RUTCOR
RESEARCH
REPORT

PROPERTIES AND CALCULATION OF MULTIVARIATE RISK MEASURES: MVAR AND MCVAR

Jinwook Lee [a] András Prékopa [b]

RRR 25-2012, OCTOBER 1, 2012

RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a] RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854; email: jinwook.lee@rutgers.edu
[b] RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854; email: prekopa@rutcor.rutgers.edu

RUTCOR RESEARCH REPORT
RRR 25-2012, OCTOBER 1, 2012

PROPERTIES AND CALCULATION OF MULTIVARIATE RISK MEASURES: MVAR AND MCVAR

Jinwook Lee   András Prékopa

Abstract. A recent paper by Prékopa (2012) presented results in connection with Multivariate Value-at-Risk (MVaR) that has been known for some time under the name of p-quantile or *p-Level Efficient Point* (pLEP) and introduced a new multivariate risk measure, called Multivariate Conditional Value-at-Risk (MCVaR). The purpose of this paper is to further develop the theory and methodology of MVaR and MCVaR. This includes new methods to numerically calculate MCVaR, for both continuous and discrete distributions. Numerical examples with recent financial market data are presented.

Keywords. Multivariate risk measure; Multivariate Value-at-Risk (MVaR); Multivariate Conditional Value-at-Risk (MCVaR); Multivariate quantile function; Multivariate stochastic order; Projection of MVaR; Stochastically dependent structure; Corporate M&A (Mergers and Acquisitions) deals; Risk of correlated assets; Low-correlation investment

1 Introduction

Value-at-Risk (VaR) has already existed in the statistical literature since the second half of the 19th century, under the name of quantile or percentile. The term Value-at-Risk was introduced at the beginning of the 1990s in the financial literature and became widely used in a short time. We refer the reader to Jorion (2006) and Saita (2007) for various topics of Value-at-Risk. Its multivariate counterpart turned up in the stochastic programming literature, primarily in the works of Prékopa (1970, 1973a, 1990, 1995, etc.). Based on this, Multivariate Conditional Value-at-Risk (MCVaR) was recently introduced by the same author (2012).

In stochastic programming one standard way to create a decision model out of one, where some of the parameters are random, is to prescribe a lower bound on the probability that the stochastic constraints are jointly satisfied. If, for example, a decision problem is an LP: min $c^T x$ subject to $Tx \geq \xi$, $Ax = b$, $x \geq 0$, where ξ is a random vector, then we may formulate the problem: $\min c^T x$ subject to $P(Tx \geq \xi) \geq p$, $Ax = b$, $x \geq 0$, or min $\{c^T x + \sum_{i=1}^{r} q_i E([\xi_i - T_i x]_+)\}$, subject to the same constraints, where T_i is the ith row of the $r \times n$ matrix T and the q_i, $i = 1, \ldots, r$ are nonnegative constants. The practical application of this model goes in such a way that first we decide on the value of x and, after that, we observe the realized value of ξ. The probability p is chosen near 1 so that the inequality $Tx \geq \xi$ should be satisfied in most cases. If ξ has continuous distribution and its c.d.f. is $F(z) = P(\xi \leq z)$, $z \in R^r$, then the probabilistic constraint can be rewritten as: $Tx \geq z$, for at least one z such that $F(z) = p$. If ξ is discrete, then we may use the *p-Level Efficient Points* (pLEP's), or briefly p-efficient points $z^{(1)}, \ldots, z^{(N)}$, and reformulate the probabilistic constraint as $Tx \geq z^{(i)}$, for at least one $i = 1, \ldots, N$. The above mentioned sets $\{z \mid F(z) = p\}$ and $\{z^{(1)}, \ldots, z^{(N)}\}$ can be regarded as multivariate quantiles. In Prékopa (2012) the term Multivariate Value-at-Risk (MVaR) was introduced as an alternative name for the collection of p-efficient points. Methods to generate elements of MVaR in the case of a continuously distributed ξ and the entire MVaR, in the discrete case, has already been existed in the literature (see, e.g., Prékopa (1995) and the references therein; Prékopa, Vizvári, Badics (1998); Boros et al. (1998); Dentcheva, Prékopa, Ruszczyński (2000), etc.).

The term Conditional Value-at-Risk (CVaR) was introduced by Rockafellar and Uryasev (2000). The same notion was named by Föllmer and Schied (2002) Average Value at Risk (AVaR). Earlier, in 1973, Prékopa has already used that risk measure in stochastic programming. If the rows of the matrix T are $T_1, \ldots, T_r$ and the components of ξ are $\xi_1, \ldots, \xi_r$, then the use of the constraints $E(\xi_i - T_i x \mid \xi_i - T_i x > 0) \leq d_i$, $i = 1, \ldots, r$ was proposed as replacement of the computationally more complicated constraint: $P(Tx \geq \xi) \geq p$, or, as a supplement to it. One major advantage of the conditional expectation constraints is that if the components of ξ have continuous distributions with logconcave p.d.f.'s, then each of them is equivalent to a linear constraint (see Prékopa (1973a, 1995)).

Let F denote the probability distribution function of the random variable $X \in R$. Then the Value-at-Risk(VaR), for some fixed probability level p, is defined as the p-quantile of the probability distribution function F:

$$\mathrm{VaR}_p(X) = F^{-1}(p), \tag{1}$$

where, by definition,

$$F^{-1}(p) = \min\{u \mid F(u) \geq p\}. \tag{2}$$

It can also be defined as the optimum value of the following two problems:

$$\begin{aligned}&\min v\\ &\text{subject to } P(X \leq v) \geq p,\end{aligned} \tag{3}$$

and

$$\begin{aligned}&\sup v\\ &\text{subject to } P(X \geq v) > 1-p,\end{aligned} \tag{4}$$

where p is some fixed probability, $0 < p < 1$. The optimum values of problems (3) and (4) are equal. Value-at-Risk (VaR) has a property, considered undesirable by many authors for a risk measure: it is not convex, in general, and it measures the frequency, not the amount of losses beyond VaR (the predicted maximum amount of losses at a fixed probability level). This motivated the development of the notion of a coherent risk measure, equal to the conditional expectation of a random variable, given that it surpasses $\text{VaR}_p(X)$. Conditional Value at Risk, designated by $\text{CVaR}_p(X)$, where X is the random variable involved and p the probability, is defined as:

$$\text{CVaR}_p(X) = E(X \mid X \geq \text{VaR}_p(X)). \tag{5}$$

Uryasev and Rockafellar (2000) and Pflug (2000) have shown that

$$\text{CVaR}_p(X) = \min_a \left\{ a + \frac{1}{1-p} E([X-a]_+) \right\} \tag{6}$$

and that $\text{CVaR}_p(X) \geq \text{VaR}_p(X)$. Equation (6) can also be used as the definition of $\text{CVaR}_p(X)$.

An optimization problem, similar to that of (6) was introduced and applied to a "chance constrained problem" by Ben-Tal and Teboulle (1986). After reformulation of the problem, a random objective function is obtained, for which a new type of "certainty equivalent" is formulated. If X is a random variable and u is a (increasing and strictly concave) utility function, then it is equal to:

$$\sup_a \{a + E[u(X-a)]\}. \tag{7}$$

In the mentioned and in other papers (see, e.g., Ben-Tal and Ben-Israel (1991), Ben-Tal et al. (1991), etc.) Ben-Tal, Ben-Israel and Teboulle expound a theory and application of the new certainty equivalent (7) and show that its negative enjoys the properties of a coherent risk measure in the sense of Artzner et al. (1999).

Coherence, however, is not such a property of a risk measure that it would be imperative to rely on it, under all circumstances. VaR is not a coherent risk measure, in general, but it is widely and successfully used in many applications, testing statistical hypotheses, sequential analysis, decision theory, stochastic programming and others. If, for example, an unfavorable event may cause huge damage that has to be avoided, then VaR may be more important than CVaR. Another point is that CVaR takes average on rare events while small probabilities are multiplied by large numbers, making the estimation of the risk measure inaccurate and we need very long trial sequences to realize the benefit of the conditional expectation in practice, where the conditioning event has very low frequency. If the population is at hand at the same time, then CVaR may have reasonable practical interpretation.

We think that risk measures and axiomatic systems for risk measures should not be regarded in an exclusive manner. As in geometry, various axiomatic systems define various geometries out of which we may choose the one most suitable for a given application, the axiomatic systems for risk measures and the risk measures themselves should provide us only with a menu to choose one or more than one for our purpose.

Multivariate risk measures, other than ours, exist in the literature (see, e.g., Cousin and Di Bernadino (2011) and the references therein.). The results in connection with them, however, are little to do with ours, especially because those risk measures are mostly vectors while ours are numbers. On the other hand we have in mind applications in stochastic optimization which require convexity statements and algorithms to calculate the numerical values of the risk measures.

The organization of this paper is as follows. In Section 2 we recall the notions of Multivariate Value-at-Risk and Multivariate Conditional Value-at-Risk, following the guidelines of Prékopa (2012). In Section 3 basic properties of both risk measures are stated. While MVaR enjoys similar properties as the univariate counterpart, MCVaR does not have the convexity property. Explanation is supplied. In Section 4 we present numerical procedures, for both the continuous and discrete cases, to approximate MCVaR, by the use of bounding, based on the binomial moment method and the Boolean bounding scheme. The practical meaning of MCVaR is illustrated on two portfolios with different correlation structures. Finally, in Section 5 we present conclusions.

2 The Notions of Multivariate Value-at-Risk(MVaR) and Multivariate Conditional Value-at-Risk(MCVaR)

While VaR and CVaR both have been around for some time, only VaR had a multivariate counterpart. However, the fact that we intend to take into account the stochastic dependence of the random variables involved, called for the introduction of the Multivariate Conditional Value at Risk or MCVaR. That was done in the recent paper by Prékopa (2012). For the sake of completeness below we recall the definitions of both MVaR and MCVaR (Definitions 1 and 2, respectively).

Definition 1. *(Prékopa 1990) Let $X \in R^r$ be a random variable and F its c.d.f. A point $s \in R^r$ is said to be a p-*Level Efficient Point*, or briefly p-efficient point, of the probability distribution, or the distribution function F, if $F(s) \geq p$ and there is no y such that $y \leq s,\ y \neq s,\ F(y) \geq p$. $MVaR_p(X)$ is the set of all p-efficient points of the random vector X.*

If X has discrete distribution on Z^r, then its support is finite or countably infinite. In both cases $\text{MVaR}_p(X)$ is a finite set by the following

Theorem 1. *If the components of the random vector ξ are integer-valued, then for any $p \in (0,1)$ the set of p-level efficient points is nonempty and finite.*

Theorem 1 is an immediate consequence of Dickson's Lemma (3, Cor. 4.48). It was mentioned, in another context, by Vizvári (1987) and Dentcheva, Prékopa, Ruszczyński (2000), for p-efficient points. The assertion of Theorem 1 is not necessarily true if the support of the random vector is countable but not part of the integer lattice. In case of an integer valued Y, there exist $N \geq 1$ and $s^{(1)}, \ldots, s^{(N)}$ such that

$$\text{MVaR}_p(Y) = \{s^{(1)}, \ldots, s^{(N)}\},$$

where $s^{(i)} \in R^r,\ i = 1, \ldots, N$. If X has continuous distribution, with strictly increasing c.d.f. ($F(s_1) > F(s_2)$ if $s_1 \geq s_2, s_1 \neq s_2$), then

$$\text{MVaR}_p(X) = \{s \mid F(s) = p\}. \tag{8}$$

The following concepts were introduced in Prékopa (2012). Suppose that the random vector X is related to losses, then the favorable event for X is defined as

$$X \in \bigcup_{s \in \text{MVaR}_p(X)} (s + R^r_-). \tag{9}$$

The complementary of the event in (9) is the unfavorable event:

$$X \in \bigcap_{s \in \text{MVaR}_p(X)} (s + R^r_-)^c. \tag{10}$$

Let us introduce the notation:

$$D_p = \bigcup_{s \in \text{MVaR}_p(X)} (s + R^r_-). \tag{11}$$

The sets D_p, D_p^c are called favorable and unfavorable sets, respectively. As illustrated in the Figure 1, the unfavorable set D_p^c is the north east shaded region and the favorable set D_p is the south west unshaded region. Sometimes we write $D_p(X), D_p^c(X)$ to indicate the dependence on X.

Definition 2. *(Prékopa (2012)) The Multivariate Conditional Value-at-Risk, or MCVaR, of the random vector X, is defined as:*

$$MCVaR_p(X) = E(\psi(X) \mid X \in \overline{D_p^c}),$$

where ψ is some r-variate function such that $E(\psi(X))$ exists. The value $0 < p < 1$ (or $1-p$) is called the level of MCVaR. The symbol $\overline{D_p^c}$ designates the closure of D_p^c.

If the probability $P(X \in \text{MVaR}_p(X))$ is negligible, then

$$E(\psi(X) \mid X \in \overline{D_p^c}) \approx E(\psi(X) \mid X \notin D_p).$$

Thus, we can write $\text{MCVaR}_p(X)$ as $E(\psi(X) \mid X \notin D_p)$ in case of $P(X \in \text{MVaR}_p(X)) \approx 0$ (see Prékopa (2012)). Let us define the function $\psi(u)$ as

$$\psi(u) = \sum_{i=1}^{r} \lambda_i u_i, \tag{12}$$

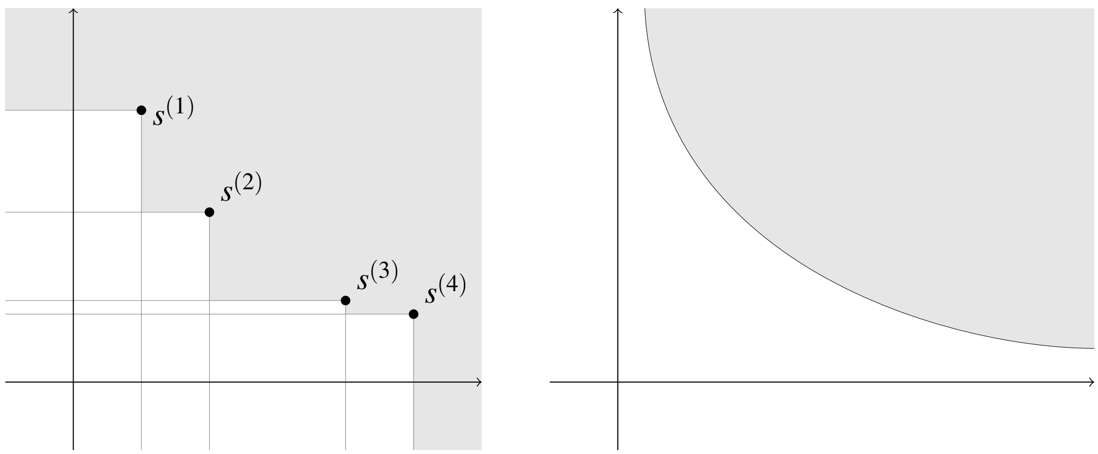

Figure 1: 2-D Illustration of the favorable set, and its complementary set where the Multivariate Conditional Value-at-Risk is defined in both types of a random vector – discrete and continuous.

X and Y are discrete and continuous random vectors, respectively. LHS: $MVaR_p(X) = \{s^{(1)}, \ldots, s^{(4)}\}$, *RHS:* $MVaR_p(Y)$ *is the boundary of the shaded region. MCVaR is defined in the shaded region (north east), i.e., the unfavorable set. The unshaded region (south west) is the favorable set.*

where $\sum_{i=1}^{r} \lambda_i = 1$ and $\lambda_1, \ldots, \lambda_r$ are nonnegative. If the components of X are losses in different portfolios, then λ_i weighs the loss in portfolio i, $i = 1, \ldots, r$.
The following equation holds true:

$$E(\psi(X)) = E(\psi(X) \mid X \notin D_p)P(X \notin D_p) + E(\psi(X) \mid X \in D_p)P(X \in D_p), \tag{13}$$

from where we derive:

$$\text{MCVaR}_p(X) = E(\psi(X) \mid X \notin D_p) = \frac{1}{P(X \notin D_p)}\Big(E(\psi(X)) - E(\psi(X) \mid X \in D_p)P(X \in D_p)\Big). \tag{14}$$

Equation (14) can be written as:

$$\text{MCVaR}_p(X) = \frac{1}{1 - P(X \in D_p)} \left(\sum_{i=1}^{r} \lambda_i m_i - \sum_{i=1}^{r} \lambda_i E(X_i | \, X \in D_p) P(X \in D_p) \right), \tag{15}$$

where $m_i = E(X_i), \ i = 1, \ldots, r$.

While Definition 2 of MCVaR applies for the general case, a simpler definition can be given for the continuous case, as follows.

Definition 3. *The Multivariate Conditional Value-at-Risk, or MCVaR, of a continuous random vector* $Z \in R^r$*, is the value:*

$$MCVaR_p(Z) = E(\lambda^T Z \mid F_Z(Z) \geq p), where\ F_Z(z) = P(Z \leq z).$$

Remark 1. *The Conditional Value-at-Risk (CVaR) measures the amount of losses beyond the Value-at-Risk(VaR). In the multivariate case, however, the elements of R^r are only partially ordered and MVaR is a set, not a single point, in general. On the other hand, we may not want to interpret the occurrence of an unfavorable event in terms of the sum of values of the r portfolios we are holding. In fact, a finance company typically creates a variety of portfolios, in an informative way, not only on their sum. The definition of MCVaR solves this problem in such a way that an unfavorable event is said to have occurred wherever X is larger than at least one element in MVaR, where "larger" means in the sense of partial order of vectors ($x > y$ iff $x \geq y, x \neq y$). The value of MCVaR is then the conditional expectation of the total loss, given that an unfavorable event occurs. In the total loss each asset has a multiplier ($\lambda_1, \ldots, \lambda_r$) to be able to value the different asset types on a common ground, by the use of a numeraire.*

3 Properties of MVaR and MCVaR

The following definition and theorem about the multivariate stochastic ordering is well-known (see, e.g., Müller and Stoyan (2002)).

Definition 4. *Let X and Y be r-variate random vectors. Then we define*

(SD1) Stochastic dominance of order 1: $X \preceq_{(1)} Y$, if $Ef(X) \leq Ef(Y)$ for all bounded increasing functions $f : R^r \to R$.

(SD2) Stochastic dominance of order 2: $X \preceq_{(2)} Y$, if $Ef(X) \leq Ef(Y)$ for all nondecreasing concave functions $f : R^r \to R$ such that the expectations exist.

Theorem 2. *The following statements are equivalent.*

(i) $X \preceq_{(1)} Y$,

(ii) $P(X \in U) \leq P(Y \in U)$ for all upper sets U,

(iii) $P(X \in U) \leq P(Y \in U)$ for all closed upper sets U.

A set U is called an upper set if $x \in U$ implies $y \in U$ for every $y \geq x$. A set L is a lower set if $x \in L$ implies $y \in L$ for every $y \leq x$. Note that the Multivariate Conditional Value-at-Risk is defined on an upper set (both shaded sets in Figure 1).

The relationship between the Multivariate Value-at-Risk (MVaR) and multivariate stochastic dominance of order 1 is illustrated in Figure 2.

We recall that the notions of logconcave p.d.f. and logconcave probability measure, designated by f and P, respectively, are defined by the inequalities

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\geq [f(x)]^{\lambda}[f(y)]^{1-\lambda}, \\ P(\lambda A + (1-\lambda)B) &\geq [P(A)]^{\lambda}[P(B)]^{1-\lambda}, \end{aligned} \tag{16}$$

where $x, y \in R^r$, $0 < \lambda < 1$, and A, B are convex subsets of R^r. Two basic theorems are as follows.

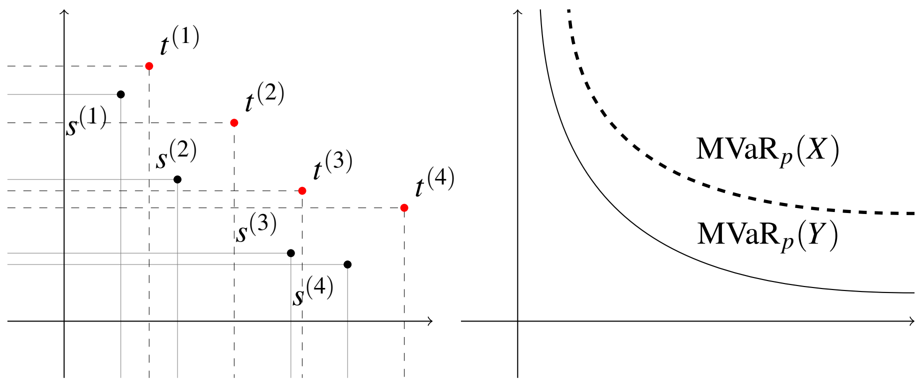

Figure 2: 2-D Illustration of the first order stochastic dominance of random vectors: $X \preceq_{(1)} Y$. *LHS: the case of discrete random vectors X and Y. $MVaR_p(X)$ is the collection of the p-efficeint points $t^{(i)}$'s and $MVaR_p(Y)$ is the collection of the p-efficient points $s^{(i)}$'s. RHS: the case of continuous random vectors X and Y. $MVaR_p(X)$ is the hypersurface $\{t \mid F_X(t) = p\}$ and $MVaR_p(Y)$ is the hypersurface $\{s \mid F_Y(s) = p\}$.*

Theorem 3. *(Prékopa 1971, 1973b) If a probability measure is generated by a logconcave p.d.f., then it is a logconcave measure.*

Theorem 4. *(Prékopa 1973b) If the probability measure P is generated by a logconcave p.d.f. that is strictly logconcave in an open set $D \subset R^r$, then the c.d.f. is also strictly logconcave in D.*

It follows that if a multivariate p.d.f. is logconcave, then its c.d.f. is also logconcave for every $0 \le p \le 1$, and the set $\{z \mid F(z) = p\}$ is convex. If $0 < p < 1$, then the set $\{z \mid F(z) = p\}$ is an $r-1$ dimensional hypersurface embedded in R^r which is illustrated by the boundary of the shaded set in Figure 3. If F is strictly logconcave, then it can also be described as a strictly concave function in which we take $r-1$ (arbitrarily chosen) variables as independent and one as dependent variable. For example, $x_r = x_r(x_1, \ldots, x_{r-1})$. This idea is used for the calculation of MCVaR in case of continuous random vectors, presented in Section 4.1.

Consider the family of sets:

$$H(p) = \{z \mid F(z) - p \ge 0\}, \tag{17}$$

depending on the parameter p ($0 < p < 1$). For every fixed p, the set $H(p)$ is convex but now we want to consider $F(z) - p$ as a function of all variables in z and p. Since $F(z) - p$ is not a logconcave function of z, p, in general, we change the parameter and look at the family of sets:

$$K(u) = \{z \mid F(z) \ge e^u\}. \tag{18}$$

If $F(z) > 0, z \in R^r$, then $K(u) = \{z \mid \log F(z) - u \ge 0\}$. For any $-\infty < u < 0$, we have $K(u) \neq \emptyset$. We have the following

Theorem 5. *$K(u)$, $-\infty < u < 0$ is a concave family of sets, i.e., if u_1, u_2 are arbitrary negative numbers and $0 < \lambda < 1$, then $K(\lambda u_1 + (1-\lambda)u_2) \supset \lambda K(u_1) + (1-\lambda)K(u_2)$.*

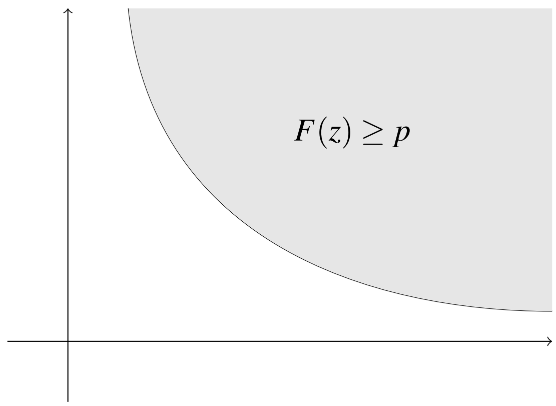

Figure 3: Illustration of the set $\{z|\ F(z) \geq p\}$.

The boundary of the shaded set is $\{z \mid F(z) = p\}$, i.e., an $r-1$ dimensional hypersurface embedded in R^r.

Proof. Let $z_1 \in K(u_1)$, $z_2 \in K(u_2)$. Then

$$\begin{aligned} F(\lambda z_1 + (1-\lambda)z_2) &\geq (F(z_1))^{\lambda}(F(z_2))^{1-\lambda} \\ &\geq (e^{u_1})^{\lambda}(e^{u_2})^{1-\lambda} = e^{\lambda u_1 + (1-\lambda)u_2}, \end{aligned}$$

which proves that

$$\lambda z_1 + (1-\lambda)z_2 \in K(\lambda u_1 + (1-\lambda)u_2).$$

□

Corollary 1. *Let G be any convex subset of R^r and $K(u)$, $-\infty < u < 0$ the concave family of sets defined in (18) with a logconcave distribution function F. Then $K(u) \cap G$, $-\infty < u < 0$ is a concave family of sets.*

Theorem 3 implies

Theorem 6. *Let $f(z)$, $z \in R^r$ be any logconcave function. Then*

$$\int_{K(u)} f(z)dz$$

is a logconcave function of $u \in (-\infty, 0)$. In other words, the function

$$\int_{F(z) \geq p} f(z)dz$$

is logconcave in $\log p$.

Definition 5. *Two random variables X and Y defined on the same probability space $(\Omega, \mathscr{F}, P)$ are said to be comonotone, if for all $\omega_1, \omega_2 \in \Omega$,*

$$[X(\omega_1) - Y(\omega_1)][X(\omega_2) - Y(\omega_2)] \geq 0 \ \ a.s.$$

We are now ready to state properties of MVaR_p and MCVaR_p.

Theorem 7. *Let $X,Y \in R^r$ be random vectors on the same probability space. Then we have the following properties:*

(1) *$MVaR_p$ is translation-equivariant: $MVaR_p(X+c) = MVaR_p(X)+c$, where $c \in R^r$.*

(2) *$MVaR_p$ is positively homogeneous: $MVaR_p(cX) = cMVaR_p(X)$, where $c \in R_+$.*

(3) *$\{z \mid P(-X \geq z) \geq p$ and it does not hold for any $y \geq z,\ y \neq z\} = -MVaR_p(X)$.*

(4) *$MVaR_p$ is monotonic w.r.t. the first order stochastic dominance, i.e.:*

$$X \preceq_{(1)} Y \text{ implies } D_p(X) \supset D_p(Y)$$

(5) *If X_i and Y_i are comonotone, X,Y have independent components, continuous and increasing distribution functions, then*

$$MVaR_p(X+Y) = \left\{ \begin{pmatrix} VaR_{\alpha_1}(X_1) \\ \vdots \\ VaR_{\alpha_r}(X_r) \end{pmatrix} + \begin{pmatrix} VaR_{\alpha_1}(Y_1) \\ \vdots \\ VaR_{\alpha_r}(Y_r) \end{pmatrix}, \alpha_1 \ldots \alpha_r = p,\ 0 < \alpha_i < 1,\ i = 1,\ldots,r \right\}.$$

(6) *For any X and $0 < p < 1$, $MVaR_p(X)$ is bounded from below.*

Proof. The proofs of (1), (2) and (4) are simple and therefore omitted.

(3)

$$\begin{aligned} & \{z \mid P(-X \geq z) \geq p \text{ and there is no } y \geq z,\ y \neq z \text{ such that } P(-X \geq y) \geq p\} \\ = & \{z \mid P(X \leq -z) \geq p \text{ and there is no } y \leq -z,\ y \neq -z \text{ such that } P(X \leq y) \geq p\} \\ = & -\mathrm{MVaR}_p(X). \end{aligned} \tag{19}$$

It can also be written as

$$\begin{aligned} -\mathrm{MVaR}_p(X) = & \{z \mid P(-X_i < z_i, \text{ for at least one } i = 1,\ldots,r) < 1-p \\ & \text{ and there is no } y \geq z,\ y \neq z \text{ such that} \\ & P(-X_i < y_i, \text{ for at lest one } i = 1,\ldots,r) < 1-p\}. \end{aligned} \tag{20}$$

(5) If X has independent components and continuous and increasing distribution functions, then

$$\mathrm{MVaR}_p(X) = \{u \mid u_i = \mathrm{VaR}_{\alpha_1}(X_i)\alpha_i,\ \alpha_1 \cdots \alpha_r = p,\ 0 < \alpha_i < 1,\ i = 1,\ldots,r\}.$$

We have the equation

$$\mathrm{MVaR}_p(X+Y) = \{u \mid F_{X_i+Y_i}(u_i) = \alpha_i,\ \alpha_1 \cdots \alpha_r = p,\ 0 < \alpha_i < 1,\ i = 1,\ldots,r\}.$$

Since X_i and Y_i are comonotone, $i = 1,\ldots,r$, it follows that

$$u_i = \mathrm{VaR}_{\alpha_i}(X_i) + \mathrm{VaR}_{\alpha_i}(Y_i),\ i = 1,\ldots,r$$

which implies (5).

(6) For every $z \in R^r$ we have the inequality $F_i(z_i) \geq F(z_1,\ldots,z_r)$, hence $F(z_1,\ldots,z_r) \geq p$ implies that $F_i(z_i) \geq p$. Since $F(\mathrm{VaR}_p(X_1),\ldots,\mathrm{VaR}_p(X_r)) \geq p$, it follows that

$$F_i(\mathrm{VaR}_p(X_i)) \geq p \text{ and } \mathrm{VaR}_p(X_i) \geq F_i^{-1}(p),\ i=1,\ldots,r.$$

□

Theorem 8. *Let $X,Y \in R^r$ be r-component random variables with finite expectations. Then $MCVaR_p$ exhibits the following properties:*

(1) $MCVaR_p$ is translation-equivariant: $MCVaR_p(X+c) = MCVaR_p(X)+c$.

(2) $MCVaR_p$ is positively homogeneous: $MCVaR_p(cY) = cMCVaR_p(Y),\ c \in R_+$.

(3) $MCVaR_p$ is subadditive when X,Y are continuously distributed and all components in X and Y are independent, i.e., we have the inequality

$$MCVaR_p(X+Y) \leq MCVaR_p(X)+MCVaR_p(Y).$$

(4) If the components of $X=(X_1,\ldots,X_r)$ are independent and have continuous distributions with logconcave p.d.f.'s, then $MCVaR_p(X)$ is logconcave in $\log p$, for $p \geq p_0$, where p_0 is a probability $(0 < p_0 < 1)$ such that $VaR_p(X_i) \geq 0,\ i=1,\ldots,r$.

Remark 2. *We can think, from Property (3) of MCVaR, about both cases of "good" and "bad" corporate M&A (Mergers and Acquisitions) deals. From a risk management perspective, Property (3) indicates that not all M&A deals would be successful, i.e., for some "bad" M&A deals, risk would not be reduced, since $MCVaR_p$ is not always subadditive. More detailed explanation is presented in Remark 5.*

Proof. Let us recall the following equation:

$$\begin{aligned} \mathrm{MCVaR}_p(X) &= E(\psi(X) \mid X \notin D_p) \\ &= \frac{1}{1-P(X \in D_p)} \left(\sum_{i=1}^r \lambda_i m_i - \sum_{i=1}^r \lambda_i E(X_i \mid X \in D_p) P(X \in D_p) \right), \end{aligned} \tag{21}$$

where $m_i = E(X_i)$ for $i=1,\ldots,r$, $\psi(X) = \sum_{i=1}^r \lambda_i X_i$ and $\sum_{i=1}^r \lambda_i = 1,\ \lambda_i \geq 0$ for $i=1,\ldots,r$.

(1) Let $D'_p = \bigcup_{s' \in \mathrm{MVaR}_p(X+c)} (s'+R^r_-),\ D_p = \bigcup_{s \in \mathrm{MVaR}_p(X)} (s+R^r_-)$. Then we have the equations:

$$\begin{aligned} \mathrm{MCVaR}_p(X+c) &= E(\psi(X+c) \mid X+c \notin D'_p) \\ &= E(\psi(X) \mid X+c \notin D'_p) + \sum_{i=1}^r \lambda_i c_i \\ &= E(\psi(X) \mid X \notin D_p) + \sum_{i=1}^r \lambda_i c_i \\ &= \mathrm{MCVaR}_p(X)+c. \end{aligned} \tag{22}$$

(2) If we use the notations D_p, D'_p with $s = \frac{s'}{c}$, then we derive:

$$\begin{aligned} \text{MCVaR}_p(cX) &= E(\psi(cX) \mid cX \notin D'_p) \\ &= E(c\psi(X) \mid cX \notin D'_p) \\ &= cE(\psi(X) \mid X \notin D_p) \\ &= c\text{MCVaR}_p(X). \end{aligned} \tag{23}$$

The third equality holds since it can easily be seen that $cX \notin D'_p$ is equivalent to $X \notin D_p$, where D'_p and D_p are defined as above. If we use the second property of MVaR then we can obtain the following equations:

$$\begin{aligned} & cX \notin D'_p = \bigcup_{s' \in \text{MVaR}_p(cX)} (s' + R^r_-) \\ \Leftrightarrow\ & cX \notin D'_p = \bigcup_{s' \in c\text{MVaR}_p(X)} (s' + R^r_-) \\ \Leftrightarrow\ & X \notin \bigcup_{\frac{s'}{c} \in \text{MVaR}_p(X)} (\frac{s'}{c} + R^r_-) \\ \Leftrightarrow\ & X \notin D_p = \bigcup_{s \in \text{MVaR}_p(X)} (s + R^r_-),\ s = \frac{s'}{c}. \end{aligned} \tag{24}$$

(3) First we remark that if $Z = (Z_1, \dots, Z_r)$ is a continuously distributed random vector and $Z_1, \dots, Z_r$ are independent, then $P(Z \in D_p)$ is independent of the distribution of Z. In fact, let F_i be the c.d.f. of $Z_i,\ i = 1, \dots, r$. Then we have

$$\begin{aligned} & P(Z \in D_p) = P(F_1(Z_1) \cdots F_r(Z_r) \geq p) = P(U_1 \cdots U_r \geq p) \\ & = P(-\log U_1 - \cdots - \log U_r \leq -\log p) = \int_0^{-\log p} \frac{z^{r-1} e^{-e}}{(r-1)!} dz, \end{aligned} \tag{25}$$

where $U_1, \dots, U_r$ are independent random variables, uniformly distributed in (0,1). Hence, the numerator counts in the second equation of (15), only. We may disregard the linear terms and it is enough to look only at $E(X_1 \mid X \notin D_X), E(X_2 \mid X \notin D^X_p), E(Y_1 \mid Y \notin D^Y_p), E(Y_2 \mid Y \notin D^Y_p), E(X_1 + Y_1 \mid X + Y \notin D^{X+Y}_p), E(X_2 + Y_2 \mid X + Y \notin D^{X+Y}_p)$, where

$$D^X_p = \bigcup_{s \in \text{MVaR}_p(X)} (s + R^r_-), \quad D^Y_p = \bigcup_{t \in \text{MVaR}_p(Y)} (t + R^r_-), \quad D^{X+Y}_p = \bigcup_{u \in \text{MVaR}_p(X+Y)} (u + R^r_-).$$

It is enough to prove that

$$E(X_1 + Y_1 \mid X + Y \notin D_{X+Y}) \leq E(X_1 \mid X \notin D_X) + E(Y_1 \mid Y \notin D_Y), \tag{26}$$

$$E(X_2 + Y_2 \mid X + Y \notin D_{X+Y}) \leq E(X_2 \mid X \notin D_X) + E(Y_2 \mid Y \notin D_Y). \tag{27}$$

If we multiply the inequalities (26), (27) by -1 and add them, then by (15) and the fact that $P(X \notin D_X)$ is independent of the random variable, the convexity proof of MCVaR will be complete.

Proof. Proof of (26) (proof of (27) is the same):

$$E(X_1+Y_1 \,|X+Y \notin D_{X+Y}) = E(X_1+Y_1 \,|F_{X_1+Y_1}(X_1+Y_1)F_{X_2+Y_2}(X_2+Y_2) \geq p). \tag{28}$$

Note that X_1+Y_1 and X_2+Y_2 are independent

$$\begin{aligned} F_{X_1+Y_1}(X_1+Y_1) &\sim U_1, \\ F_{X_2+Y_2}(X_2+Y_2) &\sim U_2. \end{aligned} \tag{29}$$

It follows that (28) is further equal to

$$E(X_1+Y_1 \mid F_{X_1+Y_1}(X_1+Y_1) \geq \frac{p}{U_2}). \tag{30}$$

Incidentally we mention that if V is any random variable, then

$$E(V \mid F_V(V) \geq q) = \frac{\int_q^\infty [1-G(v)]dv}{1-G(q)} + q, \tag{31}$$

where G is c.d.f. of V, and this, as a function of q is increasing.

Let $\delta > 0$ and introduce the notations:

$$A = \int_{q+\delta}^\infty [1-G(v)]dv, \; B = 1-G(q+\delta).$$

Then we have:

$$\begin{aligned} & \frac{\int_{q+\delta}^\infty [1-G(v)]dv}{1-G(q+\delta)} + q + \delta - \frac{\int_q^\infty [1-G(v)]dv}{1-G(q)} - q \\ = \; & \delta + \frac{A(B+G(q+\delta)-G(q)) - \left(A + \int_q^{q+\delta}(1-G(v))dv\right)B}{(1-G(q+\delta))(1-G(q))} \\ = \; & \delta + \frac{A(G(q+\delta)-G(q)) - \int_q^{q+\delta}(1-G(v))dv\, B}{(1-G(q+\delta))(1-G(q))} \\ = \; & \delta + \frac{A}{1-G(q+\delta)} - \frac{A}{1-G(q)} - \frac{\int_q^{q+\delta}(1-G(v))dv}{1-G(q)} \\ \geq \; & \delta - \frac{\int_q^{q+\delta}(1-G(v))dv}{1-G(q)} \\ \geq \; & \delta - \frac{\delta(1-G(q))}{1-G(q)} \\ = \; & 0. \end{aligned} \tag{32}$$

This implies that

$$E(X_1+Y_1 \mid F_{X_1+Y_1}(X_1+Y_1) \geq \frac{p}{U_2}) \leq E(X_1+Y_1 \mid F_{X_1+Y_1}(X_1+Y_1) \geq p). \qquad (33)$$

□

In the same way, we have

$$E(X_2+Y_2 \mid F_{X_2+Y_2}(X_2+Y_2) \geq \frac{p}{U_1}) \leq E(X_2+Y_2 \mid F_{X_2+Y_2}(X_2+Y_2) \geq p) \qquad (34)$$

and the assertion is proved.

A simple counterexample of Property (3) in the general case

Suppose that the random vectors $X, Y \in R^2$ have the following possible values with probability 0.25 for each point:

$$X = \{(1.1, 4.4)^T, (2,1)^T, (2,8)^T, (8,4)^T\} \text{ and } Y = \{(1,1)^T, (2,2)^T, (3,3)^T, (4,4)^T\},$$

as depicted in Figure 4. At $p = 0.75$, $\mathrm{MVaR}_p(X) = \{(2,8)\}$ and $\mathrm{MVaR}_p(Y) = \{(3,3)\}$. $E(X_1) = 3.275, E(X_2) = 4.35$ and $E(X_1 \mathbb{1}_{X\in D_p}) = 1.275, E(X_2 \mathbb{1}_{X\in D_p}) = 3.35$. Let $\lambda_1 = \lambda_2 = 0.5$. Plugging in those values into MCVaR formulation (15), $\mathrm{MCVaR}_p(X) = 6$. For the random vector Y, $\mathrm{MCVaR}_p(Y) = 4$ using $E(Y_1) = E(Y_2) = 2.5, E(Y_1 \mathbb{1}_{Y\in D_p}) = E(Y_2 \mathbb{1}_{Y\in D_p}) = 1.5$ and $\lambda_1 = \lambda_2 = 0.5$. Then $\mathrm{MCVaR}_p(X)+ \mathrm{MCVaR}_p(Y) = 10$.

Let $Z = X+Y$. Then $Z_1 = \begin{cases} 2.1 & \text{with } p = 0.25^2 \\ 3 & \text{with } p = (0.5)(0.25) \\ 3.1 & \text{with } p = 0.25^2 \\ 4 & \text{with } p = (0.5)(0.25) \\ 4.1 & \text{with } p = 0.25^2 \\ 5 & \text{with } p = (0.5)(0.25) \\ 5.1 & \text{with } p = 0.25^2 \\ 6 & \text{with } p = (0.5)(0.25) \\ 9 & \text{with } p = 0.25^2 \\ 10 & \text{with } p = 0.25^2 \\ 11 & \text{with } p = 0.25^2 \\ 12 & \text{with } p = 0.25^2 \end{cases}$, $Z_2 = \begin{cases} 2 & \text{with } p = 0.25^2 \\ 3 & \text{with } p = 0.25^2 \\ 4 & \text{with } p = 0.25^2 \\ 5 & \text{with } p = 2\times 0.25^2 \\ 5.4 & \text{with } p = 0.25^2 \\ 6 & \text{with } p = 0.25^2 \\ 6.4 & \text{with } p = 0.25^2 \\ 7 & \text{with } p = 0.25^2 \\ 7.4 & \text{with } p = 0.25^2 \\ 8 & \text{with } p = 0.25^2 \\ 8.4 & \text{with } p = 0.25^2 \\ 9 & \text{with } p = 0.25^2 \\ 10 & \text{with } p = 0.25^2 \\ 11 & \text{with } p = 0.25^2 \\ 12 & \text{with } p = 0.25^2 \end{cases}$.

At $p = 0.75$, we have $\mathrm{MVaR}_p(Z) = \{(6,12), (9,11), (10,10), (11,9), (12,8.4)\}$, since $F_Z(6,12) = F_Z(12,8.4) = 0.75$, $F_Z(9,11) = F_Z(11,9) = 0.7617875$ and $F_Z(10,10) = 0.765625$ as described in Figure 5.

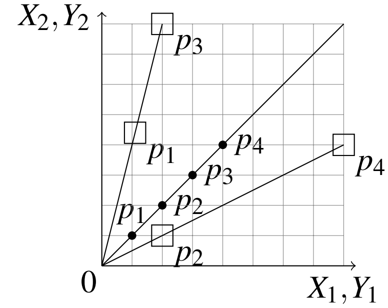

Figure 4: A counterexample of MCVaR Property (3): subadditivity, in the general case.
Each node of p_1, p_2, p_3, p_4 has probability 0.25. The length of each grid element is 1. The rectangle-nodes are the possible values of X and the circle-nodes are that of Y; $X = \{(1.1, 4.4)^T, (2,1)^T, (2,8)^T, (8,4)^T\}$ and $Y = \{(1,1)^T, (2,2)^T, (3,3)^T, (4,4)^T\}$. At probability level $p = 0.75$, $\mathrm{MCVaR}_p(X) = 6$ and $\mathrm{MCVaR}_p(Y) = 4$.

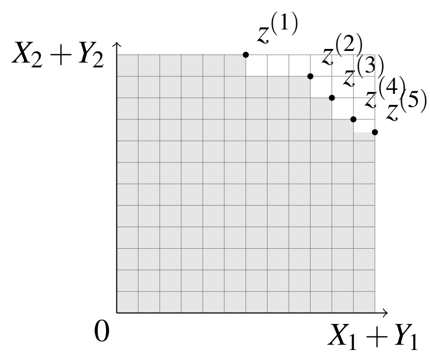

Figure 5: Illustration of $\mathrm{MVaR}_{\mathbf{p}}(\mathbf{Z})$, where $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$.
The length of each grid element is 1. Let $Z = X + Y$, $Z \in R^2$. The points $z^{(i)},\ i = 1, \ldots, 5$ are the elements of $\mathrm{MVaR}_p(Z)$, i.e., $\mathrm{MVaR}_p(Z) = \{(6,12), (9,11), (10,10), (11,9), (12,8.4)\}$. Under the probability level $p = 0.75$, $\mathrm{MCVaR}_p(Z) = 11$.

From a simple calculation, we get $P(Z \in D_p) = 0.9609375$, $E(Z_1) = 5.775, E(Z_2) = 6.85$, $E(Z_1 \mathbb{1}_{Z \in D_p}) = 5.3453125, E(Z_2 \mathbb{1}_{Z \in D_p}) = 6.4203125$ and let $\lambda_1 = \lambda_2 = 0.5$. Plugging in those values into MCVaR formulation (15), we obtain $\mathrm{MCVaR}_p(Z) = 11$. Thus $\mathrm{MCVaR}_p(Z) = 11 > 10 = \mathrm{MCVaR}_p(X) + \mathrm{MCVaR}_p(Y)$ and this is the counterexample of Property(3) of MCVaR in the general case.

(4) Since $P(X \in D_p)$ does not depend on the distribution of X, it is enough to prove that

$$\int_{F_1(z_1)\cdots F_r(z_r) \geq p} \psi(z) f(z) dz \tag{35}$$

is logconcave in $\log p$, for $p \geq p_0$. Since $\lambda_i > 0,\ i = 1, \ldots, r$, the function $\psi(z)$ and also

$\psi(z)f(z)$ is logconcave in $\{z \mid z \geq 0\}$. On the other hand, if $p \geq p_0$, then

$$\{z \mid F_1(z_1)\cdots F_r(z_r) \geq p\} \subset \{z \mid z \geq (\mathrm{VaR}_p(X_1),\ldots,\mathrm{VaR}_p(X_r))\} \subset \{z \mid z \geq 0\}. \tag{36}$$

The rest of the proof is the same as the proof of Theorem 5.

□

Remark 3. *(Relationship between VaR and MVaR) Let us define*

$$\begin{aligned} &\left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \middle| \, P(\xi_1 \leq z_1,\ldots,\xi_r \leq z_r) \geq p \right\} = \overline{D_p^c}(\xi) \subset R^r,\ \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix}, \\ &\left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} \middle| \, P(\xi_1 \leq z_1,\ldots,\xi_k \leq z_k) \geq p \right\} = \overline{D_p^c}(\eta) \subset R^k,\ \eta = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_k \end{pmatrix},\ k < r. \end{aligned} \tag{37}$$

If we create a cylinder set out of $\overline{D_p^c}(\eta)$ in such a way that we take

$$\left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \middle| \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} \in \overline{D_p^c}(\eta) \right\} = \overline{D_p^{r\,c}}(\eta),\ k < r, \tag{38}$$

then we have the relation

$$\overline{D_p^c}(\xi) \subset \overline{D_p^{r\,c}}(\eta). \tag{39}$$

It is true that the projection of $MVaR_p(X)$ to a space of a smaller number of components of X, i.e., as before in (37) , from R^r to R^k, then the lower bound of the projection in R^k of $MVaR_p(X)$ is equal to $MVaR_p((X_1,\ldots,X_k)^T)$. The projection of MVaR is illustrated in Figures 6 and 7. In Figure 6, for a random vector $X \in R^2$, the sets $\{z \mid z \geq VaR_p(X_1)\}$ and $\{z \mid z \geq VaR_p(X_2)\}$ are closures of the projections of $MVaR_p(X)$ onto the horizontal and vertical axes, respectively. The same sets are the closures of the projections of $D_p^c(X)$. In Figure 7, for a random vector $X \in R^3$, we illustrate a boundary of the set $\{(z_1,z_2,z_3)^T \mid (z_1,z_2)^T \geq MVaR_p(X_1,X_2)\}$. The set $\{z \in R^2 \mid z \geq MVaR_p(X_1,X_2)\}$ is the closure of the projection of $D_p^c(X)$ onto (X_1,X_2)-plane and also the closure of the projection of $MVaR_p(X)$ onto the same plane.

Let a random vector $X \in R^r$ denote losses from investment in a composite of portfolios, where each component $X_i,\ i = 1,\ldots,r$ is just a single portfolio. Then we know that the unfavorable set $D_p^c(X)$ is only a part of $\{z \in R^r \mid z_i \geq VaR_p(X_i), i = 1,\ldots,r\}$. It has practical meaning: if we have several portfolios put together in the random vector X, then a single portfolio may signal an unfavorable event, i.e., the realized value of X_i is greater than $VaR_p(X_i)$, while X is still in the favorable set $D_p(X)$, i.e., there is no such signal for X, the composite of individual portfolios.

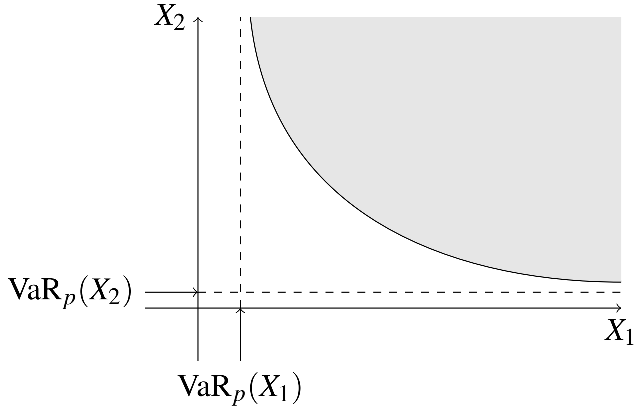

Figure 6: Projection of $\mathrm{MVaR}_{\mathbf{p}}((\mathbf{X_1},\mathbf{X_2})^{\mathbf{T}})$ from $\mathbf{R^2}$ onto the space of $\mathbf{X_1}$, $\mathbf{X_2} \in \mathbf{R}$.
The points $VaR_p(X_1)$ and $VaR_p(X_2)$ are the lower bounds of the projection of $MVaR_p(X)$, $X = (X_1,X_2)^T \in R^2$, onto the space of X_1, $X_2 \in R$ respectively.

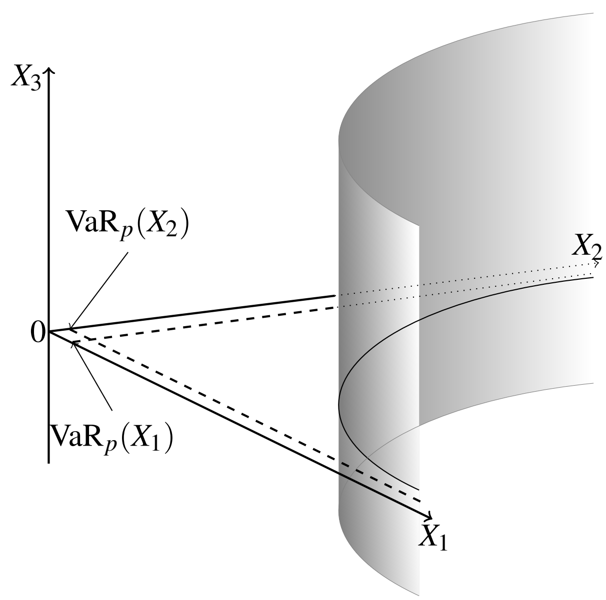

Figure 7: Projection of $\mathrm{MVaR}_{\mathbf{p}}((\mathbf{X_1},\mathbf{X_2},\mathbf{X_3})^{\mathbf{T}})$ from $\mathbf{R^3}$ onto the space of $(\mathbf{X_1},\mathbf{X_2})^{\mathbf{T}} \in \mathbf{R^2}$.
The shaded surface is the $MVaR_p((X_1,X_2)^T)$, which is a cylinder set of the lower bound of the projection of $MVaR_p(X)$, $X = (X_1,X_2,X_3)^T \in R^3$, onto the space of $(X_1,X_2)^T \in R^2$.

4 Calculation of MCVaR

4.1 The Case of a Continuous Distribution

Assume that a random vector Z has continuous distribution, its p.d.f. and c.d.f. are $f_Z(z)$ and $F_Z(z)$, respectively, where $Z = (z_1, \dots, z_r)^T$. Then

$$\begin{aligned}\mathrm{MCVaR}_p(Z) &= E(\lambda^T Z \mid F_Z(Z) \geq p) \\ &= \frac{\int \cdots \int_{D_p^c} (\lambda_1 z_1 + \cdots + \lambda_r z_r) f_Z(z_1, \dots, z_r)\, dz_1 \dots dz_r}{\int \cdots \int_{D_p^c} f_Z(z_1, \dots, z_r)\, dz_1 \dots dz_r},\end{aligned} \tag{40}$$

where $D_p^c = \{z \mid F_Z(z) \geq p\}$.

If $f_Z(z)$ is a log concave function, then so is $F_Z(z)$ (see Prékopa (1995)) and the set D_p^c is convex. Its boundary is a convex surface that can be represented in the form of a function provided that no coordinate axis is a supporting line of D_p^c. In this case we can take any $r-1$ variables out of $z_1, \dots, z_r$, the remaining variable will be a function of them and this function uniquely describes the surface of D_p^c (see Busemann (2008)). The set D_p^c can then be represented as

$$D_p^c = \{(z_1, \dots, z_r) : \mathrm{VaR}_p(Z_1) < z_1,\ l_1(z_1) < z_2,\ l_2(z_1, z_2) < z_3, \dots,\ l_{r-1}(z_1, \dots, z_{r-1}) < z_r\}, \tag{41}$$

where $l_k(z_1, \dots, z_k)$'s are k-variate lower bound functions, $k = 1, \dots, r-1$ and $l_1(z_1)$ is continuous on the domain $\{\mathrm{VaR}_p(Z_1) < z_1\}$, $l_2(z_1, z_2)$ is continuous on the 2-dimensional domain $\{\mathrm{VaR}_p(Z_1) < z_1,\ l_1(z_1) < z_2\}, \cdots, l_{r-1}(z_1, z_2, \dots, z_{r-1})$ is continuous on the "r-1"-dimensional domain $\{\mathrm{VaR}_p(Z_1) < z_1,\ l_1(z_1) < z_2,\ l_2(z_1, z_2) < z_3, \dots,\ l_{r-2}(z_1, z_2, \dots, z_{r-2}) < z_{r-1}\}$. Then (40), together with (41) can be written as

$$\mathrm{MCVaR}_p(Z) = \frac{\int_{\mathrm{VaR}_p(Z_1)}^{\infty} \int_{l_1}^{\infty} \cdots \int_{l_{r-2}}^{\infty} \int_{l_{r-1}}^{\infty} (\lambda_1 z_1 + \cdots + \lambda_r z_r) f_Z(z_1, \dots, z_r)\, dz_r dz_{r-1} \dots dz_2 dz_1}{\int_{\mathrm{VaR}_p(Z_1)}^{\infty} \int_{l_1}^{\infty} \cdots \int_{l_{r-2}}^{\infty} \int_{l_{r-1}}^{\infty} f_Z(z_1, \dots, z_r)\, dz_r dz_{r-1} \dots dz_2 dz_1}. \tag{42}$$

Each lower bound of the integrals in (42) represents MVaR_p on its corresponding multidimensional space, i.e.,

$$\begin{aligned}\{(z_1, \dots, z_r)^T \mid z_r = l_{r-1}(z_1, \dots, z_{r-1})\} &= \mathrm{MVaR}_p((Z_1, \dots, Z_r)^T), \\ \{(z_1, \dots, z_{r-1})^T \mid z_{r-1} = l_{r-2}(z_1, \dots, z_{r-2})\} &= \mathrm{MVaR}_p((Z_1, \dots, Z_{r-1})^T), \\ &\vdots \\ \{(z_1, z_2)^T \mid z_2 = l_1(z_1)\} &= \mathrm{MVaR}_p((Z_1, Z_2)^T).\end{aligned} \tag{43}$$

Generally, there is no closed form of the quantile function for a multivariate distribution. Thus, the functions of lower bounds in the set D_p^c of (41), i.e., the lower limits of the integrals of (42) can be constructed by some numerical methods, e.g., multivariate nonlinear approximation. For various methods of multivariate function fitting, we refer the readers to related books and literature (see, e.g., (Atkinson 1988), (Gasca and Sauer 2000), (Sauerbrei et al. 2006), (Strang 2007), etc.).

We generate a multidimensional grid of equally spaced points in the following set:

$$\{(z_1,\dots,z_r)^T : z_k \in I_k \text{ for } k=1,\dots,r\}, \text{ where } I_k = [\mathrm{VaR}_p(Z_k), x \text{ such that } F_{Z_k}(x) \approx 1]. \quad (44)$$

Then we generate a collection of the closest points to the $\mathrm{MVaR}_p(Z) = \{z \mid F_Z(z) = p\}$. By a nonlinear approximation based on the collection of such points, functions of lower bounds in the set D_p^c of (41) can be constructed. This is followed by (42), the calculation of MCVaR.

Two numerical examples of recent finance market data are presented. The type of financial securities is exchange-traded funds (ETFs), which can be regarded as mutual funds that can be bought and sold just like common stocks, i.e. exchange-traded products. We use 6 months of time period from February 15, 2012 to August 14, 2012 for the calculation of MCVaR with probability levels $p = 0.80$, $p = 0.90$, $p = 0.95$ and $p = 0.99$. From Yahoo Finance, online finance portal, we download the data of daily closing prices for the time period of 6 months from February 15, 2012 to August 14, 2012.

We want to show, by Examples 1 and 2, how MCVaR works on a set of "stochastically dependent" random variables. In Example 1 we have two "positively" correlated funds: Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF and Vanguard S&P 500 (VOO) ETF that closely resemble two of the US major indices, Nasdaq and S&P 500, respectively. For Example 2, we select Deutsche Bank US Dollar Index Bullish (UUP) ETF which tracks the performance of the Deutsche Bank Long US Dollar Futures index, while keeping Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF, in order to see how MCVaR measures a risk on the set of index funds "negatively" correlated with each other.

Example 1 (Positively correlated two ETFs)**.** *We choose two ETFs: Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF and Vanguard S&P 500 (VOO) ETF, "positively" correlated, as in Figure 8.*

Initial prices per share on February 15, 2012 are $115.02 for ONEQ, $61.56 for VOO. Assume that we have total available amount of $1,000,000 for the investment which is intended for the equal investment in each kind, i.e. $500,000 each. However, since there is no fractional shares for those securities in real financial market, the initial investment is $499,991.94 and $499,990.32 for ONEQ and VOO, respectively. The corresponding number of shares is 4,347 for ONEQ and 8,122 for VOO. We manipulate the data of daily closing prices into daily losses from the following equation:

$$\mathit{Loss} = \frac{\textit{initial investment - (number of shares} \times \textit{ price per share)}}{\textit{initial investment}}. \quad (45)$$

Let X_1, X_2 denote the random variables of losses from an investment in Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF and Vanguard S&P 500 (VOO) ETF, respectively. Each is assumed normally distributed. The random vector $X = (X_1, X_2)^T$ has a bivariate normal distribution with parameters $E(X_1) = \mu_{x_1} = -0.01185$, $\sigma_{x_1} = 0.02956$, $E(X_2) = \mu_{x_2} = -0.01439$,

Figure 8: Basic Chart of Fidelity Nasdaq ETF and Vanguard S&P 500 ETF.
Over 6 months from February 15 2012 to August 14 2012, it is a basic chart of Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF and Vanguard S&P 500 (VOO) ETF from Yahoo Finance.

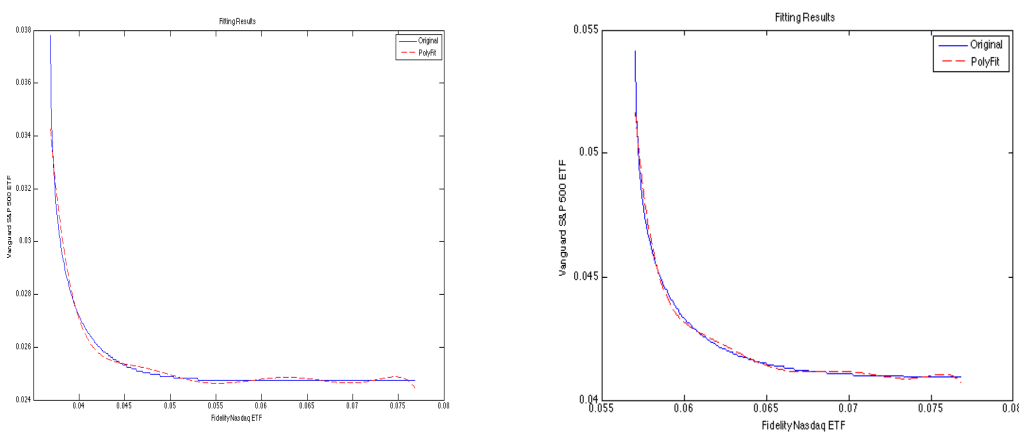

Figure 9: Multivariate Value-at-Risk, the quantile function at the probability levels $p = 0.95$ and $p = 0.99$.

At probability level $p = 0.95$ for LHS and $p = 0.99$ for RHS, over 6 months of time period, $MVaR_p(X)$ is well approximated by polynomial fitting in the sense of the least-squares (dotted curve). The random vector has components of losses from the investment in Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF and Vanguard S&P 500 (VOO) ETF from Yahoo Finance. The piecewise linear line (looks like a curve) is the set of line segments between the points (x_1, x_2) such that $F_X(x_1, x_2) = p$.

$\sigma_{x_2} = 0.02477$ and $\rho = 0.95139$. We calculate

$$\begin{aligned} MCVaR_p(X) &= E(\lambda^T X \mid F_X(X) \geq 0.95), \text{where } F_X(x) = P(X \leq x) \\ &\approx \frac{\int_{VaR_p(X_1)}^{\infty} \int_{l(x_1)}^{\infty} (\lambda_1 x_1 + \lambda_2 x_2) f_X(x_1, x_2)\, dx_2 dx_1}{\int_{VaR_p(X_1)}^{\infty} \int_{l(x_1)}^{\infty} f_X(x_1, x_2)\, dx_2 dx_1} \\ &\approx \frac{\int_{VaR_p(X_1)}^{\mu_{x_1}+7\sigma_{x_1}} \int_{l(x_1)}^{\mu_{x_2}+7\sigma_{x_2}} (\lambda_1 x_1 + \lambda_2 x_2) f_X(x_1, x_2)\, dx_2 dx_1}{\int_{VaR_p(X_1)}^{\mu_{x_1}+7\sigma_{x_1}} \int_{l(x_1)}^{\mu_{x_2}+7\sigma_{x_2}} f_X(x_1, x_2)\, dx_2 dx_1}, \end{aligned} \tag{46}$$

where $f_X(x)$ and $F_X(x)$ denote bivariate normal p.d.f. and c.d.f., respectively; $\lambda_1 = \lambda_2 = 1/2$ and

$$\begin{aligned} l(x_1) = & -12407756274.6875 x_1^7 + 5103841979.8231 x_1^6 - 893105499.6990 x_1^5 + \\ & 86172769.9900 x_1^4 - 4951126.6304 x_1^3 + 169402.1044 x_1^2 - 3196.2936 x_1 + 25.6869, \end{aligned} \tag{47}$$

with domain of $\{VaR_p(X_1) \leq x_1 \leq \mu_{x_1} + 7\sigma_{x_1}\}$, which is simply constructed by the use of Matlab "polyfit" function fitting the polynomial in the sense of the least squares. We used the same upper bounds $\mu_{x_i} + 7\sigma_{x_i},\ i = 1,2$ for the integrals in (46) since $P(X_i \geq E(X_i) + 7\sigma_{x_i}) = 0.000000019$ which is small enough. At $p = 0.95$ and $p = 0.99$, the p-quantile set $\{(x_1, x_2)^T \mid x_2 = l(x_1)\} \approx MVaR_p((X_1, X_2)^T)$ is illustrated in Figure 9.

From (46), we obtain $MCVaR_p(X) = 0.00048388$, and it means that \$483.88 is the expected loss amount beyond the $MVaR_p(X)$ at probability level $p = 0.95$. In other words, that amount of loss is expected to exceed at least one element in MVaR with $p = 0.95$. With more critical probability level $p = 0.99$, we approximate $MVaR_p(X)$ as in Figure 9 and calculate the value of $MCVaR_p(X) = 0.00082815$, i.e., we can expect \$828.15 of loss from the investment at probability level $p = 0.99$. Note that $MCVaR_p(X)$ at probability level $p = 0.99$ is clearly larger than that at $p = 0.95$.

Example 2 (Negatively correlated two ETFs)**.** *While keeping Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF, we replace S & P 500 ETF with Deutsche Bank US Dollar Index Bullish (UUP) ETF, which is "negatively" correlated to Nasdaq ETF. As we observe in Figure 10, the two index funds do not move in the same directions.*

Initial prices per share on February 15, 2012 are \$115.02 for ONEQ, \$22.21 for UUP. Like the previous example, we assume that we have total available amount of \$1,000,000 for the investment which is intended for the equal investment in each kind, i.e. \$500,000 each. And due to no fractional shares for those securities in real financial market, the initial investment is \$499,991.94 and \$499,991.52 for ONEQ and UUP, respectively. The corresponding number of shares is 4,347 for ONEQ and 22,512 for UUP. Again, we manipulate the data of daily closing prices into daily losses using the equation (45). Let Y_1, Y_2 denote the random variables of losses from an investment in Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF and

Figure 10: Basic Chart of Deutsche Bank US Dollar Index ETF and Fidelity Nasdaq ETF. *Basic chart over 6 months of Deutsche Bank US Dollar Index Bullish (UUP) ETF and Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF from Yahoo Finance.*

Deutsche Bank US Dollar Index Bullish (UUP) ETF, respectively. Each is assumed normally distributed. The random vector $Y = (Y_1, Y_2)^T$ *have a bivariate normal distribution with parameters* $\mu_{y_1} = -0.01185, \mu_{y_2} = -0.00875, \sigma_{y_1} = 0.02956, \sigma_{y_2} = 0.01705$ *and* $\rho = -0.70933$.

With $MVaR_p(Y)$ *of probability level* $p = 0.95$*, depicted in LHS of Figure 11, we calculate the following:*

$$MCVaR_p(Y) \approx \frac{\int_{VaR_p(Y_1)}^{\mu_{y_1}+7\sigma_{y_1}} \int_{l(y_1)}^{\mu_{y_2}+7\sigma_{y_2}} (\lambda_1 y_1 + \lambda_2 y_2) f_Y(y_1, y_2)\, dy_2 dy_1}{\int_{VaR_p(Y_1)}^{\mu_{y_1}+7\sigma_{y_1}} \int_{l(y_1)}^{\mu_{y_2}+7\sigma_{y_2}} f_Y(y_1, y_2)\, dy_2 dy_1}, \tag{48}$$

where $f_Y(y)$ *denote bivariate normal p.d.f.,* $\lambda_1 = \lambda_2 = 1/2$ *and*

$$\begin{aligned} l(y_1) = {} & -7207383.7882 y_1^7 + 5413617.7299 y_1^6 - 1703609.1254 y_1^5 + \\ & 290811.0271 y_1^4 - 29061.2545 y_1^3 + 1700.3294 y_1^2 - 54.01701 y_1 + 0.74111, \end{aligned} \tag{49}$$

with domain of $\{VaR_p(Y_1) \le y_1 \le \mu_{y_1} + 7\sigma_{y_1}\}$*, constructed by the use of Matlab "polyfit" function. In Figure 11, for both probability levels* $p = 0.95$ *and* $p = 0.99$*,* $\{(y_1, y_2)^T \mid y_2 = l(y_1)\} \approx MVaR_p((Y_1, Y_2)^T)$ *is illustrated.*

The result is $MCVaR_p(Y) = 0.00040766$*, which means \$407.66 is the expected amount of loss beyond the* $MVaR_p(Y)$ *with probability level* $p = 0.95$*. At probability level* $p = 0.99$*, we obtain* $MCVaR_p(Y) = 0.00060555$*, i.e. \$605.55 is the expected amount of loss beyond* $MVaR_p(Y)$*. We have conducted both Examples 1 and 2 with probability levels:* $p = 0.8, 0.9, 0.95, 0.99$ *and summarized in Table 1 for an easy and quick comparison.*

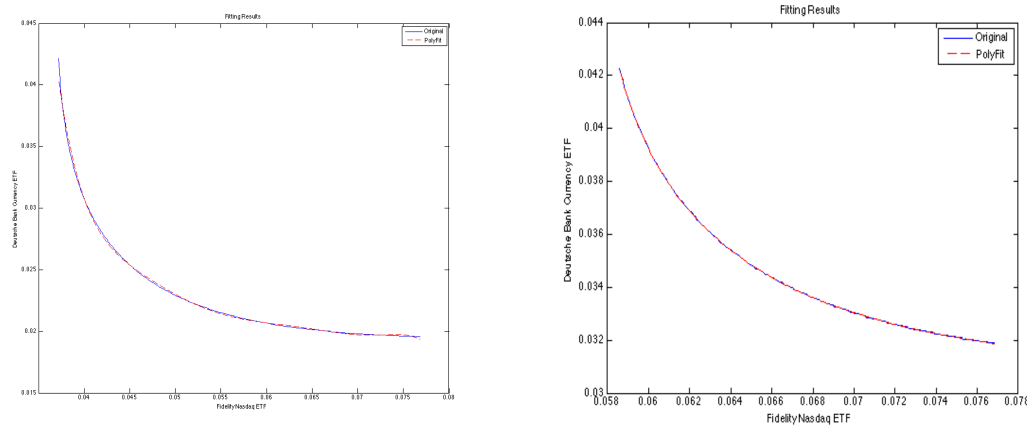

Figure 11: Multivariate Value-at-Risk, the quantile function at the probability levels $p = 0.95$ and $p = 0.99$.
At probability levels $p = 0.95$ for LHS and $p = 0.99$ for RHS, over 6 months of time period, $\text{MVaR}_p(Y)$ is well approximated by polynomial fitting in the sense of the least-squares (dotted curve). The random vector has components of the losses from the investment in Deutsche Bank US Dollar Index Bullish (UUP) ETF and Fidelity Nasdaq Composite Index Tracking (ONEQ) ETF from Yahoo Finance. The piecewise linear line (looks like a curve) is the set of line segments between the points (y_1, y_2) such that $F_Y(y_1, y_2) = p$.

Table 1: MCVaR of different stochastic dependence relationship at various probability levels.

	Nasdaq and S&P 500 ($\rho = 0.9510393$)		Nasdaq and US currency ($\rho = -0.7093342$)	
p-levels	MCVaR	$P(X \in D_p^c)$	MCVaR	$P(Y \in D_p^c)$
$p = 0.8$	0.00020031396148	0.15378401420764	0.00010897751748	0.00010152096654
$p = 0.9$	0.00033556530624	0.07132564927678	0.00020802596277	0.00000159763343
$p = 0.95$	0.00048387995078	0.03287958274500	0.00040766191359	0.00000002176898
$p = 0.99$	0.00082775999651	0.00477476826419	0.00060555466408	0.00000000000078

As in Table 1, the set of assets with negative correlation has a lower level of risk at each probability level. We also observe that, for the set of negatively correlated assets, the chance being beyond the MVaR is much smaller than the case of positive correlation among assets. That is the power of low-correlation investment. Stochastic dependence structure among assets must be taken into account and that is one of the great features of desirable multivariate risk measures. It is shown that MCVaR can be used as a risk measure on correlated assets.

Remark 4. *In the multivariate case it is reasonable to choose the value of p smaller than what we choose in the univariate case and it may depend on the number of components of a random vector.*

Remark 5. *(Corporate M&A (Mergers and Acquisitions)) Adequate measure of potential risk in a corporate M&A is essential. This is one of the most important factors in analyzing M&A*

deals from a risk management perspective. Examples 1 and 2 can also be considered as risk evaluation processes for different M&A deals. Depending on the categorization, there are many types of risk evaluation useful for M&A deals including strategic risk, compliance risk, operational risk, financial risk, reputation risk, etc. Additionally, each type of risk needs to be evaluated by a suitable risk measurement. Let us present a simple example of operational risk evaluation: suppose there are two companies of the same size (asset-based), involved in an M&A deal, and each board of directors wants to gauge its potential risk. Company A has 5 business sectors: smart phone, tablet PC, laptop computer, TV and digital camera, each with a total asset of $1 billion. Let X_i, $i = 1,\dots,5$ denote the random variable of operational loss from business sectors: smart phone, tablet PC, laptop computer, TV and digital camera, respectively. Company B has only 2 business sectors: display (LCD, LED panels) and real estate with total asset $4 billion and $1 billion, respectively. Let Y_j, $j = 1,2$ denote the random variable of operational loss from these two business sectors, respectively. Let us assume Y_1 and X_i for $i = 1,2,3,4,5$ are highly correlated and Y_2 has a low correlation with others. With given asset value of each business sector, we have weight vectors $\lambda^X = (1/5,1/5,1/5,1/5,1/5)^T$ and $\lambda^Y = (4/5,1/5)^T$ for company A and B, respectively.

For the potential operational risk evaluation of this M&A deal, we calculate

$$MCVaR_p(X_1,\dots,X_5,Y_1,Y_2) = E\left(\sum_{i=1}^{5}\lambda_i^X X_i + \sum_{i=1}^{2}\lambda_i^Y Y_i \mid H(X_1,\dots,X_5,Y_1,Y_2) \geq p\right), \tag{50}$$

where H is the c.d.f. of the random vector $(X_1,X_2,X_3,X_4,X_5,Y_1,Y_2)$. If the value of (50) is less than the sum of (51) and (52), risk measures of Company A and Company B, respectively:

$$MCVaR_p(X_1,\dots,X_5) = E\left(\sum_{i=1}^{5}\lambda_i^X X_i \mid G(X_1,\dots,X_5) \geq p\right), \tag{51}$$

where G is the c.d.f. of the random vector (X_1,X_2,X_3,X_4,X_5),

$$MCVaR_p(Y_1,Y_2) = E\left(\sum_{i=1}^{2}\lambda_i^Y Y_i \mid F(Y_1,Y_2) \geq p\right), \tag{52}$$

where F is the c.d.f. of the random vector (Y_1,Y_2), then it may be considered as a signal that this M&A deal is desirable from the point of view on managing risk. Comparison of (50) and the sum of (51) and (52) may serve the process of risk management in the M&A decision-making and it is advisable to do it for several p values to have an overview before the final decision. In the real world, the estimation of operational risk is complex and necessitates input from subject matter experts. However, MCVaR captures a stochastically dependent structure among correlated business sectors in M&A deals. In this respect, MCVaR will be able to play an important role as a risk measure in M&A analyses as well.

4.2 The Case of a Discrete Distribution

We use bounding schemes to obtain sharp lower and upper bounds for the probability $P(X \in D_p)$ as well as for the expectations $E(X_i \mathbb{1}_{X\in D_p}) = E(X_i|\, X \in D_p)P(X \in D_p)$ (see Prékopa (1988, 1990a,b, 1995, 2003)). If the lower and upper bounds are close to each other, we can use them for approximation.

4.2.1 Application of the Binomial Moment Bounding Scheme

The binomial moment problem for the probability of the union of events was introduced in Prékopa (1988). If $A_1,\dots,A_N$ are arbitrary events and

$$S_k = \sum_{1\le i_1<\cdots<i_k\le N} P(A_{i_1}\dots A_{i_k}),\ k=1,\dots,m,$$

then we solve the LP's:

$$\begin{aligned} &\min(\max)\sum_{i=1}^{N} p_i \\ &\text{subject to} \\ &\sum_{i=1}^{N}\binom{i}{k} p_i = S_k,\ k=1,\dots,m \\ &p_i\ge 0,\ i=1,\dots,N, \end{aligned} \tag{53}$$

and where m is a fixed integer, in practice $m \ll N$. Let $W_{\min},\ W_{\max}$ designate the optimum values, respectively. Then we have the sharp bounds, for the probability of the union,

$$W_{\min} \le P\left(\bigcup_{i=1}^{N} A_i\right) \le \min(W_{\max},1).$$

If the bounds are close to each other then they can be used to approximate the probability of the union. In the above formulation we are bounding probability but the method can be applied, in a straightforward manner, for subsets of an arbitrary set with finite measure.

In order to obtain lower and upper bounds for MCVaR, we take the sets $\{u \mid u\le s^{(i)}\},\ i=1,\dots,N$, the union of which is D_p and define the measure on the Borel sets of D_p, generated by functions of the type: $u_i f(u),\ u\in R^r$. Finally we construct lower and upper bounds for MCVaR, by the use of the obtained bounds.

Let $f(u)$ denote the p.d.f. of the random vector $X\in R^r$. Then

$$P(X\in D_p) = \int_{X\in D_p} f(u)du, \tag{54}$$

$$E(X_i\mathbb{1}_{D_p}) = E(X_i|\ X\in D_p)P(X\in D_p) = \int_{X\in D_p} u_i f(u)du, \tag{55}$$

where $D_p = \bigcup_{s\in \mathrm{MVaR}_p(X)} (s+R^r_-)$.

The set $A_i = s^{(i)}+R^r_-, i=1,\dots,N$ are the orthants in R^r, so are their intersections. The vertex of $A_{i_1}\cdots A_{i_k}$ is

$$\left(\min(s_1^{(i_1)},\dots,s_1^{(i_k)}),\dots,\min(s_r^{(i_1)},\dots,s_r^{(i_k)})\right).$$

Let us define k^{th} "binomial moment" S_k as follows.

$$S_k = \sum_{i_1 < \cdots < i_k} \int_{A_{i_1} \ldots A_{i_k}} g(y)dy,\ k = 1, \ldots, N. \tag{56}$$

Example 3 (Compound Poisson Processes, Insurance Claims)**.** *Suppose that various insurance claims occur according to independent, homogeneous Poisson processes. For simplicity the claims are assumed to be integer valued and independent of each other within each claim process and of the claims in the other claim processes. In the numerical example $M = 4$, and the types are: auto, health, home, life. The time period in which the claims are observed is one day. Information of daily claims to an insurance company is summarized in Table 2.*

Table 2: An insurance company's daily claims (in \$1,000) from 4 types of insurance.

ξ_1:	Auto:	$N_1 \sim \text{Poisson}(0.55)$,	$Z_{N_1} \sim U(1,2)$; the average of each claim= $\$1,500$
ξ_2:	Health:	$N_2 \sim \text{Poisson}(0.12)$,	$Z_{N_2} \sim U(1,3)$; the average of each claim= $\$2,000$
ξ_3:	Home:	$N_3 \sim \text{Poisson}(0.08)$,	$Z_{N_3} \sim U(1,5)$; the average of each claim= $\$3,000$
ξ_4:	Life:	$N_4 \sim \text{Poisson}(0.01)$,	$Z_{N_4} \sim U(1,5)$; the average of each claim= $\$3,000$

Let $N_i(t)$ and $X_i(t)$ designate the number of events and the total claim up to time t in the ith process, respectively. Then

$$\begin{aligned} P(N_i(t) = x) &= \frac{(\lambda_i t)^x}{x!} e^{-\lambda_i t},\ x = 0, 1, \ldots;\ i = 1, \ldots, M \\ X_i(t) &= Z_{i1} + Z_{i2} + \cdots + Z_{iN_i(t)},\ i = 1, \ldots, M, \end{aligned} \tag{57}$$

where Z_{ij} is the jth claim amount in claim process i.

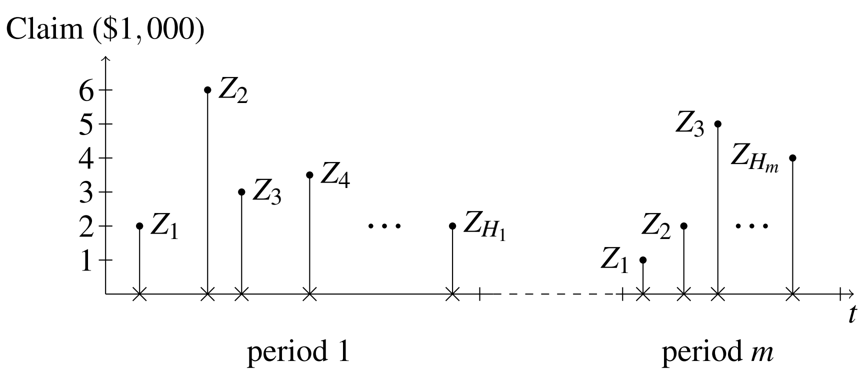

Figure 12: Illustration of Compound Poisson Distributed Losses.
H_i is the number of events incurred over the period i. The unit claim size is $\$1,000$.

The company is concerned about the loss at probability level $p = 0.9$. Let $f_i(x) = P(X_i(t) = x)$. Then, Panjer's formula (see Bowers et al. (1997)) provides us with recursions to calculate the

Table 3: $\mathrm{MVaR}_p(X)$ with $p = 0.9$, the unit claim size \$1,000.

ξ_1	ξ_2	ξ_3	ξ_4
3	4	6	1
3	7	5	6
4	3	5	3
4	3	6	1
4	4	3	1
4	6	2	6
4	7	2	5
5	2	6	1
5	3	3	4
5	3	4	1
5	4	1	1
6	2	5	5
6	3	3	1
7	2	5	3

probabilities $f_i(x),\ x = 1,2,\ldots,\ i = 1,2,3,4$*:*

$$\begin{aligned} &f_i(x) = \frac{\lambda_i}{x}\sum_{j=1}^{x} j p_i(j) f(x-j),\ x = 1,2,\ldots, \\ &f_i(0) = e^{-\lambda_i},\ i = 1,2,3,4. \end{aligned} \tag{58}$$

For MVaR, we obtained the following 14 p-efficient points, presented in Table 3.

The binomial moment bounding scheme, to obtain lower and upper bounds for $E(X_i|\ X \in D_p)P(X \in D_p)$ *is:*

$$\begin{gathered} \min(\max)\sum_{i=1}^{14} p_i \\ \textit{subject to} \\ \sum_{i=1}^{14}\binom{i}{k} p_i = S_k,\ k = 1,\ldots,m \\ p_i \geq 0,\ i = 1,\ldots,14, \end{gathered} \tag{59}$$

where

$$\begin{aligned} &S_k = \sum_{i_1<\cdots<i_k}\int_{A_{i_1}\ldots A_{i_k}} g(y)dy,\ k = 1,\ldots,m, \\ &g(y) = y_i f(y), \textit{where } y = \xi \in R^4, \\ &A_i = \{s^{(i)} + R^4_-,\ s^{(i)} \in R^4\}, i = 1,\ldots,14 \\ &\textit{and } f \textit{ is the } p.d.f. \textit{ of the random vector } X \in R^4. \end{aligned} \tag{60}$$

With $m = 14$, the constraints in (59) uniquely determine the unknowns and we obtain the values:

$$\begin{aligned} E(X_1 |\, X \in D_p)P(X \in D_p) &= 0.82115806, \\ E(X_2 |\, X \in D_p)P(X \in D_p) &= 0.23898585, \\ E(X_3 |\, X \in D_p)P(X \in D_p) &= 0.22968684, \\ E(X_4 |\, X \in D_p)P(X \in D_p) &= 0.02975693. \end{aligned} \tag{61}$$

For the bounds of $P(X \in D_p)$, we solve the LP's with $m = 14$ and integrand $g(y) = f(y)$:

$$\begin{gathered} \min(\max) \sum_{i=1}^{14} p_i \\ \textit{subject to} \\ \sum_{i=1}^{14} \binom{i}{k} p_i = S_k,\ k = 1,\ldots,m \\ p_i \geq 0,\ i = 1,\ldots,14, \end{gathered} \tag{62}$$

where

$$\begin{aligned} & S_k = \sum_{i_1<\cdots<i_k} \int_{A_{i_1}\cdots A_{i_k}} f(y)dy,\ k = 1,\ldots,m, \\ & A_i = \{s^{(i)} + R^4_-,\ s^{(i)} \in R^4\}, i = 1,\ldots,14 \\ & \textit{and } f \textit{ is the } p.d.f. \textit{of the random vector } X \in R^4. \end{aligned} \tag{63}$$

The optimum values coincide up to 8 digits and the resulting number is accepted as approximation of $P(X \in D_p)$:

$$P(X \in D_p) = 0.99832959. \tag{64}$$

Let λ_i = (the amount of premium in type i)/(total premium to all types) and assume that $\lambda_i = \frac{1}{4}$ for $i = 1,\ldots,4$. Then we have

$$\sum_{i=1}^{4} \lambda_i E(X_i|X \in D_p)P(X \in D_p) = 0.32989692. \tag{65}$$

Simple calculation gives:

$$\begin{aligned} m_1 &= E(X_1) = E(N_1)E(Z_1) = \lambda_1 E(Z_1) = 0.55 \times 1.5, \\ m_2 &= E(X_2) = E(N_2)E(Z_2) = \lambda_2 E(Z_2) = 0.12 \times 2, \\ m_3 &= E(X_3) = E(N_3)E(Z_3) = \lambda_3 E(Z_3) = 0.08 \times 3, \\ m_4 &= E(X_4) = E(N_4)E(Z_4) = \lambda_4 E(Z_4) = 0.01 \times 3, \end{aligned} \tag{66}$$

and

$$\sum_{i=1}^{4} \lambda_i m_i = 0.33375. \tag{67}$$

Plugging in the values of (64), (65) and (67) into (15), we obtain a loss amount of 2.30666989 with the unit claim size of \$1,000. The insurance company will expect the daily loss, i.e., the amount of total claims per day, of \$2,306.67 *with probability level 90%. Thus, we conclude that under that probability level, the insurance company would like to collect the amount of premium at least* \$2,306.67 *per day for accepting the risk, in order to make some underwriting profit.*

4.2.2 Application of the Boolean Bounding Scheme

The Boolean bounding scheme can also be used for bounding measures of the union of sets. Again, for the details of the Boolean bounding scheme we refer the readers to Prékopa (2003). Let us present a LP formulation for bounding the probability of the union, i.e., $P(X \in D_p)$, the probability of the union of favorable domain. In order to formulate the problem we introduce the notations:

$$\begin{aligned} a_{IJ} &= \begin{cases} 1, \text{ if } I \subset J \\ 0, \text{ if } I \not\subseteq J,\ I,J \subset \{1,\dots,n\} \end{cases}, \\ x_J &= P\Big(\big(\bigcap_{j\in J} A_j\big) \cap \big(\bigcap_{j\notin J} \bar{A}_j\big)\Big), \\ p_I &= P\Big(\bigcap_{j\in I} A_j\Big),\ I,J \subset \{1,\dots,n\}. \end{aligned}$$

The probability p_I means that all events $A_j,\ j \in K$ occur and the probability x_J means that all events $A_j,\ j \in I$ occur but the other do not occur. The Boolean probability bounding problem, or scheme for the probability of the union is the following:

$$\begin{aligned} &\min(\max) \sum_{\emptyset\neq J\subset\{1,\dots,n\}} x_J \\ &\text{subject to} \\ &\sum_{J\subset\{1,\dots,n\}} a_{IJ} x_J = p_I,\ I \subset \{1,\dots,n\}, |I| \leq m \\ &x_J \geq 0, J \subset \{1,\dots,n\}. \end{aligned} \tag{68}$$

Problem (68) has $1 + \sum_{i=1}^{m} \binom{n}{i}$ equality constraints and 2^n variables. If we remove x_0 and the equality constraint containing x_0 (meaning that the sum of the variables is equal to 1), then we obtain an equivalent Boolean problem, for bounding the probability of the union:

$$\begin{aligned} &\min(\max) \sum_{\emptyset\neq J\subset\{1,\dots,n\}} x_J \\ &\text{subject to} \\ &\sum_{\emptyset\neq J\subset\{1,\dots,n\}} a_{IJ} x_J = p_I,\ \emptyset \neq I \subset \{1,\dots,n\}, |I| \leq m \\ &x_J \geq 0,\ \emptyset \neq J \subset \{1,\dots,n\}. \end{aligned} \tag{69}$$

We also need to introduce the notations x_{J_i} and E_{I_i} as the following:

$$x_{J_i} = E\left(X_i \mathbb{1}_{\left(\bigcap_{j\in J} A_j\right)\cap\left(\bigcap_{j\notin J} \bar{A}_j\right)}\right), \tag{70}$$

$$E_{I_i} = E\left(X_i\ \mathbb{1}_{\bigcap_{j\in I} A_j}\right),\ I,J \subset \{1,\dots,n\}. \tag{71}$$

For example, let $I,J \subset \{1,2,3\},\ |I| \le 2$. Then E_{I_i} can be described as follows.

$$E_{I_i} = \begin{pmatrix} \int_{A_1} x_i f(x)dx \\ \int_{A_2} x_i f(x)dx \\ \int_{A_3} x_i f(x)dx \\ \int_{A_1A_2} x_i f(x)dx \\ \int_{A_1A_3} x_i f(x)dx \\ \int_{A_2A_3} x_i f(x)dx \end{pmatrix}, \tag{72}$$

where $f(x)$ is the p.d.f. of the random vector $X \in R^r$. Now we are ready to apply Boolean bounding scheme to the same numerical data of the examples in the previous section.

Example 4 (using the same data of Example 3 with the Boolean bounding scheme)**.** *The following LP formulation (73) with $m = 4$, $n = 14$ is for $P(X \in D_p)$:*

$$\begin{aligned} &\min(\max) \sum_{\emptyset\ne J\subset\{1,\dots,n\}} x_J \\ &\textit{subject to} \\ &\sum_{\emptyset\ne J\subset\{1,\dots,n\}} a_{IJ}x_J = p_I,\ \emptyset \ne I \subset \{1,\dots,n\}, |I| \le m \\ &x_J \ge 0,\ \emptyset \ne J \subset \{1,\dots,n\}, \end{aligned} \tag{73}$$

where $x_J = P\Big(\big(\bigcap_{j\in J} A_j\big) \cap \big(\bigcap_{j\notin J} \bar{A}_j\big)\Big)$ and $p_I = P\Big(\bigcap_{j\in I} A_j\Big),\ I,J \subset \{1,\dots,n\}$. The corresponding Boolean matrix is very large – the size of the matrix is 1480 $\times$ 16383 because $\sum_{i=1}^{4}\binom{14}{i} = 1470$ and $2^{14} - 1 = 16383$.

The LP formulation (73) provides us with the following result:

$$0.99832959468795 \le P(X \in D_p) \le 0.99832959471865. \tag{74}$$

Since the difference between lower and upper bounds in (74) is very small, let us present the bounds as one number of 8 decimal places: $P(X \in D_p) \approx 0.99832959$ which is the same as (64). In order *to calculate $E(X_i|\ X \in D_p)P(X \in D_p)$, we solve the following Boolean bounding problem again with $m = 4$, $n = 14$:*

$$\begin{aligned}
&\min(\max) \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} x_{J_i} \\
&\textit{subject to} \\
&\sum_{\emptyset \neq J \subset \{1,\ldots,n\}} a_{IJ} x_{J_i} = E_{I_i},\ \emptyset \neq I \subset \{1,\ldots,n\}, |I| \leq m \\
&x_J \geq 0,\ \emptyset \neq J \subset \{1,\ldots,n\},
\end{aligned} \tag{75}$$

*where $E_{I_i} = E\left(X_i \mathbb{1}_{\bigcap_{j\in I} A_j}\right)$, $I,J \subset \{1,\ldots,n\}$, and $x_{J_i} = E\left(X_i \mathbb{1}_{\left(\bigcap_{j\in J} A_j\right)\cap\left(\bigcap_{j\notin J} \bar{A}_j\right)}\right)$, where $A_i = \{s^{(i)} + R^4_-,\ s^{(i)} \in R^4\}$, which is an orthant with vertex $(s_1^{(i)}, s_2^{(i)}, \ldots, s_r^{(i)})$ for $i = 1, \ldots, 14$ since we have 14 p-*Level Efficient Points*, as those are enumerated in Table 3.*

From the above LP formulation the results are obtained as follows.

$$\begin{aligned}
0.821158068554149 \leq E(X_1|\ X \in D_p)P(X \in D_p) \leq 0.821158068554182, \\
0.238985857588695 \leq E(X_2|\ X \in D_p)P(X \in D_p) \leq 0.238985857631699, \\
0.229686849617116 \leq E(X_3|\ X \in D_p)P(X \in D_p) \leq 0.229686849618277, \\
0.029756933777791 \leq E(X_4|\ X \in D_p)P(X \in D_p) \leq 0.029756933782136.
\end{aligned} \tag{76}$$

Again, let us use numbers of 8 decimal places. Then all expectations are represented as one number, which are the same as (61) in Example 3. Since all the inputs are the same as one with binomial moment scheme, we obtain the same result that MCVaR for an insurance company is the loss of $\$2,306.67$. *Thus, under the probability level $p = 0.9$, the insurance company would like to collect the amount of premium at least* $\$2,306.67$ *per day for accepting the risk.*

5 Conclusions

We have explored various properties of Multivariate Value at Risk, or MVaR and Multivariate Conditional Value at Risk, or MCVaR. We have shown that many properties enjoyed by VaR and CVaR, carry over to the multivariate risk measures. In addition we have derived some properties of MVaR and MCVaR, based on multivariate logconcave theory, that do not have univariate counterpart or it is trivial. As regards the convexity of MVaR and MCVaR, none of them has that property, in general, but we have proved the convexity of MCVaR under the assumption that the components of the random vector are independent. We have proposed the numerical procedures to calculate or approximate MCVaR values. In case of a continuously distributed random vector we have used approximation and numerical integration. In case of a discrete random vector we have used the recently developed binomial moment and Boolean bounding schemes to approximate MCVaR. The results are illustrated on real life data and it is shown how MCVaR depends on the probability level and the correlation between the components of the random vector, representing different portfolios.

References

Artzner, P., F. Delbaen, J.M. Eber, D. Heath. 1999. Coherent Measures of Risk. *Mathematical Finance* **9**(3) 203–228.

Atkinson, Kendall E. 1988. *An introduction to numerical analysis, Second Edition*. John Wiley and Sons, Inc.

Ben-Tal, A., A. Ben-Israel. 1991. A recourse certainty equivalent for decisions under uncertainty. *Ann. Oper. Res.* **30** 3–44.

Ben-Tal, A., A. Ben-Israel, M. Teboulle. 1991. Certainty equivalents and generalized information measures: Duality and extremal principles. *J. Math. Anal. Appl.* **157** 211–236.

Ben-Tal, A., M. Teboulle. 1986. Expected utility, penalty functions, and duality in stochastic nonlinear programming. *Management Science* **32**(11) 1445–1466.

Boros, E., K. Elbassioni, V. Gurvich, L. Khachiyan, K. Makino. 1998. *An intersection inequality for discrete distributions and related generation problems*, vol. 2719. Berlin: Springer.

Bowers, N. L. Jr., H. U. Gerber, J. C. Hickman, D. A. Jones, C. J. Nesbitt. 1997. *Actuarial Mathematics*. The society of Actuaries, Schaumburg, Illinois.

Busemann, H. 2008. *Convex Surfaces*. Dover, Mineola, New York.

Cousin, A., E. Di Bernadino. 2011. A multivariate extension of Value-at-Risk and Conditional-Tail-Expectation. *Université Lyon 1, ISFA, Laboratoire SAF* .

Dentcheva, D., A. Prékopa, A. Ruszczyński. 2000. Concavity and efficient points of discrete distributions in probabilistic programming. *Math. Programming* **Ser. A**(89) 55–77.

Föllmer, H., A. Schied. 2002. Convex measures of risk and trading constraints. *Finance and Stochastics* **6** 429–447.

Gasca, M., T. Sauer. 2000. On the history of multivariate polynomial interpolation. *Journal of Computational and Applied Mathematics* **122** 23–35.

Jorion, Philippe. 2006. *Value at Risk: The New Benchmark for Managing Financial Risk, Third Edition*. McGraw-Hill.

Müller, A., D. Stoyan. 2002. *Comparison Methods for Stochastic Models and Risks*. Wiley, Chichester.

Pflug, G. Ch. 2000. Some remarks on the value-at-risk and conditional Value at Risk. *Probabilistic constrained optimization* 272–281.

Prékopa, A. 1970. On probabilistic constrained programming. *Proceedings of the Princeton Symposium on Mathematical Programming (Princeton University Press, NJ)* 113–138.

Prékopa, A. 1971. Logarithimic concave measures with applications to stochastic programming. *Acta Scientiarum Mathematicarum (Szeged)* **32** 301–316.

Prékopa, A. 1973a. Contributions to the theory of stochastic programming. *Math. Programming* **4** 202–221.

Prékopa, A. 1973b. On logarithmic concave measures and functions. *Acta Scientiarum Mathematicarum (Szeged)* **34** 335–343.

Prékopa, A. 1988. Boole-Bonferroni inequalities and linear programming. *Oper. Res.* **36**(1) 145–162.

Prékopa, A. 1990. Dual method for a one-stage stochastic programming problem with random rhs obeying a discrete probability distribution. *Oper. Res.* **34** 441–461.

Prékopa, A. 1990a. Sharp bounds on probabilities using linear programming. *Oper. Res.* **38**(2) 227–239.

Prékopa, A. 1990b. The discrete moment problem and linear programming. *Discrete Appl. Math.* **27** 235–254.

Prékopa, A. 1995. *Stochastic Programming*. Kluwer Academic Publishers.

Prékopa, A. 2003. Probabilistic Programming. *Hand books in Operations Research and Management Science (Ruszczyński, A. and Shapiro, A., Eds.)* **10** 267–351.

Prékopa, A. 2012. Multivariate value at risk and related topics. *Ann. Oper. Res.* **193**(1) 49–69.

Prékopa, A., B. Vizvári, T. Badics. 1998. *Programming Under Probabilistic Constraint with Discrete Random Variable*. Kluwer, Dordrecht, Boston.

Saita, Francesco. 2007. *Value at Risk and Bank Capital Management*. Elsevier Inc.

Sauerbrei, W., C. Meier-Hirmer, A. Benner, P. Royston. 2006. Multivariate regression model building by using fractional polynomials: Description of SAS, STATA and R programs. *Computational Statistics and Data Analysis* **50** 3464–3485.

Strang, Gilbert. 2007. *Computational Science and Engineering*. Wellesley-Cambridge Press.

Uryasev, S., R.T. Rockafellar. 2000. Optimization of conditional Value at Risk. *The Journal of Risk* **2** 21–41.

Vizvári, B. 1987. Beiträge zum Frobenius Problem. Dr. sc. nat. Dissertation. *Techniche Hochschule Carl Schorlemmer, Leuna-Merseburg, Germany* .