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On the existence of Nash equilibria in pure stationary strategies for the n-person positional games with perfect information, no moves of chance, and mean or total effective cost

Vladimir Gurvich, Vladimir Oudalov

Abstract. We study existence of Nash equilibria (NE) in pure stationary strategies in n-person positional games with no moves of chance, with perfect information, and with the mean or total effective cost function.
We construct a NE-free three-person game with positive local costs, disproving the conjecture suggested by Boros and Gurvich in Math. Soc. Sci. 46 (2003) 207-241. Still, the following four problems remain open:
Whether NE exist in all *two*-person games with total effective costs such that (I) all local costs are strictly positive or (II) without directed cycles of the cost zero?
If NE exist in all n-person games with the terminal (transition-free) cost functions, provided all directed cycles form a unique outcome c and (III) assuming that c is worse than any terminal outcome or (IV) without this assumption?
For $n = 3$ cases (I) and (II) are answered in the negative, while for $n = 2$ cases (III) and (IV) are proven. We briefly survey other negative and positive results on Nash-solvability in pure stationary strategies for the games under consideration.

Keywords: Stochastic games, Chess-like games, local cost, mean and total effective cost, pure and stationary strategies, Nash equilibrium

1 Introduction

1.1 Main concepts and definitions

We consider the n-person positional games with no moves of chance, with perfect information, and with the mean or total effective cost function.

Such a game is modeled by a directed graph (digraph) $G = (V, E)$ whose vertex-set V is partitioned into $n + 1$ subsets, $V = V_1 \cup \ldots \cup V_n \cup V_T$, where vertices V_i are interpreted as the positions controlled by the player $i \in I = \{1, \ldots, n\}$, while V_T is the set of all terminal (of out-degree 0) positions; V_T might be empty. In fact, we can easily make V_T empty by just adding a loop ℓ_v in each $v \in V_T$. A directed edge (arc) $e = (v, v') \in E$ is interpreted as a move of the player i whenever $v \in V_i$. A move is called *terminal* if $v' \in V_T$. Obviously, a terminal move cannot belong to any directed cycle (dicycle).

Given a local cost function $r : I \times E \to \mathbb{R}$, its value $r(i, e)$ is interpreted as an amount that the player $i \in I$ has to pay for the move $e \in e$. Respectively, $-r(i, e)$ is frequently referred to as the local reward or payoff. Let us remark that all players $i \in I$ pay for each move $e \in E$, not only that i who makes this move. Of course, costs may be 0 or negative.

The local cost function r is called *terminal* [10, 7] (or *transition-free* [20, 21]) if $c(i, e) = 0$ unless e is a terminal move.

Remark 1 *The n-person games with terminal local costs are referred to as the* transition-free [20, 21] *or* Chess-like *[2, 6] games. If $n = 2$, these games are called BW games. In this case $V_1 = V_W$ and $V_2 = V_B$ are called the White and Black positions, respectively. Allowing also Random positions, V_R, we would obtain a much more general class of the* Backgammon-like *or* BWR *games; see [32, 7] for more details. It is not difficult to demonstrate [5, 11] that BWR and the classic Gillette [25] mean cost stochastic games are in fact equivalent.*

In this paper, we restrict the players (and ourselves) by their pure stationary strategies. Such a strategy x_i of a player $i \in I$ is a mapping that assigns a move (v, v') to each position $v \in V_i$. A set of n strategies $x = (x_i \mid i \in I)$ is called a *strategy profile* or *situation*.

A situation x and an initial position $v_0 \in V$ uniquely define a directed path (dipath) $p(v_0, x)$ as follows: Position v_0 is controlled by a player $i \in I$ whose strategy x_i defines the first move (v_0, v'); position v' is controlled by a player $i' \in I$ whose strategy $x_{i'}$ defines the second move (v', v''); etc. The obtained dipath $p(v_0, x)$ is called a *play*. By construction, $p(v_0, x)$ begins in v_0 and either (i) terminates in a $v \in V_T$ or (ii) ends in a cycle that is repeated infinitely. Indeed, as soon as the play comes to a position, where it has already been before, it forms a dicycle that will be repeated infinitely, because all considered strategies are stationary. In case (i) we will call the play *terminal* and in case (ii) a *lasso*.

Remark 2 *By adding a loop ℓ_v to each terminal position $v \in V_T$, we reduce (i) to (ii), since every terminal play becomes a lasso too.*

Given a lasso L that consists of a a dicycle C repeated infinitely and an initial dipath P from v_0 to C, let us introduce the *effective costs* $R(i, L)$ as follows:

If L is a terminal play, $C = \ell_v$, let us set $r(i, \ell_v) = 0$ for each player $i \in I$ and terminal $v \in V_T$, and let $R(i, L) = \sum_{e \in P} r(i, e)$ be the sum of all local costs of P.

If C is not a terminal loop and $r(i, C) = \sum_{e \in C} r(i, e) \neq 0$, let us define $R(i, L) = \infty$ when $r(i, C) > 0$ and $R(i, L) = -\infty$ when $r(i, C) < 0$.

Such effective costs were introduced by Tijsman and Vreze [50, 51] and called *total*.

Remark 3 *Let us note that the case $r(i, C) = 0$ is more sophisticated and we will postpone it till Section 4.1, assuming that $r(i, C) \neq 0$ for any $i \in I$ and dicycle C in G, in the rest of the paper. In fact, we will construct our main counterexample in Section 2.2, assuming that (j) $r(i, e) > 0$ for all $i \in I, e \in E$, and hence, (jj) $r(i, C) > 0$ for any $i \in I$ and C in G too. (Obviously, (j) implies (jj), but in fact, these two conditions are equivalent; see [10] for a proof based on the Gallai potential transformation [23]; see also [36, 32, 7].)*

Let us repeat that loops ℓ_v, which were artificially added to all terminals $v \in V_T$, are treated differently from the dicycles of G; in particular, in the counterexample of Section 2.2 we will set $r(i, ell_v) = 0$ for all $i \in I$ and $v \in V_T$.

In Section 3 (only), we will consider also the more traditional *mean* or *average effective cost function*, defined by $R(i, L) = |C|^{-1} \sum_{e \in C} r(i, e)$, where $|C|$ is the length of the cycle C; see for example, [49, 25] and also [18, 19, 42, 43, 36, 32, 37].

The positional structure is a triplet (G, D, v_0), where $G = (V, E)$ is a digraph, the mapping $D : V \setminus V_T \to I$ defines a partition that assigns a player to every non-terminal position, and v_0 is a fixed initial position. Respectively, the *game in positional form* is defined by a positional structure and a local cost function $r : I \times E \to \mathbb{R}$.

The corresponding *normal game form* is a mapping $g : X \to \mathcal{L}$, where $X = \prod_{i \in I} X_i$ and X_i is the set of all (pure stationary) strategies of the player $i \in I$, while $\mathcal{L}$ is the set of lassos of the digraph G. Respectively, the *game in normal form* is a pair (g, R), where $R : I \times \mathcal{L} \to \mathbb{R}$ is the total or mean effective cost function defined above.

We will consider only one concept of solution: the classic *Nash equilibrium (NE)* [45, 46] defined for the normal form game (g, R) as follows: A situation $x \in X$ is called a NE if $R(i, g(x)) \leq R(i, g(x'))$ for every player $i \in I$ and each situation x' that may differ from x only in the ith coordinate. In other words, no player i can reduce his/her effective cost by replacing his/her strategy x_i by another strategy x'_i provided all other players keep their strategies $(x_j \mid j \in I \setminus \{i\})$ unchanged. Furthermore, we say that x is a NE in the positional form game (G, D, v_0, r) if it is a NE in the corresponding normal form game (g, R).

A situation x is called a *uniform* (or *subgame perfect*) NE if it is a NE in (G, D, v_0, r) for every choice of the initial position $v_0 \in V \setminus V_T$.

Remark 4 *The term "subgame perfect" is more frequent in the literature (see, for example, [20, 21]) and this is justified in case of the recursive or acyclic games. Yet, in presence of dicycles, none of the two games (G, D, v'_0, r) and (G, D, v''_0, r) is a subgame of the other whenever v' and v'' belong to a dicycle. it seems logical to call properties that hold for all possible initial positions $v_0 \in V \setminus V_T$ "uniform" rather than "subgame perfect".*

1.2 Main results and open problems

It is not difficult to construct a NE-free n-person game (G, D, v_0, r) with the total cost function and without zero-dicycles (that is, $r(i, C) = \sum_{e \in C} r(i, e) \neq 0$ for all dicycles C of G and players $i \in I$). Such an example with $n = 4$ was given in [10]. in Section 2 we will obtain a much simpler example with $n = 3$. In both these examples, digraph G has a unique dicycle C, yet, it is negative, $r(i, C) < 0$ for a player $i \in I$.

It was conjectured in [10] that such examples fail to exist if we assume additionally that all dicycles are positive, that is, (i): $r(i, C) > 0$ for all $i \in I$ and C in G. (As we already mentioned, (i) is equivalent with a seemingly stronger assumption (ii): $r(i, e) > 0$ for all $i \in I$ and $e \in E$.) In [10] this conjecture was proven for the so-called *play-once* games, in which every player controls only one position, that is, $|V_i| = 1$ for all $i \in I$.

Yet, the general the conjecture fails. In Section 2.2 we will give a counterexample.

However, it still remains open if a NE always exists in case of the terminal costs, or in other words, whether the n-person Chess-like (transition free) game structures are NS.

In [1, 7], the following two versions of this problem were considered. Obviously, no terminal move can belong to a dicycle. Hence, $r(i, e) = 0$ for every dicycle C of G, its arc $e \in C$, and player $i \in I$. Standardly [10, 1, 7] we assume that in a terminal game all dicycles form a unique outcome c, in addition to the k terminal outcomes $\{a_1, \dots, a_k\}$. We might also assume that

(A): the outcome c is the worst (most expensive) one for all players $i \in I$, that is, $R(i, c) < r(i, a_j)$ for all $i \in I$ and $j \in [k] = \{1, \dots, k\}$.

The above NS problem was answered in the positive for $k \leq 2$ [10] and for $k \leq 3$ provided (A) holds [14]. Yet in general, with or without assumption (A), NS of the n-person Chess-like (transition free) game structures remains an open problem [1, 7].

It is solved in the negative if we replace a NE by a uniform (subgame perfect) NE. The latter may not exist even under assumption (A). An example for $n = 3$ was given in [10] and it was strengthened to $n = 2$ in [7]. Moreover, it was shown recently in [13] that in both these examples, a uniform NE fails to exist not only in the pure but even in the mixed strategies.

In contrast, for $n = 2$ the above problem is solved in the positive, even without assumption (A) [10, 12]; see the last section of each paper. The proof is based on an old criterion [27, 28] stating that a game form g is NS if (and only if, of course) a NE exists in the game (g, r) for each zero-sum cost function r taking only the values ± 1. A slightly weaker form of this criterion was obtained earlier by Edmonds and Fulkerson [17] and independently in [26]; see also [31, 4, 12] for more details.

Remark 5 *The problem of NS of the two-person games structures in which every dicycle is a separate outcome (rather than all dicycles form one outcome) was considered in [12]. In this case, the NS criterion of [27, 28] implies partial results.*

Thus, the terminal two-person game structures are NS. In contrast, the following two problems related the non-terminal total costs remain open: whether NS holds for the local cost function r such that for all $i \in I$ and C in G (i): $r(i, C) > 0$ or (iii): $r(i, C) \neq 0$. Obviously, assumption (i) (which is equivalent with (ii)) is stronger than (iii).

Yet, NS certainly fails if the equalities $r(i, C) = 0$ are allowed. Moreover, in Sections 3 and 4.3 we will construct a two-person NE-free game in which $r(i, C) = 0$ for *all* $i \in I$ and C in G. This example is based on an old example of a two-person NE-free *mean* cost game from [29]; see also [30, 32].

In contrast, the zero-sum two-person games with the total cost function are NS. It was first proven by Tijsman and Vreze in [51]; ; see also [47, 48]. An alternative proof based on the well-known approach of discounted approximation was recently suggested in [8, 9].

Given a lasso L that consists of a dipath P and a zero-dicycle C such that $r(i, C) = 0$ for a player $i \in I$, it seems that the total effective cost function $R(i, L)$ (introduced in [51], see Section 4.1 for the definition) is the only one that guarantees NS, at least for the zero-sum two-person games. In Section 4.2, we will give an example showing that there may be no saddle point in the game if we "naturally" define for such a lasso:

$R'(i, L) = r'(i, P) = \sum_{e \in L} r(i, e) = \sum_{e \in P} r(i, e)$.

1.3 Two-person games

Every two-person zero-sum BW game has a saddle point in *pure stationary uniformly optimal strategies*. For the the mean effective costs, $R(i, L) = |C|^{-1} \sum_{e \in C} r(i, e)$, this was proven by Moulin [42, 43] for complete bipartite graphs, by Ehrenfeucht and Mycielski [18, 19] for all bipartite graphs, and by Gurvich, Karzanov, and Khachiyan [32] in general.

In contrast, we will show in Section 3 that saddle points may fail to exist in case of the additive but non-averaged cost functions $R'(i, L) = \sum_{e \in C} r(i, e)$.

For the BW games with the total effective cost (see Section 3) the above existence result was obtained by Tijsman and Vreze [51]. An alternative simpler proof based on the classic discounted approximation was recently given in [8, 9].

Finally, let us note that the existence of a saddle point in *pure stationary uniformly optimal strategies* for the two-person zero-sum mean cost games holds not only for BW but for a more general BWR [32, 4] model, which includes not only Black and White but also Random positions. This can be easily derived from the classic results of Shapley [49], Gillette [25], Liggett and Lippman [40], since Gillette's and BWR mean cost models are in fact equivalent [5, 11]. Yet, for the BWR games with total effective cost there is no proof.

Remark 6 *The case of the terminal (transition free) cost function was considered much earlier. Zermelo [54] was the first who proved solvability of Chess (but in fact, of any two-person Chess-like zero-sum game with perfect information) in pure strategies. Let us notice that already the digraph of Chess has dicycles, since a position can be repeated in the play. However, Zermelo did not restrict the players by their stationary strategies. This was done later, by König [39] and Kalmar [35]; see also [53] and [2] for a recent survey.*

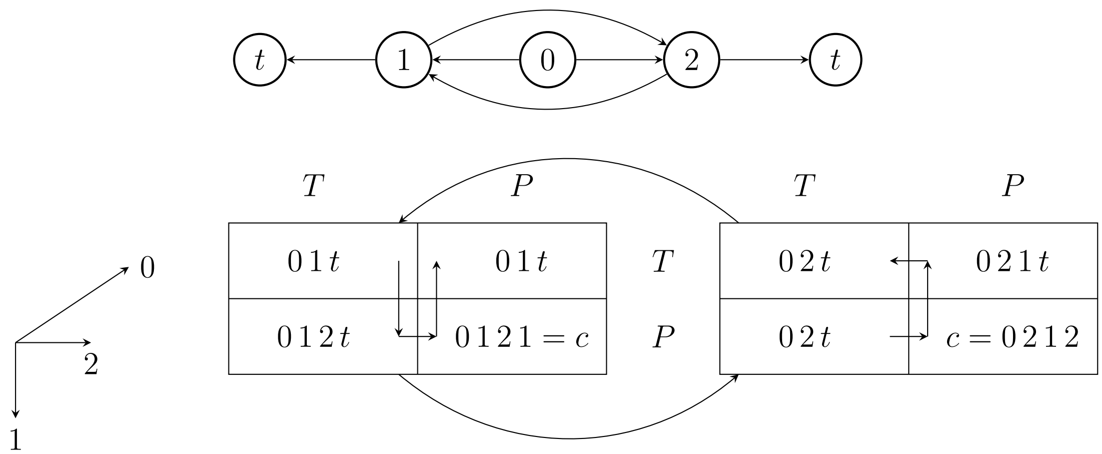

Figure 1: Not Nash-solvable three-person game form. Three players $I = \{0, 1, 2\}$ control three non-terminal positions $V \setminus V_T = V \setminus \{v_t\} = \{v_0, v_1, v_2\}$. To save space, we replace the positions just by their indices $0, 1, 2$ (which denote the corresponding players, too) and t.

In contrast, for the non-zero-sum case the existence of a NE in pure stationary (but not necessarily uniformly optimal) strategies was proven in [10] only for the two-person Chess-like (transition free) games. Two open problems for $n = 2$ along with several negative results [7, 13] are summarized in the previous section.

2 NE-free three-person games with total effective costs

2.1 Arbitrary local costs

In the beginning, let us wave the requirement of positivity. Then, a quite simple example can be constructed; see Figure 1. Players 1 and 2 control respectively positions v_1 and v_2, which form a dicycle of length 2. Each of these two players has two strategies: to terminate (T) or proceed (P). Player 0 controls the initial position v_0 and also has two strategies (1) and (2): to move to v_1 or v_2, respectively.

The corresponding normal game form g given by the $2 \times 2 \times 2$ table in Figure 1. Every its entry is a situation $x \in X$ that defines the play (lasso) $L(x)$, as shown in Figure 1.

We want to find a local cost function $r : I \times E \to \mathbb{Z}$ such that the obtained game (g, R) is NE-free, where the total effective cost $R(i, L)$ is defined for a given player $i \in I = \{0, 1, 2\}$ and a lasso $L = L(x)$ as follows: If $L = L(x)$ is a terminal play, that is, a dipath $P = P(x)$ from v_0 to the (unique) terminal position v_t (in which, by convention, we add to P a loop ℓ_{v_t} with $r(i, \ell_{v_t}) = 0$ for all $i \in I$) then $R(i, L) = \sum_{e \in P} r(i, e)$. If $L = L(x)$ is a cyclic play, which ends in a dicycle C, then $R(i, L)$ is ∞ (respectively, $-\infty$) when $r(i, C) = r(i, (v_1, v_2)) + r(i, (v_2, v_1))$ is positive (respectively, negative).

Remark 7 *In the present example, the equality $r(i, C) = 0$ holds for no $i \in I$. In Section 4.1, we will recall the definition of the total effective cost for this case and construct an example of a* two-*person NE-free game in which $r(i, C) = 0$ for* all *$i \in I$ and C in G.*

By the definition, a game (g, R) has no NE if and only if for every $x = (x_0, x_1, x_2) \in X$ there is a player $i \in I = \{0, 1, 2\}$ and a situation $x' = (x'_0, x'_1, x'_2) \in X$ that differs from x only in the coordinate i and such that $R(i, L(x')) < R(i, L(x))$. In Figure 1, for every $x \in X$ the arrow from the corresponding x' is drawn. Yet, we have to verify that these arrows (strict inequalities) are not contradictory. This should be done for each $i \in I$ separately, since there are no relations between the local costs of different players. It is easily seen that for $i \in I = \{0, 1, 2\}$ we obtain the following three systems:

$$r(0,(v_0,v_2)) + r(0,(v_2,v_t)) < r(0,(v_0,v_1)) + r(0,(v_1,v_t))$$
$$r(0,(v_0,v_1)) + r(0,(v_1,v_2)) < r(0,(v_0,v_2)) + r(0,(v_1,v_t))$$

$$r(1,(v_1,v_t)) < r(1,(v_1,v_2)) + r(1,(v_2,v_t))$$
$$r(1,(v_1,v_2)) + r(1,(v_2,v_1)) < 0$$

$$r(2,(v_2,v_1)) + r(1,(v_1,v_t)) < r(2,(v_2,v_t))$$
$$r(2,(v_1,v_2)) + r(2,(v_2,v_1)) > 0$$

Remark 8 *In this example, the total cost of a cyclic play is either ∞ or $-\infty$. Respectively, in these two cases it is $>$ or $<$ than the total cost of any terminal play, which is finite. Let us also notice that some terms of the corresponding sums are canceled in the above inequalities.*

It is easy to verify that all three systems are feasible. For example, we can set

$\mathbf{r} = (r(i,(v_0,v_1)), r(i,(v_1,v_t)), r(i,(v_0,v_2)), r(i,(v_2,v_t)); r(i,(v_1,v_2)), r(i,(v_2,v_1)))$ to

$(1,4,3,1;1,1), (1,1,1,1;1,-2)$ and $(1,1,1,3;1,1)$ for $i = 0, 1,$ and 2, respectively.

Remark 9 *Let us unite two players* 0 *and* 2 *and replace them by the single player* 2 *getting $I = \{1, 2\}$. It can be verified that the obtained two-person game structure is NS. The corresponding normal 2×4 game form g contains five distinct outcomes (plays): (v_0, v_1, v_t) and (v_0, v_2, v_t) appears twice each, (v_0, v_1, v_2, v_1) and (v_0, v_2, v_1, v_2) form the same outcome c, finally, (v_0, v_1, v_2, v_t) and (v_0, v_2, v_1, v_t) form two separate outcomes. It is easy to verify that the last two outcomes cannot simultaneously be the best responses of the player* 2*, since the corresponding system of strict linear inequalities is infeasible. It is interesting to notice that the criterion of NS of [27, 28] "a two-person game form is NS if and only if it is tight" does not seem to be applicable in this case. Indeed, the considered game form is NS but not tight. The reason is, it is not a game form in the sense of [27, 28], since the outcomes are now the plays and the total payoff being additive cannot take any values, unlike [27, 28]; see [4, 12] for the definition and more details. The above NS criterion might be extendable for such generalized two-person game forms, yet, it should become the subject of a separate research.*

A similar NE-free four-person game was constructed in [10]; see Figure 6 on page 223.

Let us notice that in both these NE-free examples $n > 2$ and there is a dicycle that is negative for at least one player. Whether similar NE-free *two-person* examples (with a positive or arbitrary local cost) exist is still an open problem.

2.2 Positive local costs

However, here we will show a computer-generated *three-person* NE-free game (G, D, u_1, r) with positive integer local costs r, thus, disproving the conjecture of [10].

Graph $G = (V, E)$ is given in Figure 2; $V = \{u_1, v_1; u_2, v_2; u_3, v_3, t\}$ is its vertex set. For each $i \in I = \{1, 2, 3\}$, the two positions with the subscript i are controlled by the player i. Let us note that this graph contains a unique dicycle $C = (v_1, v_3), (v_3, v_2), (v_2, v_1)$. The initial position is u_1. The integer positive local costs $r(i, e)$ are also given in this figure.

The out-degrees of the vertices are $3, 2; 3, 2; 2, 2$, respectively. Hence, players 1 and 2 have $3 \times 2 = 6$ strategies each, while 3 has only $2 \times 2 = 4$ strategies. The corresponding normal game form and game in normal form are given by the Tables 1 and 2 respectively. Each of these two tables consists of 4 subtables of the size 6×6 that correspond to the four strategies of the 3d player, while the strategies of the players 1 and 2 correspond to the rows and columns of each subtable, respectively. The strategy of a player $i \in I = \{1, 2, 3\}$ that recommends the moves (u_i, w_i') and (v_i, w_i'') is denoted by w_i', w_i'' for short, where w_i', w_i'' is a pair of positions of V; see Tables 1 and 2.

In Table 1 each entry is a play, while in Table 2 it is the the triplet that defines the effective total costs of players $1, 2$, and 3. In these tables all cyclic plays, as well as the corresponding total effective costs, (∞, ∞, ∞), are replaced by the word "cycle". In Table 2, all minima in each column are given in bold. Hence, by the definition, a NE would be an entry in which all three numbers are bold. Since there exists no such entry, the considered game is NE-free.

3rd player: v_1v_2						1st ↓	3rd player: v_1t					
cycle	$u_1v_1v_3v_2t$	cycle	$u_1v_1v_3v_2t$	cycle	$u_1v_1v_3v_2t$	v_1v_3	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$
u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t
cycle	$u_1u_2u_3v_1v_3v_2t$	cycle	$u_1u_2v_3v_2t$	u_1u_2t	u_1u_2t	u_2v_3	$u_1u_2u_3v_1v_3t$	$u_1u_2u_3v_1v_3t$	$u_1u_2v_3t$	$u_1u_2v_3t$	u_1u_2t	u_1u_2t
$u_1u_2u_3v_1t$	$u_1u_2u_3v_1t$	$u_1u_2v_3v_2v_1t$	$u_1u_2v_3v_2t$	u_1u_2t	u_1u_2t	u_2t	$u_1u_2u_3v_1t$	$u_1u_2u_3v_1t$	$u_1u_2v_3t$	$u_1u_2v_3t$	u_1u_2t	u_1u_2t
cycle	u_1v_2t	cycle	u_1v_2t	cycle	u_1v_2t	v_2v_3	$u_1v_2v_1v_3t$	u_1v_2t	$u_1v_2v_1v_3t$	u_1v_2t	$u_1v_2v_1v_3t$	u_1v_2t
$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t
u_3v_1	u_3t	v_3v_1	v_3t	tv_1	tt	2nd ↔	u_3v_1	u_3t	v_3v_1	v_3t	tv_1	tt
cycle	$u_1v_1v_3v_2t$	cycle	$u_1v_1v_3v_2t$	cycle	$u_1v_1v_3v_2t$	v_1v_3	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$
u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t
$u_1u_2u_3t$	$u_1u_2u_3t$	cycle	$u_1u_2v_3v_2t$	u_1u_2t	u_1u_2t	u_2v_3	$u_1u_2u_3t$	$u_1u_2u_3t$	$u_1u_2v_3t$	$u_1u_2v_3t$	u_1u_2t	u_1u_2t
$u_1u_2u_3t$	$u_1u_2u_3t$	$u_1u_2v_3v_2v_1t$	$u_1u_2v_3v_2t$	u_1u_2t	u_1u_2t	u_2t	$u_1u_2u_3t$	$u_1u_2u_3t$	$u_1u_2v_3t$	$u_1u_2v_3t$	u_1u_2t	u_1u_2t
cycle	u_1v_2t	cycle	u_1v_2t	cycle	u_1v_2t	v_2v_3	$u_1v_2v_1v_3t$	u_1v_2t	$u_1v_2v_1v_3t$	u_1v_2t	$u_1v_2v_1v_3t$	u_1v_2t
$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t
3rd player: tv_2						1st ↑	3rd player: tt					

Table 1: The normal game form corresponding to the game structure of Figure 2

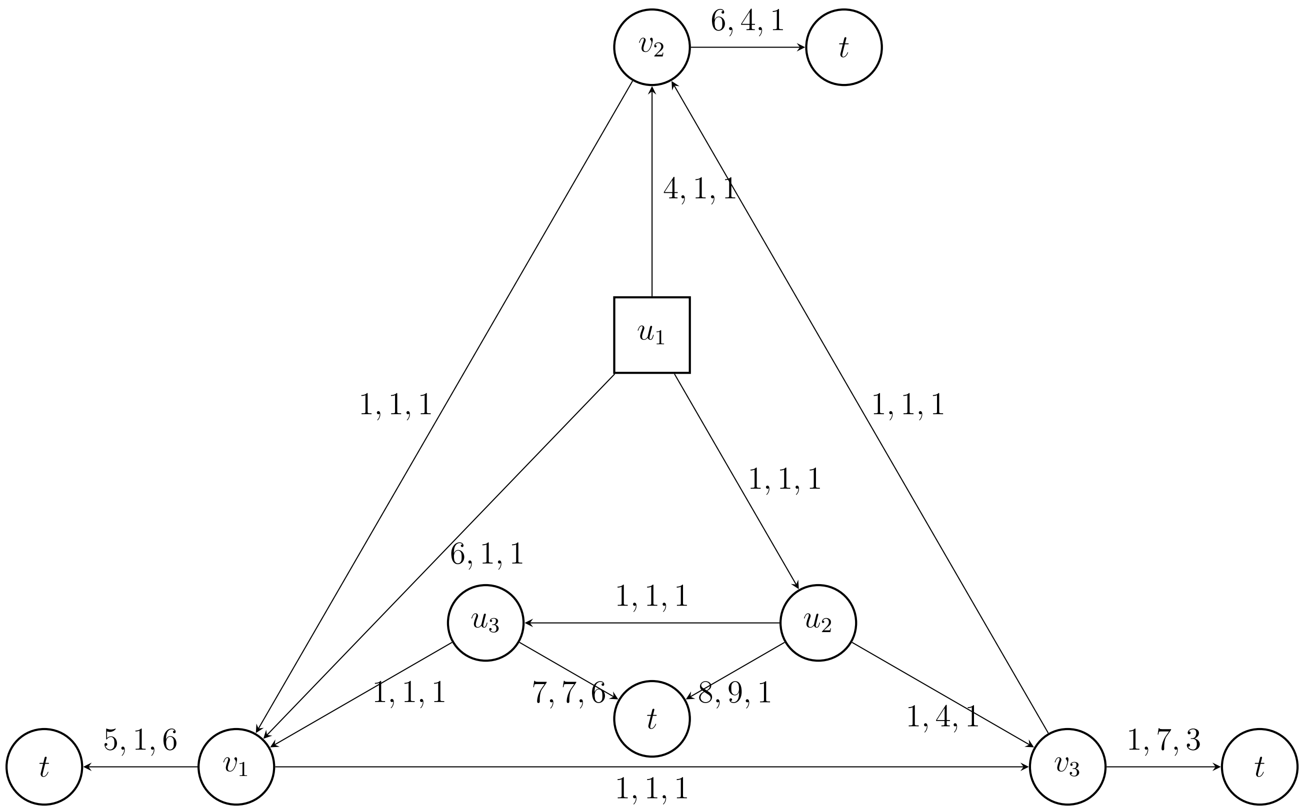

Figure 2: A NE-free three-person game with positive integer local costs

3rd player: v_1v_2						$1^{st} \downarrow$	3rd player: v_1t					
cycle	14 **7 4**	cycle	14 **7 4**	cycle	14 **7 4**	v_1v_3	8 **9 5**	8 **9** 5	8 **9 5**	8 **9** 5	8 **9 5**	**8 9** 5
11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	v_1t	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**
cycle	11 **9 6**	cycle	**9** 10 4	**9** 10 **2**	**9** 10 **2**	u_2v_3	**5** 11 **7**	**5** 11 7	**3** 12 **5**	**3** 12 5	9 **10 2**	9 **10 2**
8 4 9	**8 4** 9	**9** 8 10	**9** 10 **4**	**9** 10 **2**	**9** 10 **2**	u_2t	8 **4** 9	8 **4** 9	**3** 12 **5**	**3** 12 5	9 10 **2**	9 10 **2**
cycle	10 **5 2**	cycle	10 **5 2**	cycle	10 **5 2**	v_2v_3	7 10 **6**	10 **5 2**	7 10 **6**	10 **5 2**	**7** 10 **6**	10 **5 2**
10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	v_2t	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**
u_3v_1	u_3t	v_3v_1	v_3t	tv_1	tt	$2^{nd} \leftrightarrow$	u_3v_1	u_3t	v_3v_1	v_3t	tv_1	tt
cycle	14 **7 4**	cycle	14 **7 4**	cycle	14 **7 4**	v_1v_3	8 **9 5**	**8 9** 5	8 **9 5**	8 **9** 5	8 **9 5**	**8 9** 5
11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	v_1t	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**
9 9 8	**9 9** 8	cycle	**9** 10 4	**9** 10 **2**	**9** 10 **2**	u_2v_3	9 **9** 8	9 **9** 8	**3** 12 **5**	**3** 12 5	9 10 **2**	9 10 **2**
9 9 **8**	**9** 9 **8**	**9 8** 10	**9** 10 4	**9** 10 **2**	**9** 10 **2**	u_2t	9 **9 8**	9 **9 8**	**3** 12 **5**	**3** 12 5	9 10 **2**	9 10 **2**
cycle	10 **5 2**	cycle	10 **5 2**	cycle	10 **5 2**	v_2v_3	**7** 10 **6**	10 **5 2**	7 10 **6**	10 **5 2**	**7** 10 **6**	10 **5 2**
10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	v_2t	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**
3rd player: tv_2						$1^{st} \uparrow$	3rd player: tt					

Table 2: The corresponding NE-free game in the normal form

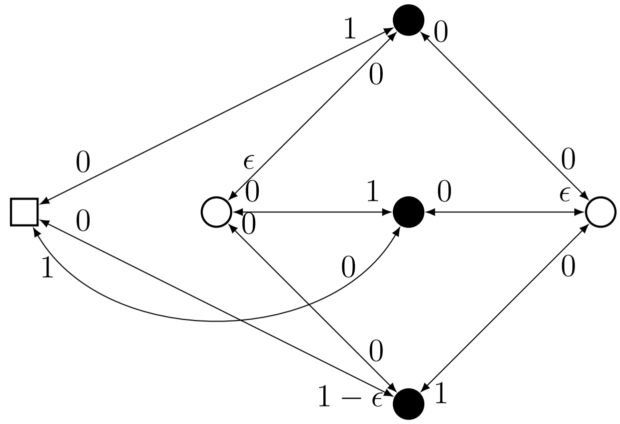

Figure 3: A NE-free two-person non-zero-sum mean effective cost game

3 Mean effective costs

Given an infinite play with the sequence of local costs $\mathbf{r} = (r_1, r_2, \dots,)$ the *mean* effective cost is defined as the Cesaro average $R_M = \lim_{k\to\infty} \sum_{j=1}^{k} r_j$.

In general, the limit may fail to exist but it surely exists when the considered play is a lasso L. In this case, the sequence $\mathbf{r}$ is pseudo-periodical $\mathbf{r} = (r'_1, \dots, r'_a(r''_1, \dots, r''_b)^\infty)$ (meaning that the second part, $(r''_1, \dots, r''_b)$, is repeated infinitely) and $R_M = \frac{1}{b}\sum_{j=1}^{b} r''_j$.

A saddle point (NE, in pure stationary strategies) exists in every two-person zero-sum mean effective cost game [25, 18, 42, 43, 19, 32]; see Introduction for more details.

However, this claim is not extended to the two-person but not necessarily zero-sum games. The following BW non-zero-sum game was constructed in [28, 32]; see Figure 3. It is defined on the complete bipartite 3×3 digraph $G = (V, E)$, that is, White and Black have three positions each and there is a move from every White to each Black position and vice versa. Every double-arrowed arcs in Figure 3 should be replaced by a pair of oppositely directed arcs on which the local costs of each player are equal.

Remark 10 *Such symmetric zero-sum BW games on complete bipartite digraphs were interpreted by Moulin as* ergodic extensions *of matrix games [42, 43]. Since the considered game is not zero-sum, it can be viewed as the ergodic extension of the next* bimatrix *game.*

$$\begin{matrix} 0 & 0 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \end{matrix} \qquad\qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-\varepsilon & 0 & 1 \end{matrix}$$

In [30], it was shown that the above example is minimal, since any BW game on a complete bipartite $2 \times k$ *digraph has a NE. The proof is based on the criterion of [27, 28].*

One could also try to "simplify" the formula $R_M = \frac{1}{b}\sum_{j=1}^{b} r''_j$ for the mean effective cost of a lasso replacing it by $R'_M = \sum_{j=1}^{b} r''_j$, that is, considering the additive rather than

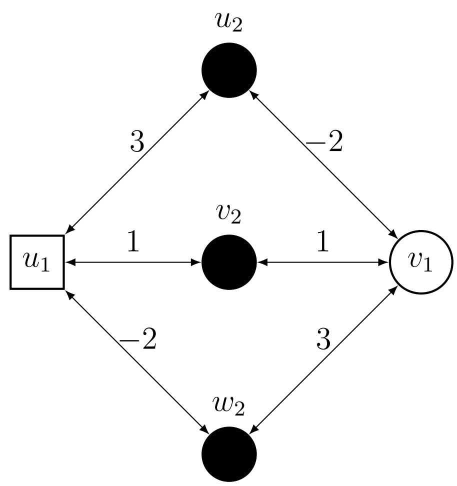

Figure 4: A saddle point free zero-sum game with the additive ("pseudo-mean") cost function

mean effective cost. However, then NS fails already for the two-person zero-sum games. The example is given in Figure 4. It represents the ergodic extension of the 2×3 matrix game

3	1	−2
−2	1	3

Again, each double-arrowed edge should be replaced by two oppositely directed arcs. White controls 2 positions and has $3^2 = 9$ strategies; Black controls 3 positions and has $2^3 = 8$ strategies; the corresponding 9×8 game form and the matrix game are given by Table 3. It is easily seen that this game has no saddle point, since $2 = \max\min < \min\max = 3$. Thus NS fails already in the two-person zero-sum case.

Remark 11 *In fact, the averaged cost function $R(i,L) = |C|^{-1}\sum_{e\in C} r(i,e)$ is in many respects "nicer" than $R'(i,L) = \sum_{e\in C} r(i,e)$. For example, the latter is NP-hard to maximize even for positive local costs, $r(i,e) > 0$, for all $e \in E$ and a given $i \in I$, since this problem generalizes the classic Hamiltonian cycle [24]. In contrast, maximizing and minimizing $R(i,L)$ can be easily reduced to LP; In [36], Karp suggested even more efficient procedures.*

4 Total effective costs

4.1 Definitions

Given an infinite play with the sequence of local costs $\mathbf{r} = (r_1, r_2, \ldots,)$ the *total* effective cost is defined as the Cesaro average of the sums $r_1, r_1 + r_2, r_1 + r_2 + r_3, \ldots,$ rather than of the local costs $r_1, r_2, r_3, \ldots$, in other words $R_T = \lim_{k\to\infty}\sum_{j=1}^{k}\frac{k-j+1}{k}r_j$; cf. to $R_M = \lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k} r_j$.

	$u_2\,u_1$ $v_2\,u_1$ $w_2\,u_1$	$u_2\,u_1$ $v_2\,u_1$ $w_2\,v_1$	$u_2\,u_1$ $v_2\,v_1$ $w_2\,u_1$	$u_2\,u_1$ $v_2\,v_1$ $w_2\,v_1$	$u_2\,v_1$ $v_2\,u_1$ $w_2\,u_1$	$u_2\,v_1$ $v_2\,u_1$ $w_2\,v_1$	$u_2\,v_1$ $v_2\,v_1$ $w_2\,u_1$	$u_2\,v_1$ $v_2\,v_1$ $w_2\,v_1$
$u_1\,u_2$ $v_1\,u_2$	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2v_1u_2$ -4	$u_1u_2v_1u_2$ -4	$u_1u_2v_1u_2$ -4	$u_1u_2v_1u_2$ -4
$u_1\,u_2$ $v_1\,v_2$	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2v_1v_2u_1$ 3	$u_1u_2v_1v_2u_1$ 3	$u_1u_2v_1v_2v_1$ 2	$u_1u_2v_1v_2v_1$ 2
$u_1\,u_2$ $v_1\,w_2$	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2v_1w_2u_1$ 2	$u_1u_2v_1w_2v_1$ 6	$u_1u_2v_1w_2u_1$ 2	$u_1u_2v_1w_2v_1$ 6
$u_1\,v_2$ $v_1\,u_2$	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1u_2u_1$ 3	$u_1v_2v_1u_2u_1$ 3	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1u_2v_1$ -4	$u_1v_2v_1u_2v_1$ -4
$u_1\,v_2$ $v_1\,v_2$	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1v_2$ 2	$u_1v_2v_1v_2$ 2	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1v_2$ 2	$u_1v_2v_1v_2$ 2
$u_1\,v_2$ $v_1\,w_2$	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1w_2u_1$ 3	$u_1v_2v_1w_2v_1$ 6	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1w_2u_1$ 3	$u_1v_2v_1w_2v_1$ 6
$u_1\,w_2$ $v_1\,u_2$	$u_1w_2u_1$ -4	$u_1w_2v_1u_2u_1$ 2	$u_1w_2u_1$ -4	$u_1w_2v_1u_2u_1$ 2	$u_1w_2u_1$ -4	$u_1w_2v_1u_2v_1$ -4	$u_1w_2u_1$ -4	$u_1w_2v_1u_2v_1$ -4
$u_1\,w_2$ $v_1\,v_2$	$u_1w_2u_1$ -4	$u_1w_2v_1v_2u_1$ 3	$u_1w_2u_1$ -4	$u_1w_2v_1v_2v_1$ 2	$u_1w_2u_1$ -4	$u_1w_2v_1v_2u_1$ 3	$u_1w_2u_1$ -4	$u_1w_2v_1v_2v_1$ 2
$u_1\,w_2$ $v_1\,w_2$	$u_1w_2u_1$ -4	$u_1w_2v_1w_2$ 6	$u_1w_2u_1$ -4	$u_1w_2v_1w_2$ 6	$u_1w_2u_1$ -4	$u_1w_2v_1w_2$ 6	$u_1w_2u_1$ -4	$u_1w_2v_1w_2$ 6

Table 3: The corresponding normal form; $2 = \max_{col} \min_{row} < \min_{row} \max_{col} = 3$

The limit may fail to exist in general but, as we already mentioned, for a lasso L sequence $\mathbf{r}$ is pseudo-periodical, $\mathbf{r} = (r'_1, \dots, r'_a(r''_1, \dots, r''_b)^\infty)$. In this case the limit exists and

$$R_T = \sum_{j=1}^{a} r'_j + \sum_{j=1}^{b} \frac{b-j}{b} r''_j. \tag{1}$$

In particular, when the play is terminal, the corresponding lasso L ends with the zero-loop ℓ_v for some $v \in V_T$; in this case $b = 1$, $\mathbf{r} = (r'_1, \dots, r'_a(0)^\infty) = (r'_1, \dots, r'_a, 0, 0, 0, \dots)$, and $R_T = r'_1 + \dots + r'_a = \sum_{j=1}^{a} r'_j$ is just the total cost of the terminal path P of L.

The same formula holds for any play L that ends in a cycle C in which all local costs are zeros; the terminal zero-loop ℓ_v is just a special case. It is also clear that $R_T(L) = \infty$ (respectively, $-\infty$) when the corresponding cycle C is positive, $\sum_{j=1}^{b} r''_j > 0$ (respectively, negative). Yet, when C is a zero but not identically zero dicycle R_T is defined by (1).

For example, the total cost function looks relevant to describe the accumulation of pension contributions. Summation $r_1 + (r_1 + r_2) + \dots + (r_1 + r_2 + \dots + r_t) + \dots$ reflects the fact that the contribution r_i works beginning with the year i. The corresponding play ends with a zero-loop at the year when the individual is retired.

Applications to shortest paths interdiction problems are considered in Section 4.4.

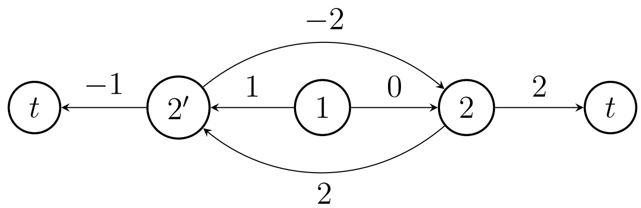

Figure 5: A saddle point free zero-sum game with a "pseudo-total" cost function

4.2 A saddle point free game with a pseudo-total costs

In this case, one could try to "simplify" (1) replacing it by $R'_T = \sum_{j=1}^{a} r'_j$, since $\sum_{j=1}^{b} r''_j = 0$. In other words, if a lasso L ends in a cycle C such that $\sum_{e \in C} r(i,e) = 0$ for all $i \in I$, it seems logical to define $R'_T(i,L) = \sum_{e \in L} r(i,e) = \sum_{e \in P} r(i,e)$.

However, the following example shows that already a two-person zero-sum game with such a *pseudo-total* cost function may have no saddle points. Such a game is given in Figure 5 and Table 4 in the positional and normal forms respectively. It is easy to verify that the game has no saddle points, since $0 = \max\min < \min\max = 1$.

	$2\,2'$ $2'\,2$	$2\,2'$ $2'\,t$	$2\,t$ $2'\,2$	$2\,t$ $2'\,t$
$1\,2$	$122'2$ 1	$122't$ 1	$12t$ 0	$12t$ 0
$1\,2'$	$12'22'$ 0	$12't$ 2	$12'2t$ 1	$12't$ 2

Table 4: The corresponding normal form; $0 = \max_{col}\min_{row} < \min_{row}\max_{col} = 1$

In contrast, any two-person zero-sum game with the total cost function defined by (1) has a saddle point. First, it was proven in 1998 by Tijsman and Vreze [51]; see also [50]. An alternative proof that is based on the well-known approach of the so-called discounted approximation was recently suggested in [8, 9].

4.3 Embedding the mean cost games into the total cost ones

Given an infinite play with the sequence of local costs $\mathbf{r} = (r_1, r_2, \dots,)$ let us replace each local cost r_j by the pair $r_j, -r_j$ getting the sequence $\mathbf{r}' = (r_1, -r_1, r_2, -r_2, \dots,)$. The partial total sums for the latter are $r_1, 0, r_2, 0, \dots$. Thus, $2R'_T = R_M$.

Similarly, given an arbitrary mean payoff game $\Gamma = (G, D, v_0, r)$, let subdivide every arc $e \in E$ into e' and e'' and set $r'(i,e') = r(i,e)$, $r'(i,e'') = -r(i,e)$ for all $e \in E$ and $i \in I$. Let us notice that in the obtained game $\Gamma' = (G', D', v_0, r')$ contains only zero-dicycles, $r(i,C) = \sum_{e \in C} r(i,e) = 0$ for all $i \in I$ and C in G'. By the above arguments the total effective cost game Γ' and the mean effective cost game Γ are equivalent. Thus, the mean cost games are embedded into the total cost games that contain only zero-dicycles.

Hence, the former (mean cost) games may be NE-free already for $n = 2$ (see example of Section 3 in Figure 3), we conclude that the two-person total cost game with only zero-dicycles may have no NE in pure stationary strategies. Yet, as we know, NS becomes an open problem if $r(i, C) \neq 0$ (or even if $r(i, C) > 0$) for all $i \in I = \{1, 2\}$ and C in G'.

4.4 Total cost games and the shortest path interdiction problem

The two-person zero-sum total cost games are closely related to the so-called *shortest path interdiction problem* (SPIP) raised by Fulkerson and Harding [22]; see also a short survey by Israely and Wood [34] for more references. The simplest version of SPIP is as follows:

Given a digraph $G = (V, E)$, with weighted arcs $r : E \to \mathbb{R}$, and two vertices $s, t \in V$, eliminate (at most) k arcs of E to maximize the length of a shortest (s, t)-path. This problem is NP-hard; moreover the inapproximability bound $10\sqrt{5} - 21 \approx 1.36$ was derived in [3] (from the same bound for the Minimum Vertex Cover Problem in graphs obtained by Dinur and Safra [16] and improving the previous bound $7/6 \approx 1.17$ given by Håstad [33]).

Unlike the above *total budget* SPIP, the following *node-wise budget* SPIP is more tractable. In this case, we are given a node-wise budget allowing to eliminate (at most) $k(v)$ outgoing arcs from each node $v \in V$.

The case of non-negative weights (local costs) was considered in [38], where an efficient interdiction algorithm was obtained. Given a digraph $G = (V, E)$, a local cost function $r : E \to \mathbb{R}_+$, constraint $k(v)$ in each node $v \in V$, and an initial node s, this algorithm finds in quadratic time an interdiction that maximizes simultaneously the lengths of all shortest paths from s to every node $v \in V$. The execution time is just slightly larger than for the classic Dijkstra shortest path algorithm.

Let us remark that after elimination of the interdicted arcs a dicycle C might be reachable from s. The algorithm of [38] maximizes the total effective cost among all lassos, including the terminal ones, which ends in the artificially added a zero-loop $\ell_v, v \in V_T$.

In case of arbitrary real local costs the SPIP is equivalent (see [38]) with solving the zero-sum mean payoff BW-games. Although the latter problem is known to be in the intersection of NP and co-NP [37], yet, it is not known to be polynomial.

It is also worth noting that the BW mean cost games are a special case of the node-wise interdiction problem corresponding to $k(v) = 0$ for $v \in V_B$ and $k(v) = outdeg(v) - 1$ for $v \in V_W$. Indeed, White, the maximizer, is entitled to choose any move in a position $v \in V_W$ and cannot restrict the choice of Black in any $v \in V_B$.

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References

[1] D. Andersson, V. Gurvich, and T. Hansen, On acyclicity of games with cycles, Discrete Applied Math. 158:10 (2010) 1049-1063.

[2] D. Andersson, K.A. Hansen, P.B. Miltersen, and T.B. Sorensen, Deterministic graphical games revisited, J. Logic and Computation, 2010.

[3] E. Boros, K. Borys, K. Elbassioni, V. Gurvich, G. Rudolf, and J. Zhao, On short paths interdiction problems: Total and node-wise limited interdiction, Theory of Computing Systems 43:2 (2008) 204-233.

[4] E. Boros, K. Elbassioni, Gurvich, and K. Makino, On effectity functions of game forms, Games and Economic Behaviour 68:2 (2010) 512–531.

[5] E. Boros, K. Elbassioni, V. Gurvich, and K. Makino, Every stochastic game with perfect information admits a canonical form, RUTCOR Research Report RRR-09-2009.

[6] E. Boros K. Elbassioni, Gurvich, and K. Makino, A pumping algorithm for ergodic stochastic mean payoff games with perfect information, Lecture Notes in Computer Science, LNCS-6080 (2010) 341-354.

[7] E. Boros, K. Elbassioni, V. Gurvich, and K. Makino, On Nash equilibria and improvement cycles in pure positional strategies for Chess-like and Backgammon-like n-person games, Discrete Math. 312:4 28 (2012) 772–788.

[8] E. Boros, K. Elbassioni, V. Gurvich, and K. Makino, On mean and total effective payoffs for the zero-sum BW games, Manuscript, RUTCOR Research report, Rutgers University, to appear.

[9] E. Boros, K. Elbassioni, V. Gurvich, and K. Makino, On k-total effective payoffs for the zero-sum BW games, Manuscript, RUTCOR Research report, Rutgers University, to appear.

[10] E. Boros and V. Gurvich, On Nash-solvability in pure strategies of finite games with perfect information which may have cycles, RUTCOR Research Report, RRR-60-2001, Rutgers University, Math. Soc. Sciences 46 (2003) 207-241.

[11] E. Boros and V. Gurvich, Why Chess and Backgammon can be solved in pure positional uniformly optimal strategies, RUTCOR Research Report 21-2009, Rutgers University.

[12] E. Boros, V. Gurvich, K. Makino, and Wei Shao, Nash-solvable two-person symmetric cycle game forms, RUTCOR Research Reports, RRR-30-2007 and 20-2009, Rutgers University; Discrete Applied Math. 159 (2011) 1461–1487.

[13] E. Boros, V. Gurvich, S. Solovieva, and E. Yamangil, Chess-like games may have no uniform Nash equilibria even in mixed strategies, RUTCOR Research Report 11-2012, Rutgers University.

[14] E. Boros and R. Rand, Terminal games with three terminals have proper Nash equilibria, RUTCOR Research Report, RRR-22-2009, Rutgers University.

[15] A. Condon, An algorithm for simple stochastic games, in Advances in computational complexity theory, volume 13 of DIMACS series in discrete mathematics and theoretical computer science, 1993.

[16] I. Dinur and S.Safra. On the hardness of approximating minimum vertex cover, Annals of Mathematics 162 (2005) 439–485.

[17] J. Edmonds and D.R. Fulkerson, Bottleneck extrema, J. of Combinatorial Theory, 8 (1970), 299-306.

[18] A. Ehrenfeucht and J. Mycielski, Positional games over a graph, Notices of the American Mathematical Society **20** (1973) A-334, Abstract.

[19] A. Ehrenfeucht and J. Mycielski, Positional strategies for mean payoff games, Int. J. Game Theory 8 (1979) 109-113.

[20] J. Flesch, J. Kuipers, J.G. Shoenmakers, and O.J. Vrieze, Pure subgame-perfect equilibria in free transition games, European J. of Operational Research 199 (2009) 442-447.

[21] J. Flesch, J. Kuipers, G. Shoenmakers, and O.J. Vrieze, Subgame Perfection in Positive Recursive Games with Perfect Information, Mathematics of Operations Research 35:1 (2010) 742–755.

[22] D.R. Fulkerson and G.C. Harding, Maximizing the minimum source-sink path subject to a budget constraint, Mathematical Programming 13 (1977) 116-118.

[23] T. Gallai, Maximum-minimum Sätze über Graphen, Acta Mathematica Academiae Scientiarum Hungaricae 9 (1958) 395–434.

[24] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, 1979.

[25] D. Gillette, Stochastic games with zero stop probabilities; in Contribution to the Theory of Games III, in Annals of Mathematics Studies 39 (1957) 179–187; M. Dresher, A.W. Tucker, and P. Wolfe eds, Princeton University Press.

[26] V. Gurvich, To theory of multi-step games, USSR Comput. Math. and Math. Phys. 13:6 (1973) 143-161.

[27] V. Gurvich, Solution of positional games in pure strategies, USSR Comput. Math. and Math. Phys. 15:2 (1975) 74-87.

[28] V. Gurvich, Equilibrium in pure strategies, Doklady Akad. Nauk SSSR 303:4 (1988) 538-542 (in Russian); English translation in

[29] V. Gurvich, A stochastic game with complete information and without equilibrium situations in pure stationary strategies, Russian Math. Surveys 43:2 (1988) 171–172.

[30] V. Gurvich, A theorem on the existence of equilibrium situations in pure stationary strategies for ergodic extensions of $(2 \times k)$ bimatrix games, Russian Math. Surveys 45:4 (1990) 170–172.

[31] V. Gurvich, Saddle point in pure strategies, Russian Acad. of Sci. Dokl. Math. 42:2 (1990) 497–501.

[32] V. Gurvich, A. Karzanov, and L. Khachiyan, Cyclic games and an algorithm to find minimax cycle means in directed graphs, USSR Computational Mathematics and Mathematical Physics 28:5 (1988) 85-91.

[33] J. Håstad, Some optimal inapproximability results, STOC '97: Proceedings of the 29th annual ACM symposium on Theory of computing, pages 1–10, New York, NY, USA, 1997, ACM Press.

[34] E. Israely and K. Wood, Shortest-path network interdiction, Networks 40(2002) 97-111.

[35] L. Kalmar, Zur Theorie der abstrakten Spiele, Acta Sci. Math. Szeged 4 (1928) 65-85.

[36] R. Karp, A characterization of the minimum cycle mean in a digraph, Discrete Math. **23** (1978), 309-311.

[37] A.V. Karzanov and V.N. Lebedev, Cyclical games with prohibition, Mathematical Programming 60 (1993) 277-293.

[38] L. Khachiyan, V. Gurvich, and J. Zhao. Extending Dijkstra's algorithm to maximize the shortest path by node-wise limited arc interdiction, Computer Science, Theory and Applications: The First Int. Comp. Science Symposium in Russia, CSR, St. Petersburg, Russia, June 8-12, 2006; Lecture Notes in Computer Science 3967 (2006) 221-234.

[39] D. König, Über eine Schlussweise aus dem Endlichen ins Unendliche, Acta Sci. Math. Szeged. 3 (1927) 121-130.

[40] T.M. Liggett and S.A. Lippman, Stochastic Games with Perfect Information and Time Average Payoff, SIAM Review 11 (1969) 604607.

[41] H. Mine and S. Osaki, Markovian decision process, American Elsevier Publishing Co, New York, 1970.

[42] H. Moulin, Prolongement des jeux à deux joueurs de somme nulle, Bull. Soc. Math. France, Memoire **45**, 1976.

[43] H. Moulin, Extension of two person zero-sum games, Journal of Mathematical Analysis and Application **55 (2)** (1976) 490-508.

[44] H. Moulin, The strategy of social choice, Advanced textbooks in economics, vol. 18, North Holland, 1983.

[45] J. Nash, Equilibrium points in n-person games, Proceedings of the National Academy of Sciences 36 (1) (1950) 48-49.

[46] J. Nash, non-Cooperative games, Annals of Mathematics 54:2 (1951) 286–295.

[47] S. D. Patek, Stochastic shortest path games: theory and algorithms, Thesis (Ph. D.) MIT, Dept. of Electrical Engineering and Computer Science, 1997.

[48] S. D. Patek and D. P. Bertsekas, Stochastic shortest path games, SIAM Journal on Control and Optimization, 37:3 (1999) 804-824.

[49] L. S. Shapley, Stochastic games, in Proc. Nat. Acad. Science, USA 39 (1953) 1095–1100.

[50] F. Thuijsman and O. J. Vrieze. The bad match, a total reward stochastic game, Operations Research Spektrum 9 (1987) 93–99.

[51] F. Thuijsman and O. J. Vrieze. Total reward stochastic games and sensitive average reward strategies. Journal of Optimization Theory and Applications 98 (1998) 175–196.

[52] O. J. Vrieze, Stochastic games with finite state and action spaces, PhD thesis, Centrum voor Wiskunde en Informatica, Amsterdam, The Netherlands, 1980.

[53] A.R. Washburn, Deterministic graphical games, J. Math. Analysis and Applications 153 (1990) 8496.

[54] E. Zermelo, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels; in: Proc. 5th Int. Cong. Math. Cambridge 1912, vol. II; Cambridge University Press, Cambridge (1913) 501-504.