	10	35	14	35	19.5
12					32	23	28	23	64.5	23.5	64.5	19.5
13									25.5	2.5	25.5	7.5
14									64.5	0	64.5	7.5
	h_1	h_2	v_1	v_2	h_1	h_2	v_1	v_2	h_1	h_2	v_1	v_2
	10	44.5	12.5	24.5	10	32	6	28	7.5	19.5	7.5	64.5

Table 7. Best-known closed loop solutions of problems with 16 and 18 cells of [Rajasekharan *et al.* 1998].

Problem	C16				C18			
Cell	x_i	y_i	a_i	b_i	x_i	y_i	a_i	b_i
1	12.5	9	12.5	5.5	22	8.5	22	5
2	33.5	25	28.5	25	29	38	29	33
3	34.5	14.5	28.5	14.5	1.5	7	7.5	7
4	32.5	5.5	28.5	5.5	11.5	1	11.5	5
5	1.5	41.5	7.5	41.5	21.5	27	21.5	33
6	3.5	5.5	7.5	5.5	3.5	28	7.5	28
7	0	17.5	7.5	17.5	47	15	39.5	15
8	0	30	7.5	30	0	17.5	7.5	17.5
9	28.5	46	28.5	41.5	35	27	39.5	27
10	12.5	0.5	12.5	5.5	12.5	10	7.5	10
11	23	0	23	5.5	45	27.5	39.5	27.5
12	11.5	18.5	7.5	18.5	7.5	37	7.5	33
13	33.5	35.5	28.5	35.5	20.5	0	20.5	5
14	23.5	13	23.5	5.5	18.5	40.5	18.5	33
15	12.5	33.5	7.5	33.5	34.5	0	34.5	5
16	25	35.5	28.5	35.5	11	20	7.5	20
17					39.5	40	39.5	33
18					11.5	29	7.5	29
	h_1	h_2	v_1	v_2	h_1	h_2	v_1	v_2
	5.5	41.5	7.5	28.5	5	33	7.5	39.5

The structure of the solution of the previous papers with the exception of the 4 and 6 cell problems of [Das 1993] are unknown. The best solutions for all problems found during this research project are listed in Tables 5 to 7.

6 Conclusions

A new MILP model is presented for closed loop layout problems with exact distances. The model was solved by the optimizer Xpress. In two cases, an optimal solution was found, and its optimality is proven. In all other cases, the obtained best feasible solutions are competitive with those generated by different heuristics. There is only one opportunity to compare the best feasible solution obtained from the MILP model to the best one obtained from other methods. In that case, the former solution is better.

The design of a layout is not a problem that must be solved in real-time mode, i.e., long CPU times are still acceptable. The size of real-life layout problems in an industrial environment is limited. One can expect that both computers and optimizers will become faster in the future. Thus, exact models and feasible solutions generated during the not completed optimization procedures remain competitive methods of solving layout problems.

In the near future, a breakthrough can occur in the computer technology. It can be a cubic processor 1,000 times faster than the recent processor [White 2011]. IBM and Intel announced the design of such processors. Another option is the spreading of quantum computers. The first one has started to work [Quantum computer]. Many problems which cannot be solved numerically for the time being, will become easy for the new computer technology. Non-real-time industrial design problems will belong to that category. This fact increases the importance of the exact models.

Acknowledgements

The authors are grateful to Junjae Chae and Brett A. Peters for providing the numerical problems.

References

[Chae-Peters 2006] Chae, J. and Peters, B.A., A simulated annealing algorithm based on a closed loop layout for facility layout design in flexible manufacturing systems. Int. J. Prod. Res., 44, 2006, 2561-2572.

[Das 1993] Das, S.K., A facility layout method for feaible manufacturing systems, International Journal of Production Research, 31(1993), No. 2, 279-297.

[Kim and Kim 2000] Kim, J.-G. and Kim, Y.-D., Layout planning for facilities with fixed shapes and input and output points. Int. J. Prod. Res., 38, 2000, 46354653.

[Lasardo and Nazzal 2011] Lasardo, V., Nazzal, D., Design of a manufacturing facility layout with a closed loop conveyor with shortcuts using queuing theory and genetic algorithm. Proceedings of the 2011 Winter Simulation Conference, 2011, 1964-1975.

[Luggen 1991] Luggen, W.W., Flexible Manufacturing Cells and Systems, Prentice-Hall, Englewood Cliffs, 1991.

[Quantum computer] http://en.wikipedia.org/wiki/Quantum_computer

[Rajasekharan *et al.* 1998] Rajasekharan, M., Peters, B.A. and Yang, T., A genetic algorithm for facility layout design in flexible manufacturing systems. Int. J. Prod. Res., 36, 1998, 95-110.

[Samarghandi *et al.* 2010] Samarghandi, H., Taabayan, P., Firouzi Jahantigh, F., A practical swarm optimization for the single row facility problem. Computer & Industrial Engineering, 58, 2010, 529-534.

[Sherali *et al.* 2003] Sherali, H.D., Fraticelli, B.M.P., Meller, R.D., Enhanced Model Formulation for Optimal Layout, Operations Research, 51(2003), 629-644.

[Tavakkoli-Moghaddam and Panahi 2007] Tavakkoli-Moghaddam, R., Panahi, H., Solving a new mathematical model of a closed-loop layout problem with unequal-sized facilities by a genetic algorithm. Proceedings of the 2007 IEEE IEEM, 2007, 327-331.

[Ting, J.-H., Tanchoco 2001] Ting, J.-H., Tanchoco, J. M. A, Optimal bidirectional spine layout for overhead material handling systems. IEEE Transactions on Semiconductor Manufacturing, 14(1), 2001, 57-64.

[Tompkins *et al.* 1996] Tompkins, J.A., White, J.A., Bozer, Y.A., Frazelle, E.H., Tanchoco, J.M.A. and Trevino, J., Facilities Planning, 2nd ed., Wiley, New York, 1996.

[White 2011] White, Ch.,Breakthrough: The Secret to Making Processors 1,000 Times Faster, http://mashable.com/2011/09/09/breakthrough-the-secret-to-making-processors-1000-times-faster-video/