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The games SEKI and D-SEKI

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The games SEKI and D-SEKI

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Abstract. Let $A : I \times J \to \mathbb{Z}_+$, be a non-negative integer $m \times n$ matrix each row and column of which contain a strictly positive entry. The game *SEKI* is defined as follows. Two players R and C alternate turns and it is specified who begins.
By one move, a player can either reduce a strictly positive entry of A by 1 (an active move) or pass. The game results in a draw after any two successive passes, one of R and one of C. Player R (respectively, C) wins when a row (respectively, a column) appears each entry of which is 0. However, such zero row and column may appear simultaneously, after a move. In this case, we assume that the player who made this last move is the winner. Yet, we will also study another version of the game, called D-SEKI, in which the above case is defined as a draw.
A matrix A is called a *seki* or a *d-seki* if it is a draw in the corresponding game, does not matter R or C begins. Furthermore, a seki or d-seki is called *complete* if each player *must* pass, that is, if (s)he makes an active move, the opponent wins.
Both games SEKI and D-SEKI are difficult. We present their complete analysis only for the 2×2 matrices, while in general we obtain only some sufficient conditions for a player to win and also some partial results and conjectures mostly related to the complete seki. These results for D-SEKI look simpler but, in return, SEKI, unlike D-SEKI, is closely connected to the so-called shared life in the classical game of GO. Both SEKI and D-SEKI are of independent interest as combinatorial games.

Key words: combinatorial games with positive incentive, draw, pass, GO, shared life, seki, complete seki, integer doubly stochastic matrix.

1 Introduction

The game SEKI was introduced by the first two authors in 1981 [4]. Some computational and theoretical results were summarized in the preprints [1, 5].

Let $A : I \times J \to \mathbb{Z}_+$, be a non-negative integer $m \times n$ matrix each row and column of which contain a strictly positive entry. The game *SEKI* is defined as follows. Two players R and C alternate turns and it is specified who begins; this player is called the *first*, while the opponent the *second*. By one move, a player can either reduce a strictly positive entry of A by 1 (an active move) or pass.

The game ends in a draw when two players pass successively. Player R (respectively, C) wins when a row (respectively, a column) appears, each entry of which is 0. However, such zero row and column may appear simultaneously, after a move. In this case, for the game SEKI we assume that the player who made this last move is the winner. Yet, we will also study another version of the game, D-SEKI, in which the above case is defined as a draw.

A matrix A is called a *seki* or a *d-seki* if it is a draw in the corresponding game, does not matter R or C is the first. Furthermore, a seki or d-seki is called *complete* if each player *must* pass, that is, if (s)he makes an active move, the opponent wins.

Both games SEKI and D-SEKI are difficult. We present their complete analysis only for the 2×2 matrices, while in general we obtain only some sufficient conditions for a player to win and also some partial results and conjectures mostly related to the complete seki.

These results for D-SEKI look simpler but in return SEKI, unlike D-SEKI, is closely connected to the so-called shared life in the classical game of GO. Both SEKI and D-SEKI are of independent interest as combinatorial games.

Remark 1 *Let us note that in fact A is defined only up to permutations of its rows and columns. This, strictly speaking, both games are played with a two-variable non-negative integer-valued function rather than a non-negative integer $m \times n$ matrix.*

The paper is organized as follows. In Section 2 we explain how the game SEKI is related to GO, while in Section 3 we compare D-SEKI and SEKI; in Section 4 we explain how both games are solved by the standard, but slightly modified, backward induction procedure; in Section 5 we show that in both games for each player to be the first is never worse than the second; in Section 6 we provide for D-SEKI and SEKI simple conditions sufficient for the first and, respectively, the second players to win; in Sections 7 we consider $2 \times n$ matrices and find all complete seki and d-seki; in Section 8 we study the direct sums for both games; finally, in Sections 9 and 10 we collect some experimental results and conjectures related to the games SEKI and D-SEKI, respectively.

2 The Game SEKI and the shared life positions in Go

The game SEKI is closely related to the classical game of GO; more precisely, to the so-called shared life in GO. Since it was the main motivation for introducing SEKI in [4], we begin

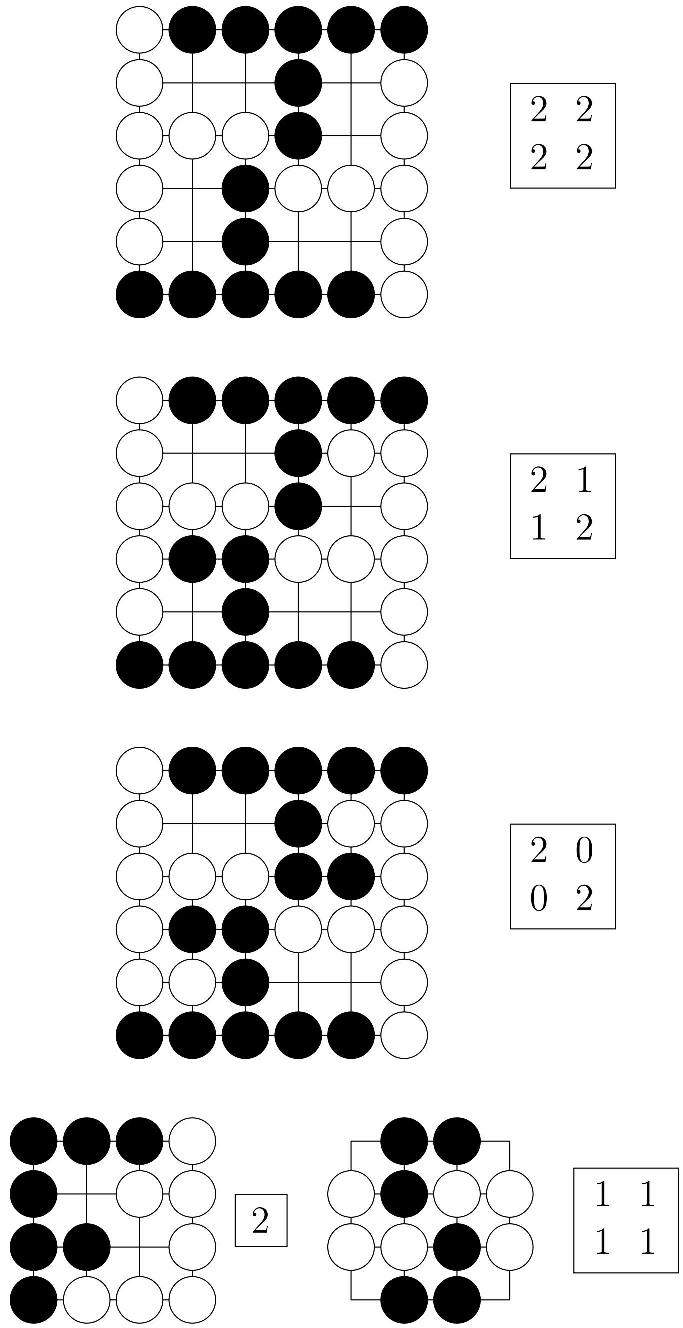
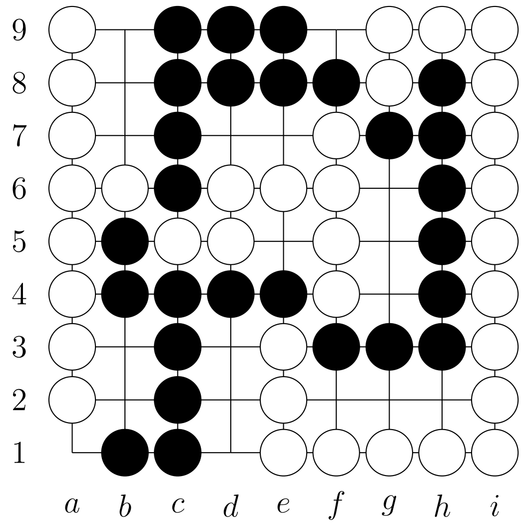


Figure 1: Five standard complete seki positions in GO and the corresponding 1×1 and four 2×2 complete seki matrices.



0	3	3
3	3	1
3	1	2

Figure 2: A complete seki position corrersponding to the matrix A_7^3.

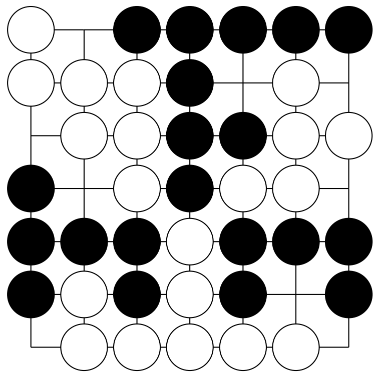
with a short explanation of this correspondence. It is addressed to the readers familiar with (at least the rules of) GO. All other could just skip this section and proceed with the rest of the paper, which explains some mathematical properties of the games SEKI and D-SEKI.

2.1 Non-negative integer matrices and shared life of eyeless groups.

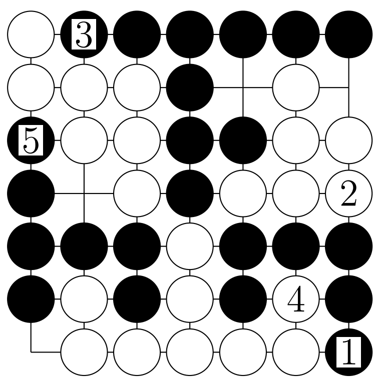
Given a non-negative integer $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, let us consider a position of GO with m white and n black groups that are indexed by I and J, respectively, and let $A(i, j)$ be the number of common free points (so-called liberties or dame) between the white group i and the black group j; see examples in Figures 1 - 4. Two players R and C in SEKI correspond to Black and White in GO. Indeed, R (respectively, C) wants to delete all positive entries of a row (respectively, of a column); accordingly, Black (respectively, White) wants to surround completely a White (respectively, a Black) group.

Remark 2 *In fact, only some special shared life positions are considered. We assume that: (i) The numbers m and n will not changed, that is, no two groups of the same color can ever be united, even if the opponent always will always passes. (ii) None of the $m + n$ involved groups has an eye and, moreover, no eye could be created, even if the opponent always passes. In particular, both (i) and (ii) holds if we require that (iii) every liberty point is adjacent two exactly two groups, one black and one white.*

Figure 1 represents five types of complete seki that are most frequent in GO. It is easy to verify that the opponent can win the game SEKI whenever R or C makes an active move, that is, reduces by one any entry of any of the five given matrices. Respectively, Black and White must pass in each of the five corresponding positions of GO. We will show that there is no other seki of size 1×1 or 2×2. Moreover, there is no seki of size $1 \times n$ for $n > 1$ or $2 \times n$ for $n > 2$ at all. Notice that only the last matrix (with four entries 1) is a d-seki.



0	1	2
1	2	0
2	0	1



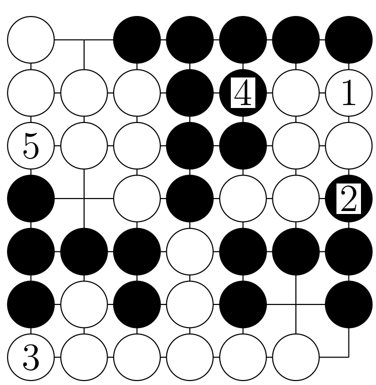
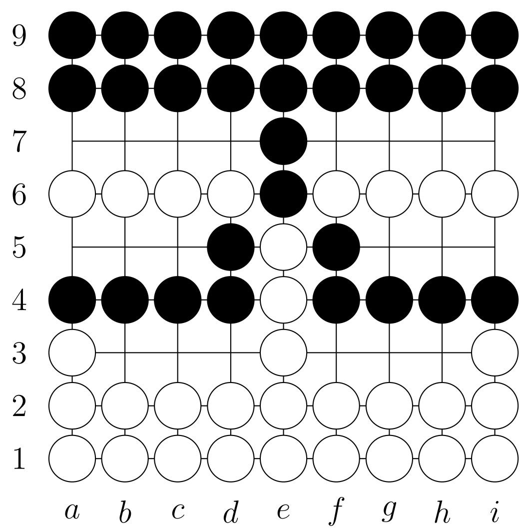
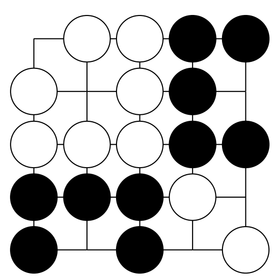


Figure 3: A non-seki position corresponding to a complete seki matrix.



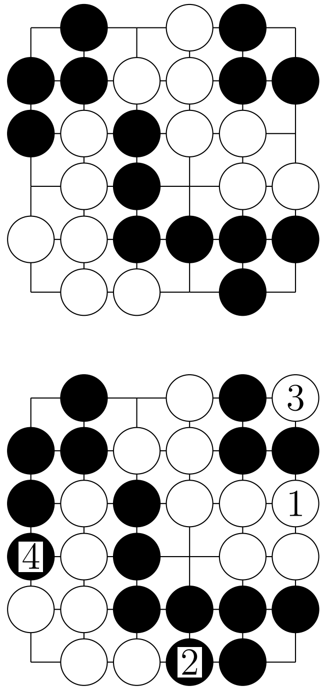
0	3	3
4	0	3
4	3	0

Figure 4: A semi-complete seki position corresponding to A_8^3.



1	1	0
1	1	

Figure 5: Complete seki with eyes.



1	0	1	1
1	1	1	0
1	1	1	

Figure 6: A complete seki position (with eyes) whose matrix is not a seki.

Figure 2 represents a non-trivial complete seki of the size 3×3. Only a long case analysis proves than both Black and White must pass, since the opponent wins after every active move; see Remark 8.

Remark 3 *Figure 3 demonstrates one more limitation of our approach. It is not difficult to verify that the given 3×3 matrix is a complete seki, while the corresponding GO position is not. Indeed, both Black and White can sacrifice a group and take a larger (or of the same size) opponent's group as a compensation. Accurate counting shows that for both players the optimal move is better than pass by two points. It is important to underline the difference: the game SEKI is over when a zero row or column appears but, in contrast, GO is not finished as soon as the first group is taken; moreover, a group might be sacrificed with a profit.*

Figure 4 demonstrate another interesting example of a "semi-complete" seki, in which R must pass but, in contrast, C has moves after which the game SEKI is still a draw. In the corresponding position of GO Black must pass but White can move further getting more points, in accordance with the Chinese rules.

2.2 Which matrices correspond to positions in GO

The standard GO is played on the planar grid 19×19. Let us notice, yet, that it can be naturally generalized for an arbitrary graph.

Let us consider the 3×3 matrix in Figures 2-4. Each of them contains at least one 0. This is not a coincidence. Indeed, the complete bipartite 3×3 graph $K_{3,3}$ is not planar.

Hence, for any position of GO satisfying (i) and (ii) of Remark 2, any 3×3 submatrix of the corresponding matrix must contain at least one 0.

The general answer to the question in the title is as follows. To a matrix A let us assign a $(0, 1)$ matrix A' such that $A'(i, j) = 1$ if $A(i, j) > 0$ and $A'(i, j) = 0$ if $A(i, j) = 0$. Then, each $(0, 1)$ matrix A' is naturally associated with a bipartite graph $B(A')$.

The following properties of a matrix A are obviously equivalent:

- (j) the matrix A is associated with a position of GO in a planar graph;
- (jj) the matrix A is associated with a position of GO in a planar grid;
- (jjj) the bipartite graph $B(A')$ is a planar.

We leave this topological exercise to the reader. Let us only remark that the 19×19 grid may not be sufficient to realize a position corresponding to A even if $B(A')$ is a planar.

The criterion of planarity is given by the famous Kuratowski theorem; see, e.g., [11].

2.3 Extending the game SEKI to groups with eyes

It is possible to extend the above correspondence including into consideration the groups with one eye, as well. To do so, let us extend the matrix A by one special $(0, 1)$ row and one special $(0, 1)$ column, in which an entry 1 indicates that the corresponding Black or White group contains an eye, while a 0 means that it does not. If a zero row or column appears in A, the game is over when the corresponding special entry is 0, but if it is 1 then one extra move is required to eliminate it. Let us remark that it cannot be eliminated before the whole corresponding row or column is.

The simplest complete seki with eyes is given in Figure 5. This corresponding position is sufficiently common in GO. Let us notice that no row of the matrix corresponds the white group with two eyes, since it does not "share life" but lives unconditionally. In this example it plays the same role as the edge of the board. Let us also notice that the other white group, corresponding to the row, consists of two isolated stones.

Another example is given in Figure 6. Here the extended matrix is *not* a seki. Indeed, C begins and eliminates the second column in just two moves. However, the corresponding black group is small and if White captures it then later Black can surround a larger White group in the opposite corner, in return. In fact, the corresponding position in GO *is* a complete seki. Similar examples that contain only eyeless groups also can be constructed; yet, they are of much larger size.

Let us finally remark that one could generalize the game even further allowing lager (than 1) entries in the special row and column. Yet, such a generalization is no longer related to GO, since in GO any group with at least two eyes lives unconditionally and, hence, cannot participate in a shared life; see Figure 5.

The examples in Figures 3 (Remark 3) and 6 show that in GO it might be good for a player to loose a group first, since it may result in a later capturing of a larger opponent's group. Yet, such situations are not that frequent.

2.4 GO and SEKI versus Bridge and Whistette

The relation between GO and SEKI is somewhat similar to the relation between Bridge and the Single-suit two-person game [6]. This game was introduced in 1929 by Emanuel Lasker who called it Whistette. It is played as follows: $2n$ cards with the numbers $1, \dots, 2n$ are shuffled and divided into two hands of n cards each, held by the players R and C. Let us say, C begins. He chooses one of his cards and put in on the table. Then R, after seeing this card, selects one of her own. The player with the higher card wins the trick and gets the lead. Two cards of the trick are removed and play continues until there are no more cards. The goal of each player is to win as many tricks as possible.

The game is not fully trivial already for $n = 3$. For example, let C and R get $6, 4, 2$ and $5, 3, 1$ (or in the terms of Bridge, A,Q,10 and K,J,9), respectively. If C leads with 6 then he will win only the first trick, yet, if C leads with 2 or 4 then he will obtain two tricks. Similarly, let C and R get $5, 4, 2$ and $6, 3, 1$ (that is, K,Q,10 and A,J,9). If C leads with 5 or 4 and R wins with 6 then she gets no more tricks, yet, if R discards 1 in the first trick then she wins the remaining two tricks. For larger n, say $n = 20$, Whistette becomes quite complicated. Many interesting results are obtained in [6]. Recently, Johan Wästlund [12] got a polynomial algorithm solving Whistette.

In some situations, Bridge is reduced to Whistette. Let us consider, for example, the six-card end-play in which North and West have $3, 6, 10, J, Q, A$ and $4, 5, 7, 8, 9, K$, respectively, one of them is on lead, and there are no trumps left. However, such situations are very rare. For example, a positions with $n > 6$ cannot appear at all, since there are only 13 cards in each suit. Thus, it is not necessary for a Bridge expert to be good at Whistette.

Similarly, the games GO and SEKI require very different skills. Shared life positions that involve three black and three white groups are extremely rare in practice.

Unlike Bridge and GO, Whistette and SEKI are pretty boring games, yet, they reveal deeper and nicer mathematical properties and they are very complicated too. At least, the corresponding positions would be difficult to analyze even for advanced Bridge and GO players; see, for examples, Figures 3 and 4.

3 Simple relations between SEKI and D-SEKI

Let us begin with a 1×1 matrix A. Every such matrix is a not complete d-seki. Indeed, in D-SEKI two players, R and C, alternating will reduce the unique entry to 0 in $A(1, 1)$ moves. Thus, D-SEKI results in a draw but no player must ever pass.

Now, let us consider SEKI. If $A(1, 1) = 1$ then the first player wins immediately. If $A(1, 1) = 2$ then A is a complete seki, since both players, R and C, must pass in this case. Finally, if $A(1, 1) \geq 3$ then A is a not complete seki.

Proposition 1 *Let S, DS, CS, and CDS stand for the sets of matrices that are seki, d-seki, complete seki, and complete d-seki, respectively. Then the following strict containments hold:*

$$CDS \subset CS \subset S \subset DS.$$

Proof. A complete seki is, in particular, a seki, by the definition.

Let us also recall that if a zero row and column appear simultaneously, after a move, then the player who made this move wins in SEKI, while the result of D-SEKI is defined as a draw. Hence, given a matrix, it is more difficult to win in D-SEKI than in SEKI. This observation implies both the right and left containments.

Already 1×1 matrices suffice to demonstrate that all three containments are strict. Indeed, if $A(1,1) = 1$ then it is a d-seki but not a seki; if $A(1,1) = 2$ then it is a complete seki but not a complete d-seki; finally, if $A(1,1) = 3$ then it is a not complete seki. □

Proposition 2 *For each $m \times n$ seki A there is a $m \times n$ complete seki A' such that $A' \leq A$, where $\leq$ means the entry-wise non-strict inequalities. Yet, for d-seki the similar claim fails.*

Proof. Indeed, by definition, both games SEKI and D-SEKI result in a draw after two consecutive passes, of R and C. The inverse also holds for SEKI, but not for D-SEKI.

Obviously, if A is a seki then a complete seki A' will result from A in several moves, provided both players act optimally.

Indeed, if any active move of any player allows the opponent to win then $A = A'$ is already a complete seki. Suppose that there is an active move, say, of R reducing A to A_1 in which C, as the first player cannot win. Yet, of course, C can make a draw, since otherwise R wins in A. Let us choose any optimal move of C in A_1 (it may be the pass) getting A_2, etc., until we get a matrix A' such that each player loses after each active move. Then A' is a complete seki, by definition. It is also clear that A and A' are of the same size and $A' \leq A$.

In contrast, any strictly positive integer 1×1 matrix is a d-seki but none of them is complete. The above arguments fail, since D-SEKI may terminate in a matrix with one zero row and column, which is not a complete seki, by convention. □

Proposition 3 *An arbitrary matrix A is a seki, d-seki, complete seki, or complete d-seki whenever the transposed transposed matrix A^T is.*

Proof. Indeed, the game just remains the same when the matrix is transposed and the players R and C are swapped. More accurately, R, as the first or second player, can guarantee some result, winning or a draw, in A if and only if C, as the first or second player, can guarantee the same in A^T, and vice versa. □

4 Solving SEKI and D-SEKI by backward induction

It is well known that any finite acyclic game can be standardly solved by backward induction [7, 8, 3]. Although, strictly speaking, SEKI and D-SEKI are not acyclic, since any pass is a loop, yet, a slightly modified backward induction procedure is applicable to both games too.

Let us generate successively all matrices, up to permutations of their rows and columns, increasing the sum of all entries one by one.

Remark 4 *The isomorphism-free exhaustive generating (without checking the isomorphism) is a difficult problem, in general [9], and in particular, no simple efficient procedure for the required matrix generation is known. Yet, for our purposes we allow some, but not too frequent, repetitions. The corresponding computer code was written for SEKI by Konrad Borys and Gabor Rudolf in 2005 and then a more efficient code, working for D-SEKI as well, was written by Diogo Andrade in 2006. In what follows we give several non-trivial examples of complete seki and d-seki; all of them were obtained by this method.*

For every matrix A we have to determine what will happen in two cases, when R and C begins. Thus, we assign to A one of the nine pairs (X,Y) = (X(A), Y(A)), where $X, Y \in \{W, L, D\}$ stand for "win", "lose", and "draw", respectively. Each symbol shows the result for the first player. For example, (W, W) means the first player, R or C, always wins, while (W,L) means that R always wins, as the first or second player.

We ignore the matrices that contain at least two zero rows or zero columns. To initialize the procedure, we assign (W,L) to any matrix A that contains one zero row but no zero column and, respectively, (L,W) if A contains one zero column but no zero row. Furthermore, if A contains one zero row i and one zero column j then we assign to it (D,D) in the case of D-SEKI, while for SEKI the corresponding pair (X,Y) remains undefined yet. If such a matrix A results from another matrix B after one move then the player who made it wins SEKI, in accordance with its rules. (Let us remark that there is a unique such B and it differs from A only in one entry: $B(i, j) = 1$, while $A(i, j) = 0$.)

In general, $X(B)$ and $Y(B)$ are defined recursively as follows. Let $\mathcal{A}(B)$ denote the set of all matrices resulting from B by one move. Let R begin. Then, $X(B) = W$ whenever $Y(A) = L$ for some $A \in \mathcal{A}(B)$. Furthermore, $X(B) = D$ if $Y(A) = L$ for no $A \in \mathcal{A}(B)$ but $Y(A) = D$ for some $A \in \mathcal{A}(B)$.

When C begins, we obtain the similar properties by just swapping X and Y. Let us also notice that X and Y must swap in the above implications too, since R ad C alternate turns.

Yet, the remaining case, when $Y(A) = W$ for all $A \in \mathcal{A}(B)$ is slightly more difficult. Then, C wins after any active move of R, yet, R may still try to pass. In this case we have: $X(B) = L$ whenever $Y(B) = W$ and $X(B) = D$ whenever $Y(B) = D$. Similarly, $Y(B) = L$ whenever $X(B) = W$ and $Y(B) = D$ whenever $X(B) = D$.

Finally, only one case remains: $X(A) = Y(A) = W$ for all $A \in \mathcal{A}(B)$. In other words, after any active move of R or C, the opponent wins. By the definition, this is exactly the complete (d-) seki. Hence, in this case we set $X(B) = Y(B) = D$.

Remark 5 *Let us notice that neither SEKI nor D-SEKI will change if we allow it to proceed after two consecutive passes. In other words, both games can be solved in pure stationary strategies. Formally, this follows from the above recursion but it is obvious too. Indeed, let us assume that a player had passed his turn several times but then switched to an active strategy to guarantee some result, D or W. Of course, the same result (s)he could guarantee applying the same strategy from the beginning, instead of passing.*

5 The importance of being first or strategy stealing

Let us show that at both games, SEKI and D-SEKI, for R or C, to be the first player is never worse than to be the second one.

Proposition 4 *If R or C can win (make a draw) as the second player then (s)he can do the same as the first player. In particular, the second player cannot win at a symmetric matrix. Also the pairs (X, Y) cannot take values (D, L), (L,D), and (L, L).*

Proof. By definition, in both games, SEKI and D-SEKI, the players are allowed to pass their turns. Hence, if R or C wins (makes a draw) as the second player, then as the first player (s)he can just pass and apply the same strategy. It is important to notice that Remark 5 is essential for the above argument.

The last claim immediately follows. For example, (L, L) and (L,D) mean that R loses as the first player but wins, respectively, makes a draw as the second player. Let us remark that the remaining six values are possible [1].

Finally, let us assume for the sake of a contradiction that the second player wins in a symmetric matrix. Then, the first player can just steal the winning strategy after passing in the first round. Indeed, becoming second, (s)he can apply the same (or, more accurately, the transposed) winning strategy. Since the matrix is symmetric, the players, R and C, become symmetric, too. Thus, we obtain a contradiction. □

The games of this type were considered in 1953 by John Milnor [10] who called them the *games with positive incentive*. This class contains SEKI, D-SEKI, and GO.

6 Conditions sufficient to win in SEKI and D-SEKI

Although it is difficult to characterize the winning positions of the games SEKI and D-SEKI, yet, the following simple sufficient conditions and their corollaries will be very instrumental.

6.1 A few definitions

Given a (non-negative integer) $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, its *height* $h = h(A)$ is defined as the maximum of its entries. The matrices of the height 1 and 2 are called the $(0, 1)$- and $(0, 1, 2)$-matrices. Furthermore, let s_i^r and s_j^c denote the sum of all entries of the row $i \in I$ and column $j \in J$, respectively. Then, A is called an *integer doubly stochastic matrix (IDSM)* with the sum $s = s(A)$ if the sum of the entries in every row and every column of A is a constant, that is, $s_i^r = s_j^c = s$ for all $i \in I$ and $j \in J$.

Obviously, any IDSM is a square matrix, $m = |I| = |J| = n$, whenever $s > 0$.

6.2 When the first player wins in D-SEKI

Theorem 1 *As the first player, R wins at D-SEKI whenever the following condition holds:*

- *S1: there is a row $i \in I$ such that $s^c_j - s^r_i \geq A(i,j)$ for every column $j \in J$.*

Proof. Let R reduce any entry of the row i and C answer arbitrarily. It is easy to check that these two moves maintain the condition S1, which will be satisfied by the same row i. In particular, $s^r_i \leq s^c_j$ always holds for any column j. Moreover, S1 implies that the row i and any column j cannot become zero simultaneously. Hence, applying the above simple strategy R guarantees that the row i will become zero before any column. □

Of course, similar statements holds for player C.

Let us also note that, in general, S1 is only sufficient but not necessary. Yet, in Section7 we will show that, in case of the $2 \times n$ matrices, S1 becomes necessary and sufficient. This will allow us to give the complete analysis of the 2×2 case.

Corollary 1 *Any $(0,1)$ IDSM A with $s(A) \geq 2$ is a complete d-seki.*

Proof. This statement follows immediately from Theorem 1. Indeed, let C (respectively, R) reduce an entry $A(i,j)$. Then the opponent wins, since the condition of Theorem 1 holds.

It is also easy to prove this claim from scratch. Indeed, R (respectively, C) easily wins, reducing the entries of the same row i (respectively, the column j) in an arbitrary order. □

Let us note, yet, that the condition $s(A) \geq 2$ is essential. Indeed, if $s(A) = 1$ then the first player makes a draw in one move. Hence, in this case A is a not complete d-seki.

6.3 When the second player wins in SEKI

Similar, just a little bit more sophisticated, statements holds for the game SEKI.

Theorem 2 *As the second player, R wins in SEKI if the following condition holds:*

- *S2: there is a row $i \in I$ such that $s^c_j - s^r_i \geq A(i,j)$ for every column $j \in J$ and this inequality is strict whenever $A(i,j) = 0$.*

Proof. Let C reduce an arbitrary entry (i',j'). Then R should reduce an entry (i,j) of the row i avoiding the column j' (that is, $j \neq j'$), whenever possible. A simple case analysis shows that the above two moves always maintain the condition S2.

Case 1: possible, $j \neq j'$. Then, s^r_i is reduced by at least 1, while s^c_j and $s^c_{j'}$ by exactly 1. The some of any other column is not changed. Thus, S2 holds for A'. Let us notice only that even when $A(i,j') = 1$ was reduced to $A'(i,j') = 0$, still in the reduced matrix A', the sum in the column j' is strictly greater than the sum in the row i. After such a move of R the row i can become zero, in contrast, the column j' (or any other column) cannot.

Case 2: not possible, $j = j'$. It means that all entries of the row i are zeros, except for $A(i,j)$. Then, $s^c_j \geq 2A(i,j)$, by S2. Although s^c_j is reduced by 2, while s^r_j only by one, S2 still holds, since $A(i,j)$ is also reduced by 1. The only exception is when $A(i,j) = 1$ becomes

$A'(i,j) = 0$. In this (and only in this) case both sums, of the row i and column j, become zeros simultaneously, thus, failing S2. Yet, by the rules of SEKI, R wins in this case, too.

It remains to note, that, by S2, no column j may become zero before the row i. Moreover, the analysis of Case 2 shows that these two events may happen simultaneously, only after a move of R, in which case (s)he still wins, in accordance with the rules of SEKI. □

Of course, similar statements holds for player C.

Again, in general, S2 is only sufficient but not necessary. Yet, in the next section we will show that, in case of the $2 \times n$ matrices, S2 becomes necessary and sufficient. This will allow us to prove Proposition 6 and give the complete analysis of the 2×2 case.

Corollary 2 *Any* $(0, 1, 2)$ *IDSM* A *with* $s(A) \geq 2$ *is a complete seki.*

Proof. By Remark 1, two cases, when R or C begin, are symmetric and, hence, equivalent. So let us assume, without loss of generality, that C begins by reducing an entry (i,j). Then, player R can reduce another entry (i,j') of the same row i (distinct from (i,j)) unless $A(i,j) = s(A) = 2$. By these two moves, the sum s_i^r is reduced by 2: the entries $A(i,j')$ and $A(i,j'')$ by 1 each, while all other entries of the row i were not changed. It is easy to check that the condition S2 holds for the obtained matrix. Now, R is the second player again and, hence, by Theorem 2, (s)he wins.

Finally, it remains to notice that if $A(i,j) = s(A) = 2$ then, after C reduced $A(i,j) = 2$ to $A'(i,j) = 1$, R still wins in one move just reducing further $A'(i,j) = 1$ to $A''(i,j) = 0$. □

Let us remark that all conditions of the corollary are essential. Indeed, if $s(A) = 1$ then the first player wins in one move. Also, if C reduces an entry $A(i,j) \geq 3$ then the conditions S2 may fail for the obtained matrix, since the reduced entry $A'(i,j)$ may still be 2.

Condition S2 of Theorem 2 means that R can take a burden of deleting $a_{i,j}$ and still win.

In Section 7.4 we will show that for $2 \times n$ matrices S2 is not only necessary but also sufficient for R to win, as the second player. As a corollary, we derive that there is no $2 \times n$ seki for $n > 2$. Furthermore, a 2×2 matrix is a seki if and only if S2 fails for both players, R and C. Furthermore, from this observation we will derive that there are only four 2×2 complete seki. A_ℓ^2, $\ell = 1, 2, 3, 4$, given above; see Proposition 8 of Section 7.6.

In, fact these results can be viewed as a solution of the $m \times n$ case for $m \leq 2$.

The $1 \times n$ case is trivial: R wins whenever $n > 1$; if $1 = n = A(1,1)$ then the first player wins in one move and we obtain a seki if $1 = n < A(1,1)$.

Let us also notice that the extra strictness requirement in S2 is essential. Indeed, if $a_{i,j} = 0$ and $s_i^r = s_j^c$ then not R but C wins. Indeed, then C can begin and delete the column j before R could eliminate the row i, or any other row.

It would be good to hove some effective sufficient conditions for R to win, as the first player, too. Let us try to replace S2 by the weaker inequality $s_j^c - s_i^r \geq a_{i,j} - 1$. Yet, the obtained condition fails to be sufficient, as, for example, the following three matrices show:

$$\begin{matrix} 1\,1\,1 \\ 1\,1\,1 \\ 1\,1\,1 \end{matrix} \qquad \begin{matrix} 1\,1\,1 \\ 1\,2\,0 \\ 1\,0\,2 \end{matrix} \qquad \begin{matrix} 1\,1 \\ 1\,1 \\ \end{matrix}$$

For each of them the inequality $s^c_j - s^r_i \geq a_{i,j} - 1$ holds for the first row, $i = 1$, and every column j. Moreover, all entries of the first row are strictly positive. Yet, these three matrices are complete seki, by Corollary 2. Hence, R cannot win even if (s)he begins.

Of course, condition S2, which is sufficient for R to win, as the second player, is also sufficient for R to win, as the first player. However, it is too strong and is not necessary already for the 2×2 matrices. In fact, the best what we can suggest is a triviality:

Corollary 3 *As the first player, R wins whenever (s)he can reduce the original matrix A to a matrix A' satisfying S2 (which is sufficient for R to win in A', as the second player).* □

Somewhat surprisingly, this statement is efficient enough. For example, let us consider

$$\begin{matrix} 1 & k \\ k & k \end{matrix}$$

This matrix is a complete seki if $k = 1$. Yet, if $k > 1$ then R begins and wins by reducing $A(1, 2) = k$ by 1. Indeed, the first row of the obtained matrix A'

$$\begin{matrix} 1 & k-1 \\ k & k \end{matrix}$$

satisfies S2, since $(k+1)-(1+(k-1)) = 1 > 0$ and $(k+(k-1))-(1+(k-1)) = k-1 > 0$. Hence, R wins, by eliminating the first row, even if C begins. More generally, if $0 < \ell < k$ then S2 holds for the matrix

$$\begin{matrix} \ell & k-\ell \\ k & k \end{matrix}$$

7 On 2×2 ($1 \times n$ and $2 \times n$) matrices

7.1 On $1 \times n$ matrices

The case $n = 1$ was already studied: there is no complete d-seki; only $A(1, 1) = 2$ is a complete seki; every $A(1, 1)$ but 1 is a seki, and each $A(1, 1)$ is a d-seki.

Proposition 5 *There is no d-seki (and hence, no seki either) of size $1 \times n$ with $n > 1$.*

Proof. Obviously, in this case, player C wins in D-SEKI (hence, in SEKI too). Actually, any active (non-passing) strategy of C is winning. Indeed, when the unique row becomes zero, all columns are zero too. Hence, one of them became zero earlier. □

7.2 Complete 2×2 seki and d-seki

It is easy to verify that the following four 2×2 matrices are complete seki:

A_1^2	A_2^2	A_3^2	A_4^2
1 1	2 0	2 1	2 2
1 1	0 2	1 2	2 2

The corresponding GO positions are shown in Figure 1.

Let us also notice that among these four, only the first one is a complete d-seki.

Proposition 6 *Except for the listed above, there is no other $1 \times n$ or $2 \times n$ complete seki or d-seki. In particular, there is no $m \times n$ seki with $m \le 2$ and $m < n$.*

The case $m = 1$ follows from Proposition 5. The rest will be proven in Section 7.6.

For example, it is easy to check that each of the following matrices is a not complete seki:

2 1	2 1	3 2	3 2	3 1	3 0
1 1	1 3	2 2	2 3	1 3	1 3

Let us finally notice that there exist many $2 \times n$ d-seki with $n > 2$. For example, neither R nor C wins in D-SEKI, even as the first player, in the following 2×3 matrix.

$$\begin{matrix} 1 & 1 & 1 \\ 2 & 3 & 3 \end{matrix}$$

Let us also notice that S1 does not hold, neither for R nor for C. Indeed, $s_1^r = s_1^c = 3$, while $A(1,1) = 1 > 0$; furthermore, $s_2^r = 7$ and $s_2^c = s_3^c = 4$.

7.3 Simple corollaries of conditions S1 and S2 for $m = 2$

Obviously, for each matrix $A : I \times J \to \mathbb{Z}_+$ we have

$$\sum_{i \in I} s_i^r = \sum_{j \in J} s_j^c = \sum_{i \in I, j \in J} A(i,j). \tag{1}$$

Condition S1 claims that there is a $i \in I$ such that $s_j^c \ge s_i^r + A(i,j)$ for each $j \in J$.

Condition S2 is stronger than S1, it requires additionally that $s_j^c > s_i^r$ whenever $A(i,j) = 0$.

In particular, if $m < n$ then S1 implies that there are $i \in I$ and $j \in J$ such that $s_i^r > s_j^c$.

In case $m = 2$ we derive a much stronger corollary. If row 1 satisfies S1 then $A(2,j) \ge s_1^r$ for each $j \in J$ and, hence,

$$s_2^r = \sum_{j=1}^{n} A(2,j) \ge n s_1^r \tag{2}$$

In other words, for $m = 2$, condition S1 implies that the sum of one row is at least n times the sum of the other. Of course, this follow from S2 too, since S2 $\Rightarrow$ S1.

7.4 For $m = 2$, both S1 and S2 become necessary and sufficient

Let us show that for $m = 2$ the conditions S1 and S2 are not only sufficient but also necessary for R to win, as the first and, respectively, as the second player.

Proposition 7 *Let A be $m \times n$ matrix with $2 = m \leq n$.*

Let C begin in SEKI. Then, R wins if and only if S2 holds. Moreover, A can be a seki only if $m = n = 2$. If $2 = m < n$ then R wins when S2 holds and C wins if it does not.

Furthermore, let R begin in D-SEKI. Then, R wins if and only if S1 holds. Yet, in this case A may be a d-seki for $2 = m = n$ and for $2 = m < n$ too.

Proof. By Theorem 1, condition S2 is always (for any m) sufficient for R to win in SEKI, as the second player. To prove that it is also necessary for $2 = m \leq n$, we assume that S2 fails and will show that C can always maintain this situation, that is, C has an active move such that, for any response of R, S2 still does not hold for the obtained matrix. Then, either A is a seki or C wins. Indeed, obviously, S2 holds for a matrix that contains a zero row but no zero column. Hence, by failing S2, player C guarantees that no zero row appears before a zero column.

Without loss of generality, let us assume that $s^r_1 \leq s^r_2$. Then S2 fails (for the row 1, i. e., there is a column $j \in J$ such that $s^c_j - s^r_1 < A(1, j)$) unless $s^r_2 \geq ns^r_1$.

The problem that C has to resolve is not too difficult; for example, the following simple strategy works: whenever possible, C reduces $A(2, j)$. It is easy to check that, after any move of R, S2 still fails for the obtained matrix.

If $A(2, j) = 0$ then C can reduce $A(1, j)$. If $A(1, j) = 1$ then (s)he wins in one move. If $A(1, j) \geq 2$ then either C eliminates column j, before R eliminates the rest of row 1, and C wins, or, if R finishes with the rest of row 1 first, then the matrix is decomposed into the direct sum of a 1×1 matrix $(1, j)$, with $A(1, j) \geq 2$ and the corresponding $1 \times (n-1)$ matrix. There are two subcases. If $n > 2$ then C wins in every $1 \times (n - 1)$ matrix and, hence, in the original matrix, too. Yet, if $n = 2$ then both matrices may be seki.

The second part is proven in the similar way. By Theorem 2, condition S1 is always (for any m) sufficient for R to win in D-SEKI, as the first player. To prove that it is also necessary for $2 = m \leq n$, we assume that S1 fails. A simple case analysis shows that for any move of R, there is a response of C that maintains this situation, that is, S1 still fails in the matrix obtained after the above two moves. Then, either A is a seki or C wins. Indeed, obviously, S1 holds for a matrix that contains a zero row but no zero column. Hence, by failing S1, player C guarantees that no zero row appears before a zero column.

However, the 2×3 d-seki given above shows that, unlike seki, d-seki may exist for for a non-square $m \times n$ matrices, even for $m = 2$. □

7.5 The standard form of a 2×2 matrix

Given a 2×2 matrix $A : I \times J \to \mathbb{Z}_+$, where $I = J = \{1, 2\}$, without any loss of generality we will assume that

$$s_1^r \le s_2^r, \ \ s_1^c \le s_2^c, \ \ s_1^r \le s_1^c \tag{3}$$

In terms of the entries of A, we can rewrite this system of inequalities as follows:

$$A(2,2) - A(1,1) \ge A(2,1) - A(1,2) \ge 0. \tag{4}$$

7.6 All complete seki and d-seki of size 2×2

By definition, the first player cannot win in a complete seki; moreover, if (s)he does not pass then the opponent wins. By Proposition 7, for 2×2 matrices condition S2 is necessary and sufficient for the second player to win. This provides a characterization of the 2×2 complete seki from which we will derive a much simpler one.

Proposition 8 *There exist only four 2×2 complete seki $A^2_\ell, \ell = 1, 2, 3, 4$, given above.*

Proof. Let A be a 2×2 complete seki. Without any loss of generality, we can assume that (4) holds for A and that C begins. Then, R can enforce S2 after every move of C; in particular, after one at $(2, 2)$. There are two cases: S2 will hold for the row 1 or 2.

Case 1: S2 holds for row 1. Then, if R begins, (s)he can still enforce S2 for the row 1 by the same move and win. Indeed, S2 for row 1 is obviously respected by any increase of the entries of row 2. Hence, A is not a complete seki and we get a contradiction.

Case 2: S2 holds for row 2. By two moves the row 2 could be reduced by at most 2.

Hence, by (2), $s_1^r \ge 2(s_2^r - 2)$ and by (3), $s_2^r \ge s_1^r$. Thus, $s_2^r \ge 2s_2^r - 4$, that is, $s_2^r \le 4$. Summarizing, we obtain that $s_1^r \le s_2^r \le 4$. There are just a few matrices satisfying these inequalities and it is easy to verify that, except for the four matrices A^2_ℓ, $\ell = 1, 2, 3, 4$ of Section 7, there is no other complete seki among them. □

Corollary 4 *The matrix A^2_1 is a unique complete d-seki among all $1 \times n$ and $2 \times n$ matrices.*

Proof. Let us recall that each complete d-seki is a complete seki, by Proposition 1, and A^2_ℓ is a complete d-seki only for $\ell = 1$. □

7.7 Complete analysis of SEKI and D-SEKI for 2×2 matrices

The above proposition immediately implies the following complete analysis of both games, SEKI and D-SEKI, in case of the 2×2 matrices.

Corollary 5 *Let A be an integer 2×2 matrix that has no zero row or zero column.*

Player R wins in SEKI if and only if (s)he has a move that enforces S2 for the obtained matrix A'. The similar (transposed) if and only if condition holds for C. The game is a draw *(seki) if and only if none of the above two conditions holds. The seki is complete only for the four matrices A^2_ℓ, $\ell = 1, 2, 3, 4$ of Section 7.*

Similarly, player R wins in D-SEKI if and only if S1 holds for A. The similar (transposed) if and only if condition holds for C. The game is a draw (d-seki) if and only if none of the above two conditions holds. The d-seki is complete for only one matrix A^2_1. □

Remark 6 *The above analysis is based on simple criteria S1 and S2. It can be represented even more explicitly in terms of of disjunctive linear programming, that is, each case takes place if and only if at least one of a given family of systems of linear inequalities holds. For SEKI this was done in Section 6 of [1]. Although there are only four variables, the entries of A, but still the obtained disjunctive linear systems do not look elegant and we will not reproduce them here.*

8 Prime seki, complete seki, and complete d-seki

Given several non-negative integer matrices $A_\ell : I_\ell \times J_\ell \to \mathbb{Z}_+$, where $\ell \in [k] = \{1, \ldots, k\}$, their direct sum $A = \oplus_{\ell \in [k]} A_\ell$ is standardly defined as follows: $A : I \times J \to \mathbb{Z}_+$, where $I = \cup_{\ell \in [k]} I_\ell$, $J = \cup_{\ell \in [k]} J_\ell$, and all $2k$ sets I_ℓ, J_ℓ, $\ell \in [k]$ are assumed pairwise disjoint. Then, $A(i, j) = A(i_\ell, j_\ell)$ if $i \in I_\ell, j \in J_\ell$ for some $\ell \in [k]$ and $A(i, j) = 0$ otherwise.

Proposition 9 *If $A = \oplus_{\ell \in [k]} A_\ell$ then A is seki, complete seki, or complete d-seki if and only if A_ℓ have the corresponding property for all $\ell \in [k]$. In particular, matrix A is a draw in the game SEKI if and only if A_ℓ is a draw for all $\ell \in [k]$. The "if" (but not "only if") part holds for D-SEKI, as well.*

Proof. Suppose, in SEKI or D-SEKI a player, R or C, does not win at A_ℓ for all $\ell \in [k]$, even being first. Then, (s)he cannot win at A either. Indeed, the opponent guarantees at least a draw always replying optimally in the same A_ℓ, where the previous move was made.

For SEKI, the inverse also holds. If a player, say R, wins in A_{ℓ_0} then A cannot be a draw. Indeed, if R is the second then (s)he always responds optimally in the same subgame A_ℓ, where the previous move was made by C. If R is the first then (s)he starts optimally in A_{ℓ_0} and after this applies the same strategy. Obviously, R, wins whenever A_{ℓ_0} is finished. Otherwise, if some other subgame A_ℓ is finished before A_{ℓ_0} then either R or C wins, but A_ℓ cannot result in a draw. Indeed, a draw in SEKI means that both players pass their turns. Yet, R will always proceed in A_{ℓ_0} rather than pass.

However, the last argument of the proof does not work for the game D-SEKI. For example, if A_ℓ is a 1×1 matrix with an entry 1 then the first player can force a draw in one move. For this reason, Proposition 9 cannot be extended to the case of d-seki.

Yet, the claim still holds for the complete d-seki, defined it terms of winning rather than a draw. A matrix is a complete d-seki if after any active move of a player the opponent wins. By the above arguments, A has this property if and only if A_ℓ have it for all $\ell \in [k]$. □

A seki, complete seki, or complete d-seki is called *prime* if it is not a non-trivial ($k > 1$) direct sum. Obviously, by this definition, each seki, complete seki, or complete d-seki is the direct sum of prime ones.

Remark 7 *In the endplay, GO is typically a sum of several independent games; see [10]. The above proposition can be interpreted in these terms; it shows that the sum of several independent seki is a seki; see for example the third diagram in Figure 1.*

9 Computations and conjectures related to SEKI

9.1 3×3 complete seki

The computer analysis shows that there are seven prime 3×3 complete seki of height 3:

A_1^3	A_2^3	A_3^3	A_4^3	A_5^3	A_6^3	A_7^3
033	133	301	320	320	320	033
303	313	022	212	203	213	331
330	331	121	023	032	033	312

Moreover, it also shows that there are no other 3×3 prime complete seki of height ≤ 10.

Remark 8 *Even without a computer, one can check that each of these seven matrices is a complete seki. owever, such verification requires a pretty complicated case analysis. We leave it to the careful reader, but to help a bit sketch it for A_3^7. This is the joint analysis with Thomas Wolf. All cases are simple but one: player C reduces $A(2,3)$ from 1 to 0. Then, the "natural" answer of R, reducing $A(1,2)$ from 3 to 2, fails. In response C must pass in this case. The optimal R's second move is reducing $A(3,1)$ from 3 to 2, instead of reducing $A(1,2)$. Further analysis is as follows: if C reduces $A(2,1)$ or $A(1,3)$ then R answers with reducing $A(2,2)$ and $A(1,2)$ respectively. These two variants are symmetric.*

Let us notice that all seven matrices are symmetric, yet, the last two are not IDSMs. Furthermore, A_6^3 (respectively, A_7^3) results from (respectively, in) an IDSM, by one move.

Hence, there is a symmetric IDSM (for example, one obtained from A_7^3 by reducing its central entry by 1) in which the first player, R or C, wins, since A_7^3 is a complete seki.

In contrast, by Proposition 4, the second player cannot win at a symmetric matrix.

9.2 Several conjectures about complete seki

Let $H(n)$ denote the maximum height of a $n \times n$ complete seki. As we already know, $H(1) = H(2) = 2$, while $H(3) \geq 3$. In Section 3.3.2 of [5], it is shown that $H(n) \geq 3$ for all $n \geq 3$; see also Section 9.4 below. As we already mentioned, computations show that either $H(3) = 3$ or $H(3) > 10$; of course, the latter is unlikely.

Conjecture 1 *Function $H(n)$ is well defined (that is, $H(n) < \infty$) for all integer $n \geq 1$; furthermore, it is monotone non-decreasing and unbounded.*

Yet, all complete seki that we know are of height at most 3. Also, they are all quadratic.

Conjecture 2 *Each seki (and in particular, each complete seki) is a square matrix, that is, $m = |I| = |J| = n$; in other words, R or C wins in the game SEKI whenever $m \neq n$.*

In the case $\min(m, n) \leq 2$, Conjecture 2 follows from Propositions 5 and 6.

There exists no $m \times n$ seki when $m \leq 2$ and $m < n$. However, it is not known whether a 3×4 seki (or complete seki, or complete d-seki) exists. Let us also recall that

(i) there are non-square (for example, 2×3) d-seki;

(ii) every seki can be reduced to a complete one of the same size;

(iii) the strict containments of Proposition 1 hold.

Conjecture 3 *Any prime $(0, 1, 2)$ complete seki is an IDSM.*

This conjecture was proven in [1] for the $(0, 1)$ matrices and it was verified by a computer code for all $n \times n$ matrices with $n \leq 5$. Yet, even being proven in general, this conjecture would not clarify the structure of the complete seki, since even a prime one of height greater than 2 may be not an IDSM, as we know.

9.3 Semi-complete seki

A *semi-complete seki* is defined as a seki in which one player must pass (otherwise the opponent wins), while the other can make an active move (such that the reduced matrix is still a seki). Only two 3×3 semi-complete seki are known:

A_8^3	A_9^3
0 3 3	0 3 5
4 0 3	3 0 5
4 3 0	4 4 0

For example, in A_8^3 player C can reduce $A(1, 2)$ from 3 to 2. If R would pass then C wins playing at $(3, 2)$. Instead, R should reduce $A(3, 1)$ from 4 to 3 getting a (not complete) seki.

Unlike C, player R must pass. Indeed, due to obvious symmetry, R has only three distinct moves: at $(3, 2)$, $(2, 1)$, and $(1, 2)$. It is not difficult to verify that, in all three cases, C wins answering, respectively, at $(1, 2)$, $(1, 2)$, and $(3, 2)$ (or, less obviously, at $(2, 1)$).

We leave the complete case analysis of A_8^3 and A_9^3 to the careful reader.

Of course, the transposed two matrices are semi-complete seki too. Our computations show that there are no other semi-complete seki among the 3×3 matrices of height $h \leq 10$. Let us also notice that neither A_8^3 nor A_9^3 is an IDSM.

In contrast, there are many 4×4 semi-complete seki and some of them are simultaneously IDSMs. For example, there are ten 4×4 IDSMs of height 3 that are semi-complete seki.

0023	0023	0033	0123	0123	0123	0133	0312	0233	1133
2300	2201	3300	2301	2310	2121	3301	2112	3302	3311
1121	2210	1122	2121	2112	2211	2122	2121	3131	2222
2111	1121	2211	2121	2121	2211	2221	2121	2222	2222

9.4 Generating complete seki recursively; $(3,1,1,-1)$- and $(3,2,2,-2)$-extensions

Let us recall four 3×3 complete seki ($A^3_i; i = 3,4,5,6$) and consider also the following three 2×2 seki ($B^2_j, j = 1,2,3$):

B^2_1	A^3_3	B^2_2	A^3_5	A^3_4	B^2_3	A^3_6
	3 0 1		3 2 0	3 2 0		3 2 0
2 2	0 2 2	2 3	2 0 3	2 1 2	3 3	2 1 3
2 2	1 2 1	3 2	0 3 2	0 2 3	3 3	0 3 3

It is easy to notice that A^3_3, A^3_4, A^3_5 and A^3_6 can be viewed as extensions of B^2_1, B^2_2, $(B^2_2)^T$, and B^2_3, respectively. More precisely, given an $m \times n$ matrix $A : I \times J \to \mathbb{Z}_+$, we define two new $(m+1) \times (n+1)$ matrices A' and A'' as follows. Let us add one new row i_0 to I and one new column j_0 to J and set $A'(i_0, j_0) = A''(i_0, j_0) = 3$. Then, let us choose an entry (i^*, j^*) in A such that $A(i^*, j^*) \geq 1$ (respectively, $A(i^*, j^*) \geq 2$) and reduce it by 1, that is, set $A'(i^*, j^*) = A(i^*, j^*) - 1$ (respectively, by 2, that is, set $A''(i^*, j^*) = A(i^*, j^*) - 2$). All other entries of A remain the same, that is, $A(i,j) = A'(i,j) = A''(i,j)$ whenever $i \in I \setminus \{i^*\}$ or $j \in J \setminus \{j^*\}$. Finally, let us define $A'(i^*, j_0) = A'(i_0, j^*) = 1$ (respectively, $A''(i^*, j_0) = A''(i_0, j^*) = 2$) and $A'_{i,j_0} = A'_{i_0,j} = A''(i, j_0) = A''(i_0, j) = 0$ for all $i \in I \setminus \{i^*\}$ and $j \in J \setminus \{j^*\}$. The obtained two matrices A' and A'' will be called the $(3,1,1,-1)$- and $(3,2,2,-2)$-*extension* of A at (i^*, j^*), respectively.

For example, $A'' = A^3_5$ is the $(3,2,2,-2)$-extension of B^2_2 at $(1,1)$ and $A' = A^3_3$ is the $(3,1,1,-1)$-extension of B^2_1 at $(2,2)$. Let us notice that both last matrices are complete seki.

The $(3,1,1,-1)$-extension was suggested by Andrey Gol'berg in 1981. Being applied to a complete seki, it frequently, but not always, results in another complete seki. For example, computations show that *all* $(3,1,1,-1)$-extension of the five 3×3 complete seki, A^3_i, $i = 1,2,3,4,5$ result in complete seki.

Furthermore, the $(3,1,1,-1)$-extensions of A^3_6 at (i^*, j^*) are complete seki whenever $i^* + j^* \geq 4$, that is, at $(1,3)$, $(1,3)$, $(2,3)$, $(3,1)$, $(3,2)$, and $(2,2)$. However, at $(1,1)$ it is not; our computations show that the first player wins in the obtained matrix. Finally, the $(3,1,1,-1)$-extension of A^3_6 is not defined at $(1,2)$ and at $(2,1)$, since $A^3_6(1,2) = A^3_6(2,1) = 0$.

Similarly, the $(3,1,1,-1)$-extensions of A^3_7 at (i^*, j^*) are complete seki whenever $\min(i^*, j^*) \geq 2$, that is, at $(2,2)$, $(2,3)$, $(3,2)$, and $(3,3)$. Yet, applied at $(1,2)$ and $(2,1)$ it results in a semi-complete seki; at $(1,3), (3,1)$ it is not a complete seki either.

In [4] it was erroneously announced that the $(3, 1, 1, -1)$-extension of a complete seki is always a complete seki; more precisely, if A' is a $(3, 1, 1, -1)$-extension of A at (i^*, j^*) then A' is a complete seki whenever $s^r_{i^*} \geq 4$, $s^c_{j^*} \geq 4$ and A is a complete seki.

The above analysis of the CSMs A^3_6 and A^3_7 shows that this claim was an overstatement. Yet, among all 2×2 and 3×3 complete seki there are only the above five counterexamples: $(3, 1, 1, -1)$-extension of A^3_7 at $(1, 2)$, $(1, 3)$, $(2, 1)$, $(3, 1)$ and of A^3_6 at $(1, 1)$.

Let us notice that the required conditions $s^r_{i^*} \geq 4$, $s^c_{j^*} \geq 4$ are absolutely necessary. Indeed, if A is a complete seki and $s^r_{i^*} \leq 3$ (respectively, $s^c_{j^*} \leq 3$) then, being first, R (respectively C) wins in A' in at most 3 moves.

Now, let us proceed with the $(3, 2, 2, -2)$-extensions. As we already mentioned, all $(3, 2, 2, -2)$-extensions of the (not complete 2×2 seki) B^2_2 and B^2_3 are complete seki.

Let us apply the $(3, 1, 1, -1)$- and $(3, 2, 2, -2)$)-extensions at $(1, 1)$ recursively beginning, respectively, with the 2×2 matrices $A^2 = B^2_1$ and $B^2 = B^2_2$. We obtain the following two infinite matrix sequences A^n, B^n for $n = 2, 3, \ldots$

$$\begin{matrix}2&2\\2&2\end{matrix}\quad
\begin{matrix}3&1&0\\1&1&2\\0&2&2\end{matrix}\quad
\begin{matrix}3&1&0&0\\1&2&1&0\\0&1&1&2\\0&0&2&2\end{matrix}\quad
\begin{matrix}3&1&0&0&0\\1&2&1&0&0\\0&1&2&1&0\\0&0&1&1&2\\0&0&0&2&2\end{matrix}\quad
\begin{matrix}3&1&0&0&0&0\\1&2&1&0&0&0\\0&1&2&1&0&0\\0&0&1&2&1&0\\0&0&0&1&1&2\\0&0&0&0&2&2\end{matrix}\quad\ldots$$

$$\begin{matrix}3&2\\2&3\end{matrix}\quad
\begin{matrix}3&2&0\\2&1&2\\0&2&3\end{matrix}\quad
\begin{matrix}3&2&0&0\\2&1&2&0\\0&2&1&2\\0&0&2&3\end{matrix}\quad
\begin{matrix}3&2&0&0&0\\2&1&2&0&0\\0&2&1&2&0\\0&0&2&1&2\\0&0&0&2&3\end{matrix}\quad\ldots$$

Proposition 10 *Except for $B^2 = B^2_2$, both sequences contain only complete seki.*

Proof. By construction, these two sequences contain only IDSMs with the sums $s(A^n) = 4$ and $s(B^n) = 5$, for all $n \geq 2$. Moreover, except for $A(1, 1) = 3$ and $B(1, 1) = B(n, n) = 3$, all other entries take only values $0, 1$, and 2. Thus, conditions of Theorem 2 and Corollary 2 "almost" hold. Moreover, by the same arguments we can show that after "almost" every active move of a player the opponent wins in 3 (respectively, in 4) moves. However, there are two important exceptions when the opponent cannot win immediately; instead (s)he can reduce A^n (respectively, B^n) to the $(n-1)\times(n-1)$ matrix from the same sequence, in which one entry is also reduced by 1. Thus, we can finish the proof by induction on n, since in both sequences the 3×3 matrices are known complete seki: $A^3 = A^3_3$ and $B^3 = A^3_4$, respectively.

Case 1. If a player, say R, begins at $(1, 1)$ then C answers at $(2, 1)$. It is easily seen that R must play at $(1, 2)$. (In case of the second sequence, C will repeat by playing at $(2, 1)$ once more and again R must answer at $(1, 2)$. As a result of this exchange of moves, the the original $n \times n$ matrix, A^n or B^n, is reduced to the direct sum of two matrices: a 1×1 complete seki with the entry 2 and the $(n - 1) \times (n - 1)$ matrix from the same sequence in which the first entry is 2 rather than 3. Since C has to move in the obtained game, (s)he can enforce the same reduction again, etc.

Case 2. If a player, say R, begins at $(2, 1)$ then C answers at $(1, 1)$, threatening to eliminate the first column by the next 2 (respectively, 3) moves. The only defense of R is to play at $(1,2)$. (In case of the second sequence, C will proceed by eliminating of $(2,1)$ and R must eliminate $(1,2)$ in return). The above exchange of moves results in exactly the same reduction as in Case 1. Thus, C wins again.

Let us notice that the matrices of the second sequence are double-symmetric, that is, $B(i,j) = B(j,i)$ and $B(i,j) = B(n-i+1, n-j+1)$ for all $n \geq 3$ and $i, j \in [n] = \{1, \ldots, n\}$. Hence, we can substitute $(1,2)$,$(2,1)$, and $(1,1)$ by $(n-1,n)$,$(n,n-1)$, and (n,n), respectively.

It remains to verify that for any other move of a player, the opponent can win in 3 (respectively, in 4) moves by applying the strategy suggested in the proof of Corollary 2. □

Corollary 6 *For each $n \geq 3$ there is a $n \times n$ complete seki of height ≥ 3; in other words, $H(n) \geq 3$ for $n \geq 3$.*

Proof. For every $n \geq 2$, each of the above two sequences contains a (unique) $n \times n$ matrix of height 3, which is a complete seki when $n \geq 3$. □

10 Results and conjectures about the complete d-seki

The family of all known complete d-seki looks much simpler than the family of all known complete seki. By Corollary 1, a $(0,1)$ IDSM A is a complete d-seki whenever $s(A) \geq 2$.

Yet, there are other complete d-seki. The following $(0,2)$-matrices are called 2-*cycles*:

$$\begin{matrix} 2&2&0 \\ 2&0&2 \\ 0&2&2 \end{matrix} \qquad \begin{matrix} 2&2&0&0 \\ 2&0&2&0 \\ 0&2&0&2 \\ 0&0&2&2 \end{matrix} \qquad \begin{matrix} 2&2&0&0&0 \\ 2&0&2&0&0 \\ 0&2&0&2&0 \\ 0&0&2&0&2 \\ 0&0&0&2&2 \end{matrix}$$

More precisely, a (unique) 2-cycle C^k is defined for each $k \geq 3$ as the $k \times k$ matrix A such that the first and the last entries of its main diagonal, as well as all entries of the two neighbor diagonals of A, equal 2 (that is, $A(1,1) = A(k,k) = A(i,i+1) = A(i+1,i) = 2$ for $i = 1, \ldots, k-1$, while all remaining entries of A equal 0.

Theorem 3 *Every 2-cycle is a prime complete d-seki.*

Proof. Obviously, a 2-cycle is a prime. Let us prove that it is a complete d-seki.

To do so, we have to demonstrate that after any move of R or C, the opponent wins. Yet, there are many possible moves in a 2-cycle that seem different. To avoid this case analysis, let us show that all first moves are in fact equivalent. More precisely, every 2-cycle C^k is *transitive*; in other words, for every two of its non-zero entries (i,j) and (i',j'), there is a transformation of C^k that maps (i,j) to (i',j') and the set of all non-zero entries into itself; here a transformation is an arbitrary permutation of the rows and columns with possible transposing. As we know, both operations do not change the games SEKI or D-SEKI.

Proposition 11 *Every 2-cycles is transitive.*

Proof. For the beginning, let us consider the case $j = i' = i+1, j' = i$, in other words, $(i,j) = (i,i+1), (i',j') = (i+1,i)$. Then the matrix should be just transposed.

Since permutations form a group, it remains to transform $(i,i+1)$ to $(1,1)$ keeping C^k.

The following (unique) permutations of the rows and columns solve the problem:

$$(f(i), f(i+2), f(i-2), f(i+4), f(i-4), \ldots);\ (f(i+1), f(i-1), f(i+3), f(i-3), f(i+5), \ldots),$$

where the first and second permutations correspond to the rows and columns, respectively, and the function $f : \mathbb{Z} \to [k]$ is defined by the formula:

$$f(\ell) = \begin{cases} \ell, & \text{if } \ell \in [k]; \\ 2k+1-\ell, & \text{if } \ell > k; \\ 1-\ell, & \text{if } \ell < 1. \end{cases} \tag{5}$$

For example, the permutations $(3,5,1,7,2,6,4)$ and $(4,2,6,1,7,3,5)$ of, respectively, the rows and columns of C^7 transform $(3,4)$ to $(1,1)$ and keep C^7 itself. □

Now, we are ready to show that C^k is a complete d-seki, that is, for each active move of a player there is a winning response of the opponent in D-SEKI. By Proposition 11, without loss of generality, we can assume that C plays at $(1,1)$. Let us show that R wins naturally replying at $(1,2)$. Obviously, by this (s)he threatens to eliminate the first row in two moves: $(1,2)$ and $(1,1)$. Still, C could try a (unique) defense, playing at $(2,1)$. Now, against R's $(1,2)$, C would just repeat $(2,1)$ forcing a draw. Yet, R has a stronger move, at $(2,3)$! Now, (s)he has two threatening two-move sequences: $(2,3)$, $(2,1)$ and $(1,2)$, $(1,1)$. For example, If C would pass, both sequences work:

(i) R plays at $(1,2)$ and against C's $(2,1)$ wins at $(2,3)$.

(ii) R plays at $(2,3)$ and against C's $(1,1)$ wins at $(1,2)$.

In fact, C has a defence against each of these two threats but not against both.

Indeed, C's $(3,2)$ destroys (i) but (ii) still works. Also C can play at $(4,3)$ if $k > 3$ or, respectively, at $(3,3)$ if $k = 3$; this is a defence against (ii) but (i) still works. □

Remark 9 *Let us note, yet, that the above arguments does not work for $k = 2$. Moreover, by Remark 9, the 2×2 IDSM A_4^2 is a complete seki but not a complete d-seki. Indeed, it is easily seen that after any move of any player, D-SEKI is still a draw. Respectively, A_4^2 is not a 2-cycle. By definition, a 2-cycle is a $(0,2)$ IDSM of sum 4 and of size at least 3×3.*

Proposition 12 *Every $(0,2)$ IDSM of sum 4 can be transformed, by permutations of its rows and columns, to the direct sum of 2-cycles and 2×2 IDSMs A_4^2.*

Proof. If A is a $(0,2)$ IDSM of sum $s(A) = 4$ them, obviously, every row and column of A contain exactly two 2-entries, while all other entries are zeros. Hence, each 2-entry (i,j) has exactly two neighbor 2-entries (i',j) and (i,j'). Then obviously, all 2-entries are partitioned into several $(t \geq 1)$ cycles of the form

$((i_1,j_1),(i_2,j_1),(i_2,j_2),\ldots,(i_{k-1},j_{k-1}),(i_k,j_{k-1}),(i_k,j_k),(i_1,j_k),(i_1,j_1)),$

where $\{i_1,\ldots,i_k\} \subseteq I$ and $\{j_1,\ldots,j_k\} \subseteq J$ are the corresponding k rows and columns.

Obviously, $k \geq 2$; moreover, if $k = 2$ then the obtained $k \times k$ matrix is A_4^2.

Let us show that this matrix is isomorphic to a 2-cycle C^k whenever $k \geq 3$.

For each $\ell \in [k] = \{1,\ldots,k\}$, let us assign to the row i_ℓ and column j_ℓ new numbers $g(2\ell-2)$ and $g(2\ell-1)$, respectively, where function g is defined by formula

$$g(\ell') = \begin{cases} \ell', & \text{if} \quad \ell' \in [k]; \\ 2k+1-\ell', & \text{if} \quad \ell' > k; \\ 1, & \text{if} \quad \ell' = 0. \end{cases} \qquad (6)$$

For example, the rows $(i_1,\ldots,i_6)$ and columns $(j_1,\ldots,j_6)$ get numbers $(1,2,4,6,5,3)$ and $(1,3,5,6,4,2)$, respectively, when $k = 6$. It is not difficult to verify that the considered $k \times k$ cycle is transformed to the standard 2-cycle C^k; in particular (i_1,j_1) becomes $(1,1)$. $\Box$

It seems that Corollary 1 and Theorem 3 can be reversed as follows.

Conjecture 4 *Each complete d-seki is the direct sum of* $(0,1)$ *IDSMs with* $s \geq 2$ *and* $(0,2)$ *IDSMs with* $s = 4$ *of size at least* 3×3 *(2-cycles). In other words, all prime complete d-seki are listed by Corollary 1 and Theorem 3. In particular, each of them is a square matrix.*

This conjecture was verified for all $m \times n$ matrices of height at most 3 and $\max(m,n) \leq 5$.

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