

R U T C O R
R E S E A R C H
R E P O R T

KUSUOKA REPRESENTATIONS OF
COHERENT RISK MEASURES IN
FINITE PROBABILITY SPACES

Nilay Noyan^a Gábor Rudolf^b

RRR 33-2012, DECEMBER, 2012

RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
<http://rutcor.rutgers.edu/~rrr>

^aManufacturing Systems/Industrial Engineering Program, Sabancı University, 34956 Istanbul, Turkey, nnoyan@sabanciuniv.edu

^bManufacturing Systems/Industrial Engineering Program, Sabancı University, 34956 Istanbul, Turkey, grudolf@sabanciuniv.edu

RUTCOR RESEARCH REPORT
RRR 33-2012, DECEMBER, 2012

KUSUOKA REPRESENTATIONS OF COHERENT RISK MEASURES IN FINITE PROBABILITY SPACES

Nilay Noyan and Gábor Rudolf

Abstract. Kusuoka representations provide an important and useful characterization of law invariant coherent risk measures in atomless probability spaces. However, the applicability of these results is limited by the fact that such representations do not always exist in probability spaces with atoms, such as finite probability spaces. We introduce the class of functionally coherent risk measures, which allow us to use Kusuoka representations in any probability space. We show that this class contains every law invariant risk measure that can be coherently extended to a family containing all finite discrete distributions. Thus, it is possible to preserve the desirable properties of law invariant coherent risk measures on atomless spaces without sacrificing generality. We also specialize our results to risk measures on finite probability spaces, and prove that in such spaces the family of risk measures with finite Kusuoka representations is dense among all law invariant coherent risk measures over a bounded class of random variables.

Acknowledgements: The first author has been supported by The Scientific and Technological Research Council of Turkey (TUBITAK) Career Award # 111M543 and the second author has been funded by TUBITAK-2216 Research Fellowship Programme.

1 Introduction

Risk measures are functionals that represent the risk associated with a random variable by a scalar value. Desirable properties of risk measures, such as *coherence*, *comonotone additivity*, and *law invariance* have been axiomatized starting with the work of Artzner et al. (1999).

The seminal paper Kusuoka (2001), in conjunction with the additional technical details provided in Jouini et al. (2006), establishes two important representation theorems for risk measures on bounded variables in atomless probability spaces. The first theorem states that the class of law invariant coherent and comonotone additive risk measures coincides with the class of spectral risk measures, and thus can be represented as a mixture of CVaR measures. The second theorem states that all law invariant coherent risk measures can be represented as the infimum of a family of spectral measures. These results were later extended to \mathcal{L}^p spaces of random variables on atomless spaces for any $p \in [1, \infty]$, see e.g. Shapiro et al. (2009).

Despite the significant recent interest in these representations (see, e.g., Dentcheva et al., 2010; Shapiro, 2012; Pichler and Shapiro, 2012), to the best of our knowledge no analogous characterizations exist for probability spaces that are not atomless (although Bertsimas and Brown (2009) prove the related result that, in finite probability spaces where every outcome has equal probability, the integrals in Kusuoka representations can be replaced by finite sums). The chief difficulty in extending Kusuoka's theorems lies in the fact that in the presence of atoms there exist certain "pathological" law invariant coherent risk measures for which representations of the desired form are not possible (a constructive example is given in Pflug and Römisch, 2007). To exclude these ill-behaved risk measures, in this paper we introduce the class of *functionally coherent* risk measures, and show that they coincide with the class of risk measures with Kusuoka representations even in probability spaces with atoms. We also provide an analogous result that shows the equivalence between the class of functionally coherent and comonotone additive risk measures, and the class of spectral risk measures. In addition, we show that the assumption of functional coherence is not overly restrictive, as every law invariant coherent risk measure that can be evaluated on all finite discrete distributions is functionally coherent.

For finite probability spaces we prove that the mixtures of CVaR measures in Kusuoka representations can be replaced by finite convex combinations. In the equal probability case these combinations take a special form, where the number of terms is bounded from above by the size of the state space. We also examine the class of risk measures where the infimum in the Kusuoka representation can be replaced by a finite minimum, and show that, over bounded families of random variables, they are dense among functionally coherent risk measures.

The rest of the paper is structured as follows. In Section 2 we establish some notation and conventions. In Section 3 we recall some basic facts about the quantile function and related concepts, while in Section 4 we review some existing representations (including Kusuoka representations) for various classes of coherent risk measures. We introduce the class of functionally coherent risk measures in Section 5, show that they have Kusuoka rep-

representations, then specialize these results for finite probability spaces. Finally, in Section 6 we describe a general framework for coherently extending mappings on families of distribution functions, then use this framework to argue the generality of the class of functionally coherent risk measures.

2 Notation and conventions

Throughout this paper, let $\mathcal{S} = (\Omega_{\mathcal{S}}, \mathcal{A}_{\mathcal{S}}, \Pi_{\mathcal{S}})$ denote a standard atomless probability space (for instance, the interval $[0, 1]$ equipped with the Borel sigma algebra and the Lebesgue measure). Let us denote the set of all CDFs by

$$\mathfrak{F} = \left\{ F : \mathbb{R} \rightarrow [0, 1] \left| \lim_{v \rightarrow -\infty} F(v) = 0, \lim_{v \rightarrow \infty} F(v) = 1, F \text{ is right-continuous and non-decreasing} \right. \right\}.$$

Note that, as standard uniform random variables exist on any atomless space (see, e.g., Föllmer and Schied, 2004), using probability integral transforms (as outlined in Section 3) we obtain $\mathfrak{F} = \{F_V \mid V : \Omega \rightarrow \mathbb{R} \text{ is measurable}\}$. Let us denote the set of p -integrable random variables on \mathcal{S} for some $p \in [1, \infty]$ by $\mathcal{L}^p(\mathcal{S})$, and the corresponding set of CDFs by $\mathfrak{F}^p = \{F_V \mid V \in \mathcal{L}^p(\mathcal{S})\}$. Note that \mathfrak{F}^p does not depend on the choice of \mathcal{S} .

We say that random variables V_1, V_2 are equal in distribution if we have $F_{V_1} = F_{V_2}$, and denote this fact by $V_1 \sim V_2$. The relation $V_1 \leq V_2$ denotes inequality everywhere, i.e., that $V_1(\omega) \leq V_2(\omega)$ holds for all $\omega \in \Omega$.

The extended real line is denoted by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. For a value $x \in \mathbb{R}$ we denote the Dirac delta measure concentrated on x by δ_x .

Throughout our paper larger values of random variables are considered to be preferable. In this context, risk measures are often referred to as *acceptability functionals*, since higher values indicate less risky, i.e., more acceptable random outcomes. In the literature the opposite convention (where small values are preferred) is also widespread, especially when dealing with loss functions. When citing such sources, the definitions and formulas are altered to reflect this difference.

3 Basic concepts

Let us recall some well-known definitions, starting with two important risk measures that serve as the basis of the representations we consider in the following sections.

- Let V be a random variable with a cumulative distribution function (CDF) denoted by F_V . The *value-at-risk* (VaR) at confidence level $\alpha \in (0, 1]$ is defined as

$$\text{VaR}_{\alpha}(V) = \inf\{\eta : F_V(\eta) \geq \alpha\}. \quad (1)$$

- The *conditional value-at-risk* at confidence level α is given by

$$\text{CVaR}_\alpha(V) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\gamma(V) d\gamma. \quad (2)$$

CVaR is also known in the literature as *average value-at-risk* and *tail value-at-risk*.

- We formally define $\text{VaR}_0(V) = \text{CVaR}_0(V) = \text{ess inf}(V)$.

$\text{VaR}_\alpha(V)$ is also known as the α -quantile of the random variable V . Accordingly, the first quantile function $F_V^{-1} : (0, 1] \rightarrow \mathbb{R}$ of V is defined as $F_V^{-1}(\alpha) = \text{VaR}_\alpha(V)$. We recall the following well-known facts:

- The mapping $F_V \mapsto F_V^{-1}$ is order reversing, i.e., the relation $F_{V_1} \leq F_{V_2}$ implies $F_{V_1}^{-1} \geq F_{V_2}^{-1}$.
- *Probability integral transform*: For any random variable V and standard uniform random variable U we have $V \sim F_V^{-1}(U)$.

The *generalized Lorenz curve* $G_V : (0, 1] \rightarrow \mathbb{R}$ of a random variable V is defined as $G_V(\alpha) = \alpha \text{CVaR}_\alpha(V)$. Note that according to (2) the generalized Lorenz curve is continuous, since it is the integral of the first quantile function. The next lemma will be critical in establishing certain finiteness results.

LEMMA 3.1 *Given a probability space $(\Omega, \mathcal{A}, \Pi)$, let $\mathcal{K} = \{\Pi(S) : S \in \mathcal{A}, \Pi(S) > 0\}$ denote the set of all non-zero probabilities of events, and consider an interval $I \subset [0, 1]$ such that $(\text{int } I) \cap \mathcal{K} = \emptyset$, where $\text{int } I$ denotes the interior of I .*

1. *The quantile function F_V^{-1} is constant on the interval I .*
2. *The generalized Lorenz curve G_V is affine on the interval I . Furthermore, if the left endpoint of I is 0, then G_V is linear on I .*

PROOF. Consider a value $\alpha \in I$. As the level sets of the random variable V are measurable, both of the values

$$\alpha_- = \Pi(V < \text{VaR}_\alpha(V)) \quad \text{and} \quad \alpha_+ = \Pi(V \leq \text{VaR}_\alpha(V))$$

are in $\mathcal{K} \cup \{0\}$. By the definition of VaR we have $\alpha_- \leq \alpha \leq \alpha_+$, implying $I \subset [\alpha_-, \alpha_+]$. Since $\text{VaR}_\gamma(V) = \text{VaR}_\alpha(V)$ holds for any $\gamma \in (\alpha_-, \alpha_+]$, part 1 immediately follows. Part 2 is now a direct consequence of (2). \square

4 Coherent risk measures and Kusuoka representations

For $p \in [1, \infty]$, let $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{A}, \Pi)$ denote the family of p -integrable random variables on some probability space $(\Omega, \mathcal{A}, \Pi)$. We note that for finite probability spaces \mathcal{L}^p coincides with the set $\mathcal{V} = \mathcal{V}(\Omega, \mathcal{A}, \Pi)$ of *all* random variables. Following the influential work of [Artzner et al. \(1999\)](#), we say that a mapping $\rho : \mathcal{L}^p \rightarrow \overline{\mathbb{R}}$ is a *coherent risk measure* if it has the following properties (for all $V, V_1, V_2 \in \mathcal{L}^p$):

- *Normalized*: $\rho(0) = 0$.
- *Monotone*: $V_1 \leq V_2 \Rightarrow \rho(V_1) \leq \rho(V_2)$.
- *Superadditive*: $\rho(V_1 + V_2) \geq \rho(V_1) + \rho(V_2)$.
- *Positive homogeneous*: $\rho(\lambda V) = \lambda \rho(V)$ for all $\lambda > 0$.
- *Translation invariant*: $\rho(V + \lambda) = \rho(V) + \lambda$.

A pair V_1, V_2 of random variables is said to be *comonotone*, if the following condition holds.

$$(V_1(\omega_1) - V_1(\omega_2))(V_2(\omega_1) - V_2(\omega_2)) \geq 0 \quad \Pi(d\omega_1) \otimes \Pi(d\omega_2)\text{-almost surely.} \quad (3)$$

It is easy to verify that if V_1 and V_2 are comonotone then they are *quantile additive* (see, e.g., [McNeil et al., 2005](#)):

$$F_{V_1}^{(-1)}(\alpha) + F_{V_2}^{(-1)}(\alpha) = F_{V_1+V_2}^{(-1)}(\alpha) \quad \text{for all } \alpha \in (0, 1]. \quad (4)$$

If the probability space is atomless, (3) holds if and only if there exists a standard uniform random variable U such that we have $(V_1, V_2) \sim (F_{V_1}^{(-1)}(U), F_{V_2}^{(-1)}(U))$. We say that a risk measure ρ is *comonotone additive* if it has the following property.

- *comonotone additive*: V_1, V_2 comonotone $\Rightarrow \rho(V_1 + V_2) = \rho(V_1) + \rho(V_2)$.

Let us denote the family of CDFs for p -integrable random variables by $\mathcal{F}^p = \mathcal{F}^p(\Omega, \mathcal{A}, \Pi) = \{F_V : V \in \mathcal{L}^p(\Omega, \mathcal{A}, \Pi)\}$, and again note that for a finite probability space $\mathcal{F}^p(\Omega, \mathcal{A}, \Pi)$ coincides with the family $\mathcal{F}(\Omega, \mathcal{A}, \Pi)$ of the CDFs of all random variables on $(\Omega, \mathcal{A}, \Pi)$. We say that a mapping $\rho : \mathcal{L}^p \rightarrow \overline{\mathbb{R}}$ is *law invariant* if the value $\rho(V)$ depends only on the distribution of the random variable V , i.e., if there exists a mapping $\varphi_\rho : \mathcal{F}^p \rightarrow \overline{\mathbb{R}}$ such that $\rho(V) = \varphi_\rho(F_V)$ holds for all $V \in \mathcal{L}^p$. Note that in this case φ_ρ is uniquely determined by ρ .

It is well known ([Pflug, 2000](#)) that CVaR is a law invariant coherent risk measure. Moreover, CVaR can be viewed as a basic building block of coherent risk measures, as the following fundamental theorem shows. The result for $p = \infty$ is due to [Kusuoka \(2001\)](#) and [Jouini et al. \(2006\)](#), while the proof for $p \in [1, \infty)$ can be found in [Shapiro et al. \(2009\)](#).

THEOREM 4.1 (KUSUOKA REPRESENTATIONS) *Consider the atomless space \mathcal{S} and a value $p \in [1, \infty]$.*

1. *A mapping $\rho : \mathcal{L}^p(\mathcal{S}) \rightarrow \overline{\mathbb{R}}$ is a law invariant coherent risk measure if and only if there exists a family $\mathcal{M} \subset \mathcal{P}([0, 1])$ such that*

$$\rho(V) = \inf_{\mu \in \mathcal{M}} \int_{[0,1]} \text{CVaR}_\alpha(V) \mu(d\alpha) \quad \text{for all } V \in \mathcal{L}^p. \quad (5)$$

2. *A mapping $\rho : \mathcal{L}^p(\mathcal{S}) \rightarrow \overline{\mathbb{R}}$ is a law invariant coherent and comonotone additive risk measure if and only if there exists a probability measure $\mu \in \mathcal{P}([0, 1])$ on the interval $[0, 1]$ such that the following representation holds:*

$$\rho(V) = \int_{[0,1]} \text{CVaR}_\alpha(V) \mu(d\alpha) \quad \text{for all } V \in \mathcal{L}^p. \quad (6)$$

Mappings with a representation of the form (6) are also known as *spectral risk measures*, and can be expressed as

$$\rho(V) = \int_{[0,1]} F_V^{(-1)}(\alpha) \nu(d\alpha) \quad \text{for all } V \in \mathcal{L}^p,$$

where the measure $\nu \in \mathcal{P}([0, 1])$ is defined as

$$\nu(A) = \lambda \delta_0(A) + (1 - \lambda) \int_A \phi(\alpha) d\alpha \quad (7)$$

for some $\lambda \in [0, 1]$ and some non-increasing function $\phi : [0, 1] \rightarrow \mathbb{R}_+$ satisfying $\|\phi\|_1 = 1$. We mention that in finite probability spaces we have $F_V^{(-1)} \Big|_{[0, \epsilon]} \equiv F_V^{(-1)}(\epsilon)$ for $\epsilon = \min \{\Pi(\omega) \mid \omega \in \Omega, \Pi(\omega) > 0\} > 0$. Accordingly, in this case we can assume without loss of generality that $\lambda = 0$ in the formula (7).

5 Functionally coherent risk measures

In probability spaces that have atoms Kusuoka representations do not always exist, as the next example shows.

EXAMPLE 5.1 *Let $\Omega = \{\omega_1, \omega_2\}$ with $\Pi(\omega_1) > \frac{1}{2}$. It is easy to verify that the risk measure ρ defined by $\rho(V) = V(\omega_1)$ is coherent (and even comonotone additive). In addition, the equality $\rho(V) = \sup \{v : F_V(v) < \frac{1}{2}\}$ shows that ρ is law invariant. However, it can be proven (Pflug and Römisch, 2007) that ρ has no Kusuoka representation of the form (5).*

The risk measure ρ in the above example is “pathological” in the sense that its corresponding mapping φ_ρ cannot be coherently extended to CDFs from other probability spaces. We now formalize this intuitive notion.

DEFINITION 5.1 *Given a not necessarily atomless probability space $(\Omega, \mathcal{A}, \Pi)$ and a value $p \in [1, \infty]$, a law invariant mapping $\rho : \mathcal{L}^p(\Omega, \mathcal{A}, \Pi) \rightarrow \overline{\mathbb{R}}$ is called a functionally coherent risk measure if there exists a law invariant coherent risk measure $\bar{\rho} : \mathcal{L}^p(\mathcal{S}) \rightarrow \overline{\mathbb{R}}$ such that φ_ρ is a restriction of $\varphi_{\bar{\rho}}$, i.e., we have $\varphi_\rho = \varphi_{\bar{\rho}}|_{\mathcal{F}^p(\Omega, \mathcal{A}, \Pi)}$.*

If there exists some $\bar{\rho} : \mathcal{L}^p(\mathcal{S}) \rightarrow \overline{\mathbb{R}}$ that, in addition to the above properties, is also comonotone additive, then we say that ρ is a functionally coherent and comonotone additive risk measure.

We mention that the above notion is essentially equivalent to the concept of *regular coherent risk measures* discussed in [Pichler and Shapiro \(2012\)](#). It is easy to see that in atomless probability spaces the class coincides with the class of law invariant coherent risk measures. However, functionally coherent risk measures allow us to use Kusuoka representations even in probability spaces that are not atomless, as the next proposition shows.

PROPOSITION 5.1 *Consider a (not necessarily atomless) probability space $(\Omega, \mathcal{A}, \Pi)$, and a value $p \in [1, \infty]$.*

1. *A mapping $\rho : \mathcal{L}^p \rightarrow \overline{\mathbb{R}}$ is a functionally coherent risk measure if and only if it has a Kusuoka representation of the form (5) for some family $\mathcal{M} \subset \mathcal{P}([0, 1])$.*
2. *A mapping $\rho : \mathcal{L}^p \rightarrow \overline{\mathbb{R}}$ is a functionally coherent and comonotone additive risk measure if and only if it has a Kusuoka representation of the form (6) for some measure $\mu \in \mathcal{P}([0, 1])$.*
3. *The equivalences established in parts 1 and 2 still hold if we restrict ourselves to measures supported on the set $\bar{\mathcal{K}} = \{\Pi(S) : S \in \mathcal{A}, \Pi(S) > 0\}$, i.e., on the topological closure of the set \mathcal{K} introduced in Lemma 3.1.*

PROOF. If ρ is functionally coherent, then there exists a law invariant coherent risk measure $\bar{\rho}$ on an atomless space such that $\varphi_\rho = \varphi_{\bar{\rho}}|_{\mathcal{F}^p(\Omega, \mathcal{A}, \Pi)}$ holds. By Theorem 4.1 the risk measure $\bar{\rho}$ has a Kusuoka representation of the form (5), which trivially also provides a representation for ρ . On the other hand, if ρ has a Kusuoka representation, then the same representation defines a law invariant mapping $\bar{\rho} : \mathcal{L}^p(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\Pi}) \rightarrow \overline{\mathbb{R}}$ on any atomless probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\Pi})$. By Theorem 4.1 the mapping $\bar{\rho}$ is a coherent risk measure, which proves part 1. The proof of part 2 is essentially identical.

It is sufficient to prove part 3 for representations of the form (6). For a confidence level $\alpha \in [0, 1]$ let us define $\alpha_- = \sup\{\gamma \in \mathcal{K} \cup \{0\} : \gamma \leq \alpha\} \in \bar{\mathcal{K}} \cup \{0\}$ and $\alpha_+ = \inf\{\gamma \in \mathcal{K} : \gamma \geq \alpha\} \in \bar{\mathcal{K}}$. According to Lemma 3.1, the mapping $\gamma \mapsto \gamma \text{CVaR}_\gamma$ is affine on the interval $[\alpha_-, \alpha_+]$, therefore CVaR_α can be expressed as a convex combination $\lambda_\alpha^- \text{CVaR}_{\alpha_-} + \lambda_\alpha^+ \text{CVaR}_{\alpha_+}$ for some coefficients λ_α^- and λ_α^+ . In addition, if $\alpha_- = 0$, then the mapping $\gamma \mapsto \gamma \text{CVaR}_\gamma$ is linear on $[0, \alpha_+]$, therefore we can set $\lambda_\alpha^- = 0$, $\lambda_\alpha^+ = 1$.

Let us define the probability measure $\bar{\mu}$ on the set $\bar{\mathcal{K}}$ by

$$\bar{\mu}(A) = \int_{\{\alpha \in (0, 1] : \alpha_- \in A\}} \lambda_\alpha^- \mu(d\alpha) + \int_{\{\alpha \in (0, 1] : \alpha_+ \in A\}} \lambda_\alpha^+ \mu(d\alpha).$$

It is easy to verify that $\int_0^1 \text{CVaR}_\alpha(V) \mu(d\alpha) = \int_{\mathcal{K}} \text{CVaR}_\alpha(V) \bar{\mu}(d\alpha)$ holds for all $V \in \mathcal{L}^p(\Omega, \mathcal{A}, \Pi)$, which proves our claim. \square

COROLLARY 5.1 *Consider a functionally coherent and comonotone additive risk measure ρ on some finite probability space $(\Omega, 2^\Omega, \Pi)$, and let $n = |\Omega|$.*

1. *The risk measure ρ can be written as a convex combination of finitely many CVaR measures, i.e., it has a representation*

$$\rho(V) = \sum_{i=1}^M \mu_i \text{CVaR}_{\alpha_i}(V) \quad \text{for all } V \in \mathcal{V}, \quad (8)$$

for some $\alpha_1, \dots, \alpha_M \in (0, 1]$ and $\mu_1, \dots, \mu_M \in \mathbb{R}_+$ satisfying $\sum_{i=1}^M \mu_i = 1$.

2. *If every elementary event is equally likely, then ρ can be expressed in the form*

$$\rho(V) = \sum_{i=1}^n \mu_i \text{CVaR}_{\frac{i}{n}}(V) \quad \text{for all } V \in \mathcal{V}, \quad (9)$$

for some $\mu_1, \dots, \mu_n \in \mathbb{R}_+$ satisfying $\sum_{i=1}^n \mu_i = 1$.

PROOF. We trivially have $|\mathcal{K}| < 2^n < \infty$, which also implies $\bar{\mathcal{K}} = \mathcal{K}$. Furthermore, in the equal probability case $\mathcal{K} = \{\frac{1}{n}, \dots, \frac{n}{n}\}$ holds. \square

We mention that risk measures of the form (8) are also known as *mixed CVaR* (Rockafellar, 2007). Combining Corollary 5.1 and Proposition 5.1, we see that functionally coherent risk measures on finite probability spaces have Kusuoka representations of the form

$$\rho(V) = \inf_{\boldsymbol{\mu} \in \mathcal{M}} \sum_{i=1}^M \mu_i \text{CVaR}_{\alpha_i}(V) \quad \text{for all } V \in \mathcal{V}, \quad (10)$$

for some integer M , confidence levels $\alpha_1, \dots, \alpha_M \in (0, 1]$, and a family of vectors \mathcal{M} in the unit simplex $S_M = \{\boldsymbol{\mu} \in [0, 1]^M : \|\boldsymbol{\mu}\|_1 = 1\}$. If a risk measure has a representation where the set \mathcal{M} is finite, then we say that it is a *finitely representable coherent risk measure*. The next result shows that, over bounded families of random variables, these risk measures are dense among functionally coherent risk measures.

PROPOSITION 5.2 *Consider a family $\bar{\mathcal{V}}$ of random variables on a finite probability space, and assume that $\bar{\mathcal{V}}$ is bounded in the \mathcal{L}^1 -norm. Then, for any functionally coherent risk measure ρ and $\epsilon > 0$ there exists a finitely representable coherent risk measure $\bar{\rho}$ such that $|\rho(V) - \bar{\rho}(V)| < \epsilon$ holds for all $V \in \bar{\mathcal{V}}$.*

PROOF. The risk measure ρ has a Kusuoka representation of the form (10). Let us now select an integer $k > \frac{1}{\epsilon} M \sup \{\|V\|_1 : V \in \bar{\mathcal{V}}\}$, and consider the finite set

$$S_M^{(k)} = \left\{ \boldsymbol{\mu} \in \left\{ \frac{0}{n}, \frac{1}{n}, \dots, \frac{n}{n} \right\}^M : \sum_{i=1}^M \mu_i = 1 \right\} \subset S_M.$$

For every vector $\boldsymbol{\mu} \in S_M$ there exists a *controlled rounding* $\bar{\boldsymbol{\mu}} \in S_M^{(k)}$ such that $|\mu_i - \bar{\mu}_i| < \frac{1}{k}$ holds for all $i = 1, \dots, M$ (see, e.g., [Bacharach, 1966](#)). Then, using the fact that $|\text{CVaR}_\alpha(V)| \leq \|V\|_1$ holds for all $V \in \mathcal{V}$, $\alpha \in (0, 1]$, we have

$$\left| \sum_{i=1}^M \mu_i \text{CVaR}_{\alpha_i}(V) - \sum_{i=1}^M \bar{\mu}_i \text{CVaR}_{\alpha_i}(V) \right| \leq M \frac{1}{k} \sup \{ \|V\|_1 : V \in \bar{\mathcal{V}} \} < \epsilon \quad \text{for every } \boldsymbol{\mu} \in S_M, V \in \bar{\mathcal{V}}. \quad (11)$$

Let us introduce the finite set $\bar{\mathcal{M}} = \{\bar{\boldsymbol{\mu}} : \boldsymbol{\mu} \in \mathcal{M}\} \subset S_M^{(k)}$ and the finitely representable coherent risk measure

$$\bar{\rho}(V) = \inf_{\boldsymbol{\mu} \in \bar{\mathcal{M}}} \sum_{i=1}^M \mu_i \text{CVaR}_{\alpha_i}(V) = \inf_{\boldsymbol{\mu} \in \mathcal{M}} \sum_{i=1}^M \bar{\mu}_i \text{CVaR}_{\alpha_i}(V), \quad V \in \mathcal{V}.$$

Then, recalling (10) and (11), we have

$$|\rho(V) - \bar{\rho}(V)| = \left| \inf_{\boldsymbol{\mu} \in \mathcal{M}} \sum_{i=1}^M \mu_i \text{CVaR}_{\alpha_i}(V) - \inf_{\boldsymbol{\mu} \in \bar{\mathcal{M}}} \sum_{i=1}^M \bar{\mu}_i \text{CVaR}_{\alpha_i}(V) \right| < \epsilon \quad \text{for every } V \in \bar{\mathcal{V}},$$

which proves our claim. □

6 Coherent extensions

Consider a family $\mathcal{F} \subset \mathfrak{F}$ of CDFs, and a mapping $\varphi : \mathcal{F} \rightarrow \bar{\mathbb{R}}$. We say that φ is a *coherent mapping* if the following conditions hold for any random variables V, V_1, V_2 on \mathcal{S} .

- *Normalized*: $F_0 \in \mathcal{F} \implies \varphi(F_0) = 0$.
- *Monotone*: $F_{V_1}, F_{V_2} \in \mathcal{F}, F_{V_1} \geq F_{V_2} \implies \varphi(F_{V_1}) \leq \varphi(F_{V_2})$.
- *Superadditive*: $F_{V_1}, F_{V_2}, F_{V_1+V_2} \in \mathcal{F} \implies \varphi(F_{V_1+V_2}) \geq \varphi(F_{V_1}) + \varphi(F_{V_2})$.
- *Positive homogeneous*: $\alpha > 0, F_V, F_{\alpha V} \in \mathcal{F} \implies \varphi(F_{\alpha V}) = \alpha \varphi(F_V)$.
- *Translation invariant*: $\lambda \in \mathbb{R}, F_V, F_{V+\lambda} \in \mathcal{F} \implies \varphi(F_{V+\lambda}) = \varphi(F_V) + \lambda$.

If, in addition to the above properties, the condition

- *Comonotone additive*: V_1, V_2 comonotone, $F_{V_1}, F_{V_2}, F_{V_1+V_2} \in \mathcal{F} \implies \varphi(F_{V_1+V_2}) = \varphi(F_{V_1}) + \varphi(F_{V_2})$

also holds, then φ is a *coherent and comonotone additive mapping*.

DEFINITION 6.1 *We say that a family $\mathcal{F} \subset \mathfrak{F}$ is closed if for any random variables V, V_1, V_2 on \mathcal{S} we have the following.*

- $F_0 \in \mathcal{F}$.
- $F_V \in \mathcal{F} \implies F_{\alpha V} \in \mathcal{F}$ for all $\alpha \in \mathbb{R}_+$.
- $F_V \in \mathcal{F} \implies F_{\lambda+V} \in \mathcal{F}$ for all $\lambda \in \mathbb{R}$.
- $F_{V_1}, F_{V_2} \in \mathcal{F} \implies F_{V_1+V_2} \in \mathcal{F}$.

We call a family $\mathcal{F} \subset \mathfrak{F}$ *p-dense* if for any $F \in \mathfrak{F}^p$ there exists a sequence $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ such that $\lim_{n \rightarrow \infty} \|F^{-1} - F_n^{-1}\|_p = 0$.

THEOREM 6.1 Consider a value $p \in [1, \infty]$, a closed family $\mathcal{F} \subset \mathfrak{F}^p$, and a mapping $\varphi : \mathcal{F} \rightarrow \overline{\mathbb{R}}$.

1. The mapping φ is coherent if and only if it has a Kusuoka representation of the form (5).
2. If the family \mathcal{F} is *p-dense*, then φ is a coherent and comonotone additive mapping if and only if it has a Kusuoka representation of the form (6).

PROOF. According to Theorem 4.1, to prove part 1 it is sufficient to show that there exists a law invariant coherent risk measure ρ on the atomless space \mathcal{S} such that φ_ρ is an extension of φ .

For the family $\mathcal{Z} = \{Z \in \mathcal{L}^p(\mathcal{S}) \mid F_Z \in \mathcal{F}\}$ of random variables we have $\mathcal{F} = \{F_Z \mid Z \in \mathcal{Z}\}$. Let us now define the law invariant mapping $\rho : \mathcal{L}^p(\mathcal{S}) \rightarrow \overline{\mathbb{R}}$ by

$$\rho(V) = \sup \{\varphi(F_Z) \mid Z \in \mathcal{Z}, F_Z \geq F_V\} \quad \text{for all } V \in \mathcal{L}^p(\mathcal{S}). \quad (12)$$

We first show that $\varphi_\rho : \mathfrak{F}^p \rightarrow \overline{\mathbb{R}}$, defined by

$$\varphi_\rho(F_V) = \rho(V) \quad \text{for all } V \in \mathcal{L}^p,$$

is an extension of φ , i.e., that $\varphi = \varphi_\rho|_{\mathcal{F}}$ holds. If we consider an arbitrary function $F_V \in \mathcal{F}$ for some $V \in \mathcal{Z}$, we immediately have $\varphi_\rho(F_V) = \rho(V) = \sup \{\varphi(F_Z) \mid Z \in \mathcal{Z}, F_Z \geq F_V\} \geq \varphi(F_V)$. On the other hand, due to the monotone property of φ , the inequality $\varphi(F_Z) \leq \varphi(F_V)$ holds for any $Z \in \mathcal{Z}$, $F_Z \geq F_V$ featured in the above supremum, implying $\varphi_\rho(F_V) \leq \varphi(F_V)$.

We next prove that ρ is a coherent risk measure. It is easy to verify that ρ is normalized, positive homogeneous, and translation invariant. For example, the translation invariance of ρ follows from the next simple chain of equalities.

$$\rho(V + \lambda) = \sup \{\varphi(F_Z) \mid Z \in \mathcal{Z}, F_Z \geq F_{V+\lambda}\} \quad (13)$$

$$= \sup \{\varphi(F_{\bar{Z}+\lambda}) \mid \bar{Z} \in \mathcal{Z}, F_{\bar{Z}+\lambda} \geq F_{V+\lambda}\} \quad (14)$$

$$= \sup \{\varphi(F_{\bar{Z}}) + \lambda \mid \bar{Z} \in \mathcal{Z}, F_{\bar{Z}+\lambda} \geq F_{V+\lambda}\} \quad (15)$$

$$= \sup \{\varphi(F_{\bar{Z}}) \mid \bar{Z} \in \mathcal{Z}, F_{\bar{Z}} \geq F_V\} + \lambda \quad (16)$$

$$= \rho(V) + \lambda. \quad (17)$$

Here equalities (13) and (17) are provided by the definition of ρ . Equality (14) reflects the fact that, as \mathcal{F} is a closed family, for $Z = \bar{Z} + \lambda$ the conditions $Z \in \mathcal{Z}$ and $\bar{Z} \in \mathcal{Z}$ are equivalent. Equality (15) follows from the translation invariance of φ , while the trivial equivalence $F_{\bar{Z}+\lambda} \geq F_{V+\lambda} \Leftrightarrow F_{\bar{Z}} \geq F_V$ implies (16). To show that ρ is monotone it is sufficient to observe that if $V_1 \leq V_2$ holds for random variables $V_1, V_2 \in \mathcal{L}^p$, then the supremum that defines $\rho(V_1)$ in (12) is taken over a subset of the set featured in the supremum that defines $\rho(V_2)$. It remains to show that ρ is superadditive. Let us consider two random variables $V_1, V_2 \in \mathcal{L}^p$. According to Sklar's Theorem on copulas (see, e.g., Nelsen, 1999), there exist standard uniform random variables U_1 and U_2 on $(\Omega, \mathcal{A}, \Pi)$ for which $(F_{V_1}^{-1}(U_1), F_{V_2}^{-1}(U_2))$ has the same joint distribution as (V_1, V_2) . Introducing the notation $\bar{Z}_1 = F_{V_1}^{-1}(U_1)$, $\bar{Z}_2 = F_{V_2}^{-1}(U_2)$, we have the following chain of implications:

$$\begin{aligned} & Z_1, Z_2 \in \mathcal{Z}, \quad F_{Z_1} \geq F_{V_1}, \quad F_{Z_2} \geq F_{V_2} & (18) \\ \implies & Z_1, Z_2 \in \mathcal{Z}, \quad F_{Z_1}^{-1}(U_1) \leq F_{V_1}^{-1}(U_1), \quad F_{Z_2}^{-1}(U_2) \leq F_{V_2}^{-1}(U_2) & (19) \\ \implies & \bar{Z}_1, \bar{Z}_2 \in \mathcal{Z}, \quad \bar{Z}_1 \leq F_{V_1}^{-1}(U_1), \quad \bar{Z}_2 \leq F_{V_2}^{-1}(U_2) & (20) \\ \implies & \bar{Z}_1 + \bar{Z}_2 \in \mathcal{Z}, \quad \bar{Z}_1 + \bar{Z}_2 \leq F_{V_1}^{-1}(U_1) + F_{V_2}^{-1}(U_2) & (21) \\ \implies & \bar{Z}_1 + \bar{Z}_2 \in \mathcal{Z}, \quad F_{\bar{Z}_1 + \bar{Z}_2} \geq F_{V_1 + V_2}. & (22) \end{aligned}$$

Here (19) follows from the order-reversing property of the mapping $F \mapsto F^{-1}$. The definitions of Z_1, Z_2 and \mathcal{Z} , together with the relations $Z_1 \sim \bar{Z}_1, Z_2 \sim \bar{Z}_2$ immediately yield (20). To verify (21), we note that since \mathcal{F} is a closed family, \mathcal{Z} is closed under addition. Finally, (22) holds because $(F_{V_1}^{-1}(U_1), F_{V_2}^{-1}(U_2)) \sim (V_1, V_2)$ implies $F_{V_1}^{-1}(U_1) + F_{V_2}^{-1}(U_2) \sim V_1 + V_2$. The superadditivity of ρ , and thus part 1, now follows from the next chain of equalities.

$$\begin{aligned} \rho(V_1 + V_2) &= \sup \{ \varphi(F_Z) \mid Z \in \mathcal{Z}, F_Z \geq F_{V_1 + V_2} \} & (23) \\ &\geq \sup \{ \varphi(F_{\bar{Z}_1 + \bar{Z}_2}) \mid Z_1, Z_2 \in \mathcal{Z}, F_{Z_1} \geq F_{V_1}, F_{Z_2} \geq F_{V_2} \} & (24) \\ &\geq \sup \{ \varphi(F_{\bar{Z}_1}) + \varphi(F_{\bar{Z}_2}) \mid Z_1, Z_2 \in \mathcal{Z}, F_{Z_1} \geq F_{V_1}, F_{Z_2} \geq F_{V_2} \} & (25) \\ &= \sup \{ \varphi(F_{Z_1}) \mid Z_1 \in \mathcal{Z}, F_{Z_1} \geq F_{V_1} \} + \sup \{ \varphi(F_{Z_2}) \mid Z_2 \in \mathcal{Z}, F_{Z_2} \geq F_{V_2} \} & (26) \\ &= \rho(V_1) + \rho(V_2) & (27) \end{aligned}$$

Here (23) and (27) are again provided by the definition of ρ . The inequality (24) follows from the implication (18)-(22), while (25) is a consequence of the superadditivity of φ . Finally, the relations $Z_1 \sim \bar{Z}_1, Z_2 \sim \bar{Z}_2$ imply (26).

To prove the non-trivial implication in part 2, assume that φ is a coherent and comonotone additive mapping. Let us again define ρ as in (12), and note that according to part 1 it is a law invariant coherent risk measure. It only remains to show that ρ is comonotone additive. Let us consider two comonotone random variables $V_1, V_2 \in \mathcal{L}^p(\mathcal{S})$, and recall that there exists a standard uniform random variable U such that $(V_1, V_2) \sim (F_{V_1}^{(-1)}(U), F_{V_2}^{(-1)}(U))$ holds. Since the family \mathcal{F} is p -dense, there exist sequences $\{F_{i,n}\}_{n \in \mathbb{N}} \subset \mathcal{F}$ such that we have $\lim_{n \rightarrow \infty} \|F_{V_i}^{-1} - F_{i,n}^{-1}\|_p = 0$ for $i \in \{1, 2\}$. Introducing the notation $Z_{i,n} = F_{i,n}^{-1}(U)$, our

statement follows from the next chain of equalities.

$$\rho(V_1) + \rho(V_2) = \rho(F_{V_1}^{(-1)}(U)) + \rho(F_{V_2}^{(-1)}(U)) \quad (28)$$

$$= \lim_{n \rightarrow \infty} \rho(Z_{1,n}) + \rho(Z_{2,n}) \quad (29)$$

$$= \lim_{n \rightarrow \infty} \varphi(F_{Z_{1,n}}) + \varphi(F_{Z_{2,n}}) \quad (30)$$

$$= \lim_{n \rightarrow \infty} \varphi(F_{Z_{1,n} + Z_{2,n}}) \quad (31)$$

$$= \lim_{n \rightarrow \infty} \rho(Z_{1,n} + Z_{2,n}) \quad (32)$$

$$= \rho(F_{V_1}^{(-1)}(U) + F_{V_2}^{(-1)}(U)) \quad (33)$$

$$= \rho(F_{V_1 + V_2}^{(-1)}(U)) \quad (34)$$

$$= \rho(V_1 + V_2) \quad (35)$$

Equalities (28) and (35) hold due to the law invariance of ρ . We note that, as ρ is a coherent risk measure, it is continuous in the \mathcal{L}^p -norm (Ruszczyński and Shapiro, 2006). Then, if $p \in [1, \infty)$, for $i \in \{1, 2\}$ we have

$$\lim_{n \rightarrow \infty} \|F_{V_i}^{-1}(U) - Z_{i,n}\|_p = \lim_{n \rightarrow \infty} \|F_{V_i}^{-1}(U) - F_{i,n}^{-1}(U)\|_p = \lim_{n \rightarrow \infty} \left(\int_0^1 |F_{V_i}^{-1}(u) - F_{i,n}^{-1}(u)|^p \, du \right)^{\frac{1}{p}} \quad (36)$$

$$= \lim_{n \rightarrow \infty} \|F_{V_i}^{-1} - F_{i,n}^{-1}\|_p = 0. \quad (37)$$

Similarly, if $p = \infty$, we have

$$\lim_{n \rightarrow \infty} \|F_{V_i}^{-1}(U) - Z_{i,n}\|_\infty = \lim_{n \rightarrow \infty} \|F_{V_i}^{-1} - F_{i,n}^{-1}\|_\infty = 0. \quad (38)$$

Equality (29) immediately follows. By the definition of ρ we obtain (30) and (32), while (31) follows from the comonotone additivity of φ . Similarly to (36)-(38), for any $p \in [1, \infty]$ we have

$$\lim_{n \rightarrow \infty} \|F_{V_1}^{-1}(U) + F_{V_2}^{-1}(U) - (Z_{1,n} + Z_{2,n})\|_p = 0,$$

therefore equality (33) again follows from the fact that ρ is continuous in the p -norm. Finally, recalling (4), the quantile additivity of comonotone random variables yields (34). \square

For a random variable V on \mathcal{S} let $F_V^{-1}([0, 1]) = \{F_V^{-1}(u) \mid u \in [0, 1]\}$ denote the range of the quantile function F_V^{-1} , and let $\mathfrak{F}^D = \{F_V : V \in \mathcal{L}^1(\mathcal{S}), |F_V^{-1}([0, 1])| < \infty\}$ denote the family of CDFs of finite discrete distributions. As \mathfrak{F}^D is clearly a closed family, the next result is an easy consequence of part 1 of the above theorem.

COROLLARY 6.1 *Law invariant risk measures that can be coherently extended to a family containing all finite discrete distributions are functionally coherent, and thus have Kusuoka representations of the form (5). More precisely, consider a mapping $\rho : \mathcal{V}(\Omega, 2^\Omega, \Pi) \rightarrow \overline{\mathbb{R}}$ on a finite probability space. If there exists a coherent mapping $\varphi : \mathfrak{F}^D \rightarrow \overline{\mathbb{R}}$ such that $\varphi|_{\mathcal{F}(\Omega, 2^\Omega, \Pi)} = \varphi_\rho$ holds, then ρ is a functionally coherent risk measure.*

It is also easy to verify that \mathfrak{F}^d is a p -dense family for any $p \in [1, \infty]$. Indeed, consider an arbitrary CDF $F_V \in \mathfrak{F}^p$, for some $V \in \mathcal{L}^p(\mathcal{S})$, and define the discretized random variables $V_n = \frac{\lfloor nV \rfloor}{n}$. As the inequality $|V - V_n| < \frac{1}{n}$ holds everywhere, we have $\|F_V^{-1} - F_{V_n}^{-1}\|_p = \|V - V_n\|_p \leq \frac{1}{n}$ for every $n \in \mathbb{N}$. Therefore, part 2 of Theorem implies the next result.

COROLLARY 6.2 *Law invariant risk measures that can be extended to a family containing all finite discrete distributions in a coherent and comonotone additive fashion are functionally coherent and comonotone additive, and thus have Kusuoka representations of the form (6). More precisely, consider a mapping $\rho : \mathcal{V}(\Omega, 2^\Omega, \Pi) \rightarrow \overline{\mathbb{R}}$ on a finite probability space. If there exists a coherent and comonotone additive mapping $\varphi : \mathfrak{F}^D \rightarrow \overline{\mathbb{R}}$ such that $\varphi|_{\mathcal{F}(\Omega, 2^\Omega, \Pi)} = \varphi_\rho$ holds, then ρ is a functionally coherent and comonotone additive risk measure.*

References

- Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3):203–228.
- Bacharach, M. (1966). Matrix rounding problems. *Management Science*, 12(9):732–742.
- Bertsimas, D. and Brown, D. B. (2009). Constructing uncertainty sets for robust linear optimization. *Operations Research*, 57(6):1483–1495.
- Dentcheva, D., Penev, S., and Ruszczycki, A. (2010). Kusuoka representation of higher order dual risk measures. *Annals of Operations Research*, 181:325–335.
- Föllmer, H. and Schied, A. (2004). *Stochastic finance*. Number 27 in De Gruyter studies in mathematics. de Gruyter, Berlin, 2., rev. and extended edition.
- Jouini, E., Schachermayer, W., and Touzi, N. (2006). Law-invariant risk measures have the Fatou property. *Advances in Mathematical Economics*, 9(1):49–71.
- Kusuoka, S. (2001). On law invariant coherent risk measures. *Advances in Mathematical Economics*, 3:83–95.
- McNeil, A., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton Series in Finance. Princeton University Press.
- Nelsen, R. B. (1999). *An Introduction to Copulas*. Springer, New York.
- Pflug, G. C. (2000). Some remarks on the value-at-risk and the conditional value-at-risk. In Uryasev, S., editor, *Probabilistic Constrained Optimization: Methodology and Applications*. Kluwer Academic Publishers, Dordrecht.
- Pflug, G. C. and Römisch, W. (2007). *Modelling, managing and measuring risk*. World Scientific publishing, Singapore.

- Pichler, A. and Shapiro, A. (2012). Uniqueness of kusuoka representations. http://www.optimization-online.org/DB_FILE/2012/10/3660.pdf.
- Rockafellar, R. (2007). Coherent approaches to risk in optimization under uncertainty. In *Tutorials in Operations Research*, pages 38–61. INFORMS.
- Ruszczynski, A. and Shapiro, A. (2006). Optimization of risk measures. In Calafiore, G. and Dabbene, F., editors, *Probabilistic and Randomized Methods for Design under Uncertainty*, pages 119–157. Springer London.
- Shapiro, A. (2012). On kusuoka representation of law invariant risk measures. *Mathematics of Operations Research, Online First (November 2012)*. [dx.doi.org/10.1287/moor.1120.0563](https://doi.org/10.1287/moor.1120.0563).
- Shapiro, A., Dentcheva, D., and Ruszczyński, A. (2009). *Lectures on stochastic programming: modeling and theory*. The society for industrial and applied mathematics and the mathematical programming society, Philadelphia, USA.