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Graphs whose complement and square are isomorphic (extended version)

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Martin Milanič   Anders Sune Pedersen   Daniel Pellicer
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Abstract. We study square-complementary graphs, that is, graphs whose complement and square are isomorphic. We prove several necessary conditions for a graph to be square-complementary, describe ways of building new square-complementary graphs from existing ones, construct infinite families of square-complementary graphs, and characterize square-complementary graphs within various graph classes. The bipartite case turns out to be of particular interest.

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1 Introduction

A graph is self-complementary (s.c.) if it is isomorphic to its complement. The study of self-complementary graphs was initiated by Sachs ([27]) and later, but independently, by Ringel ([26]). Self-complementary graphs have been studied extensively in the literature. There exist small self-complementary graphs, for example the one-vertex graph K_1 (which we will also call the *singleton*), the 4-vertex path P_4, and the 5-vertex cycle C_5. Moreover, it is known that there exists a self-complementary graph on n vertices if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

Given two graphs G and H, we say that G is the *square* of H (and denote this by $G = H^2$) if their vertex sets coincide and two distinct vertices x, y are adjacent in G if and only if x, y are at distance (the number of edges on a shortest path connecting them) at most two in H. A number of results about squares of graphs and their properties are available in the literature (see, e.g., Section 10.6 in the monograph [4]).

The aim of this paper is to introduce and study the following notion, similar to that of self-complementary graphs.

Definition 1. *A graph G is said to be square-complementary (squco for short) if its square is isomorphic to its complement. That is, $G^2 \cong \overline{G}$, or, equivalently, $G \cong \overline{G^2}$.*

Trivially, the singleton K_1 is squco. There also exist nontrivial squco graphs:

Observation 1.1. *The 7-cycle C_7 is a squco graph.*

We will prove in Section 5 that C_7 is the unique smallest nontrivial squco graph. Notice that if G is self-complementary then its complement $\overline{G}$ is also self-complementary, but if G is square-complementary then $\overline{G}$ need not be square-complementary. For instance, C_7 is squco but $\overline{C_7}$ is not squco.

Our next example of a squco graph is a cubic vertex-transitive bipartite graph on 12 vertices, known as the Franklin graph. It is depicted (in three different ways) in Figure 1.

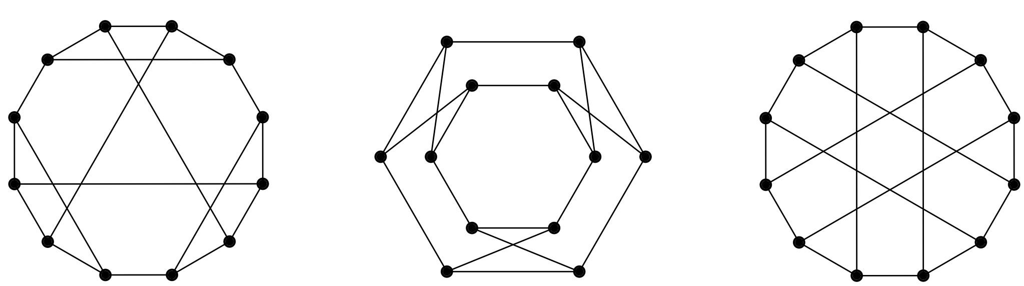

Figure 1: Three drawings of the Franklin graph.

Observation 1.2. *The Franklin graph is a squco graph.*

While Observation 1.2 can be verified directly, a proof of it will be given in a much more general context in Section 6 (cf. Theorem 6.2, Theorem 6.8 and comments following Lemma 6.9).

Motivated by the interesting notion of self-complementary graphs and by the examples given above, we initiate in this paper the study of square-complementary graphs. After giving the necessary definitions used throughout the paper in Section 2, we describe in Section 3 a way of building new squco graphs from existing ones, allowing us to construct infinite families of squco graphs. In Section 4 we prove several necessary conditions for graphs to be squco, both in terms of connectivity conditions and in terms of various graph invariants such as the radius, the diameter, the girth, the maximum degree and the independence number. Section 5 is devoted to characterizations of squco graphs in particular graph classes. We consider several subclasses of perfect graphs, graphs of independence number at most three, graphs of maximum degree at most three, and graphs on at most 11 vertices. A complete characterization of natural numbers n for which there exists a square-complementary graph of order n is given. In Section 6, we give a characterization of bipartite squco graphs, and exemplify it with a construction (based on bipartite bi-Cayley graphs over cyclic groups) of an infinite family of bipartite vertex-transitive squco graphs generalizing the Franklin graph. We conclude the paper with a discussion on possible generalizations and open problems in Section 7.

2 Notation and Definitions

Basic graph theoretic notions. Unless stated otherwise, all graphs considered in the paper will be finite, simple and undirected. Let G be a graph. A *k-vertex* (of G) is a vertex of degree k in G. An *n-vertex graph* is a graph of order n, that is, a graph on exactly n vertices. We denote by $n(G)$ the number of vertices of G and by $m(G)$ the number of its edges. Given a vertex v in a graph G, we denote by $\deg(v, G)$ its *degree*, that is, the size of its neighborhood $N_G(v) := \{u \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of v is the set $N_G[v] := N_G(v) \cup \{v\}$. By $\Delta(G)$ (resp. $\delta(G)$) we denote the maximum (resp. minimum) degree of a vertex in G. For two vertices u, v in a graph G, we denote by $d_G(u, v)$ the *distance between u and v*, that is, the number of edges on a shortest path connecting u and v; if there is no path connecting the two vertices, then the distance is defined to be infinite. For a positive integer i, we denote by $N_i(v, G)$ the set of all vertices u in G such that $d_G(u, v) = i$, and by $N_{\geq i}(v, G)$ the set of all vertices u in G such that $d_G(u, v) \geq i$. By $B_i(v, G)$ (or just $B_i(v)$ if the graph is clear from the context), we denote the ball of radius i centered at v, that is, the set of all vertices u in G such that $d_G(u, v) \leq i$. The *eccentricity* $\mathrm{ecc}_G(u)$ of a vertex u in a graph G is maximum of the numbers $d_G(u, v)$ where $v \in V(G)$. The *radius* of a graph G, denoted $\mathrm{radius}(G)$, is the minimum of the eccentricities of the vertices of G. The *diameter* of a graph G, denoted $\mathrm{diam}(G)$, is the maximum of the eccentricities of the vertices of G, or, equivalently, the maximum distance between any two vertices in G. The *girth* of a graph G, denoted $\mathrm{girth}(G)$, is the length of a shortest cycle in G (or infinity, if G has no cycles).

Given two graphs G and H, an *isomorphism between G and H* is a bijective mapping $\varphi : V(G) \to V(H)$ such that for every two vertices $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$. If there exists an isomorphism between graphs G and H, we say that G and H are *isomorphic*, and denote this relation by $G \cong H$. An *automorphism* of a graph G is an isomorphism between G and itself. The automorphism group of G is said to *act transitively* on a subset $S \subseteq V(G)$ if for every two vertices $u, v \in S$ there exists an automorphism φ of G such that $\varphi(u) = v$. A graph G is said to be *vertex-transitive* if its automorphism group acts transitively on $V(G)$. The *complement* of a graph G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$, in which two distinct vertices are adjacent if and only if they are not adjacent in G. An *independent* (or *stable*) set in a graph G is a set of pairwise non-adjacent vertices. The maximum size of an independent set in G is called the *independence* (or *stability*) *number* of G and is denoted by $\alpha(G)$. An *clique* in a graph G is a set of pairwise adjacent vertices. A *block* of G is a maximal subgraph of G without cut vertices. A graph with at least three vertices is 2-*connected* if it has no cut vertices. For a graph H, we say that a graph G is *H-free* if no induced subgraph of G is isomorphic to H. The $(H_1, \ldots, H_k)$-free graphs are defined similarly. As usual, P_n, C_n and K_n denote the path, the cycle, and the complete graph on n vertices, respectively, and $2K_2$ denotes the disjoint union of two K_2's. For other graph theoretical terminology, we refer the reader to [7].

Graph classes. A graph is a *cograph* if it can be reduced to an edgeless graph by repeatedly taking complements within components. Corneil, Lerchs, and Burlingham ([6]) proved that the cographs are exactly the P_4-free graphs. A graph is *chordal* if it is C_k-free for each $k \geq 4$. We will also consider three particular subclasses of chordal graphs: split graphs, interval graphs and block graphs. A graph is *split* if its vertex set can be partitioned into two disjoint (possibly empty) sets K and I such that K is a clique and I is independent. It has been shown by Földes and Hammer that split graphs are precisely the $(C_4, C_5, 2K_2)$-free graphs ([11]). Consequently, a graph is split if and only if both G and its complement $\overline{G}$ are chordal. A graph is a *interval graph* if it is the intersection graph of closed intervals on the real line. It is easy to see that every interval graph is chordal. A graph is a *block graph* if every block of it is complete. For more details on graph classes, see [4].

3 Examples and constructions

In this section, we provide some further examples of squco graphs, and discuss ways of building new squco graphs from known ones. In particular, this will imply the existence of arbitrarily large squco graphs.

As noted already in the introduction, the 7-cycle is a squco graph. It is also a *circulant*, that is, a Cayley graph over a cyclic group. Further examples of squco graphs can be found among circulants. For an integer $n \geq 3$ and $D \subseteq [\lfloor \frac{n}{2} \rfloor] := \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, the *circulant graph* $C_n(D)$ is the graph with vertex set $[0, n-1] := \{0, 1, \ldots, n-1\}$ in which two vertices $i, j \in [0, n-1]$ are adjacent if and only if $|i - j| \in D$ or $n - |i - j| \in D$. In this notation, the 7-cycle C_7 is the circulant graph $C_7(\{1\})$.

A computer search revealed the following small pairwise non-isomorphic squco circulants:

$$C_{17}(\{1,4\})\,,\quad C_{24}(\{1,3,11\})\,,\quad C_{40}(\{1,3,5,9,17\})\quad C_{40}(\{1,3,5,13,19\})$$

$$C_{40}(\{1,3,5,17,19\})\,,\quad C_{40}(\{1,5,9,11,19\})\quad C_{41}(\{4,5,8,10\})\,.$$

Proofs that these graphs are squco can be found in A.

Given an n-vertex graph G with vertices labelled $v_1,\dots,v_n$ and positive integers $k_1,\dots,k_n$, we denote by $G[k_1,k_2,\dots,k_n]$ the graph obtained from G by replacing each vertex v_i of G with a set U_i of k_i (new) vertices and joining vertices $u_i \in U_i$ and $u_j \in U_j$ with an edge if and only if v_i and v_j are adjacent in G. If $k_1 = \cdots = k_n = k$, then we write $G[k]$ instead of $G[k_1,\dots,k_n]$.

Proposition 3.1. *For every nontrivial squco graph G and positive integer k, the graph $G[k]$ is a squco graph.*

Proof. Let G denote a nontrivial squco n-vertex graph. First, we observe that G does not have any isolated vertices. Indeed, if v is an isolated vertex in G, then v would be adjacent to all other vertices in $\overline{G^2}$, implying that $\overline{G^2}$ would have no isolated vertices, contradicting $G \cong \overline{G^2}$. With notation as above, let the vertices of U_i for each $v_i \in V(G)$, be labelled $u_i^1, u_i^2, \dots, u_i^k$. Let $\varphi : V(G) \to V(G)$ denote an isomorphism from G to $\overline{G^2}$. We extend φ to a function $\psi : V(G[k]) \to V(G[k])$ as follows: suppose φ maps v_i to v_j, then ψ maps u_i^ℓ to u_j^ℓ for each $\ell \in [k] := \{1,\dots,k\}$. Of course, ψ is an bijection of $V(G[k])$ onto itself since φ is a bijection of $V(G)$ onto itself. In order for ψ to be an isomorphism between $G[k]$ and $\overline{G[k]^2}$, it must be the case that u_i^p and u_j^q are adjacent in $G[k]$ if and only if $\psi(u_i^p)$ and $\psi(u_j^q)$ are adjacent in $\overline{G[k]^2}$. Consider two vertices $u_i^p, u_j^q \in V(G[k])$ and set $v_r = \varphi(v_i)$ and $v_s = \varphi(v_j)$. Then, we have the following chain of equivalencies:

$$\begin{aligned}
& u_i^p \text{ and } u_j^q \text{ are adjacent in } G[k]\\
\iff\; & v_i \text{ and } v_j \text{ are adjacent in } G\\
\iff\; & \varphi(v_i) \text{ and } \varphi(v_j) \text{ are adjacent in } \overline{G^2}\\
\iff\; & v_r \text{ and } v_s \text{ are not within distance 2 of each other in } G\\
\overset{*}{\iff}\; & u_r^p \text{ and } u_s^q \text{ are not within distance 2 of each other in } G[k]\\
\iff\; & \psi(u_i^p) \text{ and } \psi(u_j^q) \text{ are not within distance 2 of each other in } G[k]\\
\iff\; & \psi(u_i^p) \text{ and } \psi(u_j^q) \text{ are adjacent in } \overline{G[k]^2}\,.
\end{aligned}$$

Let us justify the fourth equivalence $\overset{*}{\iff}$; all the other ones are straightforward. Suppose first that v_r and v_s are within distance 2 of each other in G. We distinguish two cases. If $r \neq s$ then u_r^p and u_s^q are within distance 2 of each other in $G[k]$ by construction. If $r = s$ then $u_r^p, u_s^q \in U_r$. Since G has no isolated vertices, vertex v_r has a neighbor in G, say v_t. Vertex u_t^1 is a common neighbor in $G[k]$ of u_r^p and u_s^q, implying that u_r^p and u_s^q are within distance 2 of each other in $G[k]$. For the converse direction, suppose that u_r^p and u_s^q are within distance 2 of each other in $G[k]$. If $r = s$ then there is nothing to show. Otherwise, any shortest path connecting u_r^p and u_s^q in $G[k]$ can be mapped to an equally long path connecting v_r and v_s in G. This proves $\overset{*}{\iff}$ and shows that ψ is an isomorphism between $G[k]$ and $\overline{G[k]^2}$. □

There is nothing in the above argument which requires the sets $U_1, \ldots, U_n$ to be finite, it suffices that they have the same cardinality. Hence there exist infinite square-complementary graphs. However, there are no locally finite infinite squco graphs, since each vertex in the complement of the square of a locally finite infinite graph has infinite degree.

In the particular case when G is the 7-cycle, in order to obtain a squco graph it is not necessary that all the 7 sizes of the sets U_i are the same. Denoting the 7 vertices of the C_7 in the cyclic order $v_1, \ldots, v_7$, it suffices that the sizes $k_1, \ldots, k_7$ of the sets $U_1, \ldots, U_7$ satisfy the conditions $k_2 = k_3 = k_5$ and $k_4 = k_6 = k_7$.

Proposition 3.2. *For positive integers k_1, k_2, k_3, the graph $C_7[k_1, k_2, k_2, k_3, k_2, k_3, k_3]$ is a squco graph.*

Proof. Let $G := C_7[k_1, k_2, k_2, k_3, k_2, k_3, k_3]$ and denote by $U_1, \ldots, U_7$ the corresponding seven sets of vertices partitioning $V(G)$ (in particular, $|U_1| = k_1$, $|U_2| = |U_3| = |U_5| = k_2$ and $|U_4| = |U_6| = |U_7| = k_3$). Fixing bijections ψ_{ij} between U_i and U_j for all $(i, j) \in \{(2, 5), (3, 2), (4, 6), (5, 3), (6, 7), (7, 4)\}$, it is a matter of routine verification to check that an isomorphism between G and $\overline{G^2}$ is given by

$$\varphi(v) = \begin{cases} v, & \text{if } v \in U_1; \\ \psi_{25}(v), & \text{if } v \in U_2; \\ \psi_{32}(v), & \text{if } v \in U_3; \\ \psi_{46}(v), & \text{if } v \in U_4; \\ \psi_{53}(v), & \text{if } v \in U_5; \\ \psi_{67}(v), & \text{if } v \in U_6; \\ \psi_{74}(v), & \text{if } v \in U_7. \end{cases} .$$

□

Corollary 3.3. *For every integer $n \geq 7$, there exists a squco graph with n vertices.*

Proof. Apply Proposition 3.2 with $k_1 = n - 6$ and $k_2 = k_3 = 1$. □

The following proposition gives a sufficient condition for a squco graph G to be extendable to a larger squco graph by adding to it one more vertex.

Proposition 3.4. *Let G be a graph. If there is an isomorhism φ from G to $\overline{G^2}$ and a subset S of $V(G)$ such that*

(i) *$d_G(u, v) \leq 2$ for every $u, v \in S$ and*

(ii) *$\varphi(S) = V(G) \setminus N_G[S]$,* *then the graph H obtained from G by adding a new vertex w and joining w to every vertex in S is a squco graph and the mapping $\psi : V(H) \to V(H)$ defined as*

$$\psi(x) = \begin{cases} \varphi(x) & \text{if } x \in V(G); \\ x & \text{if } x = w. \end{cases}$$

is an isomorphism from H to $\overline{H^2}$.

Proof. Let G be a graph, and suppose that there is a mapping φ and a set S as described in the hypothesis of the desired result. Moreover, let H and ψ be as described above. Of course, ψ is a bijection of $V(H)$, and so we just need to show that, for every $x, y \in V(H)$,

($\star$) x and y are adjacent in H if and only if $\psi(x)$ and $\psi(y)$ are adjacent in $\overline{H^2}$.

First, observe that by (i) the subgraph of $\overline{H^2}$ induced by $V(G)$ is identical to $\overline{G^2}$. Hence, ($\star$) holds if both x and y are in $V(G)$. Thus, we may, w.l.o.g., assume $x = w$.

Now, we distinguish between two cases. In each case we use the fact that w is mapped to itself by ψ.

Suppose that w and y are adjacent in H, that is, $y \in S$. Then we need to prove that $\psi(w)$ and $\psi(y)$ are adjacent in $\overline{H^2}$. Now, by (ii), $\psi(y) \in V(G) \setminus N_G[S]$, in particular, $\psi(y)$ is not within distance 2 of $\psi(w)$ in H, and so, as desired, $\psi(w)$ and $\psi(y)$ are adjacent in $\overline{H^2}$.

Suppose that w and y are not adjacent in H, that is, $y \in V(G) \setminus S$. Then, by (ii), $\psi(y) \in N_G[S]$, in particular, $\psi(y)$ is within distance 2 of $\psi(w)$ in H, and so, as desired, $\psi(w)$ and $\psi(y)$ are not adjacent in $\overline{H^2}$. □

Example 1. *Let G denote the Franklin graph with vertices labelled as indicated in Figure 2.*

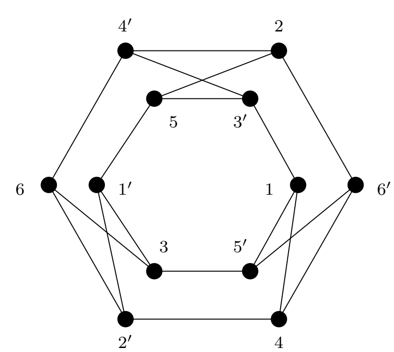

Figure 2: The Franklin graph with a labelling of the vertices.

Now the bijection φ on $V(G)$ defined in Table 1 is easily verified to be an isomorphism between G and $\overline{G^2}$. Moreover, the hypothesis of Proposition 3.4 holds with $S = \{3', 4, 6\}$. Note that (i) $d_G(u, v) \le 2$ for every $u, v \in S$ and (ii) $V(G) \setminus N_G[S] = \{1', 2, 5'\} = \varphi(S)$. Thus, by Proposition 3.4, the graph depicted in Figure 3 is a squco graph.

x	1	2	3	4	5	6	$1'$	$2'$	$3'$	$4'$	$5'$	$6'$
$\varphi(x)$	1	$3'$	5	$1'$	3	$5'$	4	$4'$	2	$2'$	6	$6'$

Table 1: An isomorphism between G and $\overline{G^2}$.

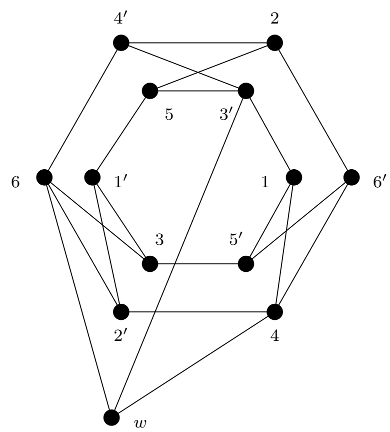

Figure 3: A supergraph of Franklin graph which, by Proposition 3.4, is a squco graph. Note that the supergraph was not obtained by simply copying vertices of the Franklin graph.

We now derive some further consequences of Proposition 3.4. Given a graph G with vertices labelled $v_1, \ldots, v_n$ and a positive integer $k \geq 1$, a *k-inflation* of v_1 is the graph $G[k, 1, \ldots, 1]$, denoted in what follows also by $\mathit{inf}(G, k, v_1)$.

Proposition 3.5. *Let G be a nontrivial squco graph, and let $\varphi : V(G) \to V(G)$ be an isomorphism from G to $\overline{G^2}$ with a fixed point, say $\varphi(v) = v$. Then, any k-inflation $\mathit{inf}(G, k, v)$ of v is a squco graph, and there exists an isomorphism from $\mathit{inf}(G, k, v)$ to the complement of its square having each copy of v as a fixed point.*

Proof. Induction on k shows that it suffices to prove the claim for the case $k = 2$. We will apply Proposition 3.4 to the set $S := N_G(v)$. By construction, property (i) from Proposition 3.4 holds for S. Moreover, since $\varphi(v) = v$ and φ is an isomorphism from G to $\overline{G^2}$, it follows that

$$\varphi(S) = \varphi(N_G(v)) = N_{\overline{G^2}}(\varphi(v)) = N_{\overline{G^2}}(v) = N_{\geq 3}(v, G) = V(G) \setminus N_G[S]\,,$$

hence condition (ii) from Proposition 3.4 is also satisfied for S. Notice that the assumption that G is nontrivial is needed for the last equality in the above chain, $N_{\geq 3}(v, G) = V(G) \setminus N_G[S]$, which only holds true if S is non-empty.

Let H denote the 2-inflation $\mathit{inf}(G, 2, v)$ obtained from G by adding a new vertex w and joining w to every vertex in S. By Proposition 3.4, H is a squco graph and the mapping

$\psi : V(H) \to V(H)$ defined as

$$\psi(x) = \begin{cases} \varphi(x) & \text{if } x \in V(G); \\ x & \text{if } x = w. \end{cases}$$

is an isomorphism from H to $\overline{H^2}$ having both copies of v as a fixed point. □

Corollary 3.6. *Let G be a nontrivial vertex-transitive squco graph and v a vertex of G. Then, for every $k \geq 1$, the k-inflation of v results in a squco graph.*

Proof. Let G be a nontrivial vertex-transitive squco graph, and let $v \in V(G)$ be arbitrary. Fix an isomorphism $\varphi : V(G) \to V(G)$ from G to $\overline{G^2}$. Let $w = \varphi(v)$ be the image of v under φ. Since G is vertex-transitive, so is $\overline{G^2}$, and there exists an automorphism ψ of $\overline{G^2}$ such that $\psi(w) = v$. Then, the composed mapping $\psi \circ \varphi : V(G) \to V(G)$ is an isomorphism from G to $\overline{G^2}$ such that $(\psi \circ \varphi)(v) = v$, and so the result follows from Proposition 3.5. □

4 Properties of squco graphs

In this section we reveal several necessary conditions that every square-complementary graph must satisfy. We start with some connectivity- and distance-related conditions.

Proposition 4.1. *Every squco graph G is connected and its complement $\overline{G}$ is connected.*

Proof. Let G denote a squco graph.

Suppose that G is disconnected, and let C denote a connected component of G. Then every vertex of $V(C)$ is adjacent to every vertex of $V(G) \setminus V(C)$ in $\overline{G^2}$, and so, in particular, $\overline{G^2}$ is connected, which contradicts the fact that G and $\overline{G^2}$ are isomorphic.

On the other hand, if $\overline{G}$ is disconnected, then G has at least two vertices, G^2 is a complete graph and hence $\overline{G^2}$ is the edgeless graph, a contradiction with the fact that G is connected. □

The following two propositions describe some distance-related restrictions that every squco graph must satisfy. They will be used extensively in the rest of the paper.

Proposition 4.2. *Let G be a squco graph. For every non-empty proper subset S of $V(G)$, there is a vertex $u \in S$ and a vertex $v \in V \setminus S$ such that $d_G(u, v) \geq 3$.*

Proof. Let G denote a squco graph, and let S denote a non-empty proper subset of $V(G)$. Suppose that there is no pair (u, v) of vertices with $u \in S$ and $v \in V(G) \setminus S$ such that $d_G(u, v) \geq 3$. Then every vertex of S is adjacent to every vertex in $V(G) \setminus S$ in G^2, and so no component of $\overline{G^2}$ intersects S and $V(G) \setminus S$, in particular, $\overline{G^2}$ is disconnected. This contradicts Proposition 4.1, and so the desired result follows. □

Proposition 4.3. *If G is a nontrivial squco graph, then*

$$3 \leq \text{radius}(G) \leq \text{diam}(G) \leq 4\,.$$

Proof. Let G be a nontrivial squco graph. For every $v \in V(G)$, applying Proposition 4.2 to the set $S = \{v\}$ we see that $\text{ecc}_G(v) \geq 3$. Hence, G is of radius at least 3. For every graph G it holds that its radius is a lower bound on its diameter. Hence, it only remains to show that the diameter of every squco graph is at most 4.

Suppose for a contradiction that G is a squco graph with diameter at least 5. Let u and v be two vertices at distance 5 in G. We will verify that the eccentricity of u in $\overline{G^2}$ is at most 2. First, every vertex $x \in V(G)$ such that $d_G(u,x) \geq 3$ is adjacent to u in $\overline{G^2}$. Second, if $x \in V(G)$ is a vertex at distance 1 or 2 from u in G, then v is a common neighbor of u and x in $\overline{G^2}$. Consequently, $d_{\overline{G^2}}(x,u) = 2$. This contradicts the fact that radius of G is at least 3. □

As there exist (arbitrarily large) s.c. graphs with a cut vertex ([15]), a natural question arises whether it is possible to construct squco graphs with a cut vertex. It turns out that this is not the case.

Theorem 4.4. *If G is a squco graph, then G has no cut vertex.*

Proof. Suppose for a contradiction that a squco graph G has a cut vertex x. By Proposition 4.3, there exists a vertex, say y, such that $d_G(x,y) > 2$. Let C be the connected component of $G - x$ containing y, let $A = N_G(x) \cap C$, let $R = V(G) \setminus (C \cup \{x\})$, and let $B = R \cap N_G(x)$. Then $B \neq \emptyset$ since x is a cut vertex.

Suppose first that $R = B$. Choose a vertex $b \in B$. We claim that the eccentricity of b in the graph $\overline{G^2}$ is at most 2. Equivalently, we will verify that in the graph G^2, every vertex v other than b is either non-adjacent to b or has a common non-neighbor with b. Let $v \in V(G) \setminus \{b\}$. If $v \in R \cup \{x\}$, then y is a common non-neighbor of v and b in G^2. If $v \in C \setminus A$, then v is non-adjacent to b in G^2. Finally, if $v \in A$, then, since the eccentricity of v is at least 3 in G, there is a vertex, say c, such that $d_G(v,c) \geq 3$. It is easy to see that $c \in C \setminus A$, hence c is a common non-neighbor of v and b in G^2. Since the eccentricity of b in the graph $\overline{G^2}$ is at most 2, Proposition 4.3 implies that $\overline{G^2} \not\cong G$. This contradiction shows that $B \neq R$, that is, there exists a vertex $r \in R \setminus B$.

But now, we can show that the eccentricity of y in the graph $\overline{G^2}$ is at most 2. We claim that in the graph G^2, every vertex v other than y is either non-adjacent to y or has a common non-neighbor with y. Let $v \in V(G) \setminus \{y\}$. If $v \in R \cup \{x\}$, then v is non-adjacent to y. If $v \in C$, then r is a common non-neighbor of v and y in G^2. Hence, the radius of $\overline{G^2}$ is at most 2, and therefore $\overline{G^2} \not\cong G$ by Proposition 4.3. The proof is now complete. □

A classic result by Fleischner ([9]) (see also [24]) states that the square of every 2-connected graph has a Hamiltonian cycle. That result and Theorem 4.4 imply the following.

Corollary 4.5. *For every nontrivial squco graph G, the graph $\overline{G}$ has a Hamiltonian cycle.*

On the other hand, a nontrivial squco graph G may fail to possess a Hamiltonian cycle. For example, the graph $C_7[2, 1, 1, 1, 1, 1, 1]$ is squco (by Proposition 3.2), but since it has two vertices of degree two with common neighbors, it does not possess a Hamiltonian cycle.

Lemma 4.6. *If G is a nontrivial squco graph, then $\delta(G) \geq 2$.*

Proof. This follows from Theorem 4.4 and the observation that every connected nontrivial graph G with minimum degree less than 2 either has a cut vertex, or is K_2. In either case, G cannot be squco. □

Now, we turn our attention to conditions on the girth.

Proposition 4.7. *If G is a nontrivial squco graph with girth$(G) \geq 7$, then G is the 7-cycle.*

Proof. Let G be a nontrivial squco graph with girth$(G) \geq 7$. Since G is squco, we have $G \cong \overline{G^2}$ and hence there exists a vertex v which has degree $\delta(G)$ in $\overline{G^2}$, in other words, $|N_{\geq 3}(v, G)| = \delta(G)$.

Since girth$(G) \geq 7$, every vertex in $N_{\geq 3}(v, G)$ is adjacent to at most one vertex not in $N_{\geq 3}(v, G)$. Since $\delta(G)$ is the minimal degree of G, it follows that the induced graph on $N_{\geq 3}(v, G)$ in G is a complete graph. On the other hand, G is triangle-free and hence $|N_{\geq 3}(v, G)| \leq 2$. By Lemma 4.6, we have $\delta(G) \geq 2$ and hence $\delta(G) = |N_{\geq 3}(v, G)| = 2$. It follows that $|N_4(v, G)| = 0$ and $|N_3(v, G)| = 2$. Write $N_3(v, G) = \{u_3, v_3\}$.

Since girth$(G) \geq 7$, u_3 is adjacent to a unique vertex from $N_2(v, G)$, say u_2. Similarly, v_3 is adjacent to a unique vertex from $N_2(v, G)$, say v_2. Recall that u_3 and v_3 are adjacent and hence $u_2 \neq v_2$, otherwise (u_2, u_3, v_3) forms a 3-cycle. Our assumption also implies that each vertex in $N_2(v, G)$ is adjacent to a unique vertex from $B_2(v, G)$. Since $\delta(G) = 2$, it follows that $N_2(v, G) = \{u_2, v_2\}$.

Since girth$(G) \geq 7$, u_2 is adjacent to a unique vertex from $N_1(v, G)$, say u_1. Similarly, v_2 is adjacent to a unique vertex from $N_1(v, G)$, say v_1. Clearly, $u_1 \neq v_1$, otherwise $(u_1, u_2, u_3, v_3, v_2)$ is a 5-cycle. Our assumption also implies that each vertex in $N_1(v, G)$ is adjacent to a unique vertex from $B_1(v, G)$. Since $\delta(G) = 2$, it follows that $N_1(v, G) = \{u_1, v_1\}$. This shows that G is a 7-cycle, concluding the proof. □

Proposition 4.8. *If G is a regular squco graph, then it cannot have girth 6.*

Proof. Let G be a regular squco graph with girth 6. Let k be the degree of G and let v be a vertex of G. Since girth$(G) = 6$, we have that $|N_2(v, G)| = k(k-1)$ and each vertex in $N_2(v, G)$ is adjacent to a unique vertex in $B_2(v, G)$. It follows that the number of edges from $N_2(v, G)$ to $N_3(v, G)$ is $k(k-1)^2$.

On the other hand, since G is squco, we have $G \cong \overline{G^2}$ and hence $|N_{\geq 3}(v, G)| = k$. It follows that the number of edges from $N_2(v, G)$ to $N_3(v, G)$ is at most k^2 and hence $k(k-1)^2 \leq k^2$. Therefore $k \leq 2$ and Proposition 4.6 implies that $k = 2$. By Proposition 4.1, G is a cycle, and since G has girth 6, G must be the 6-cycle, which can be easily seen to be non-isomorphic to the complement of its square. □

The following result describes a necessary condition for squco graphs of girth at least 4.

Observation 4.9. *If G is a triangle-free squco graph, then G does not contain three distinct vertices u, v, and w such that the closed neighborhoods $N_G[u]$, $N_G[v]$, and $N_G[w]$ are mutually disjoint.*

Proof. Suppose that G is a triangle-free squco graph with three distinct vertices u, v, and w such that the closed neighborhoods $N_G[u]$, $N_G[v]$, and $N_G[w]$ are mutually disjoint. Then the distance between each of pair of vertices from $\{u, v, w\}$ is at least 3, and so the vertices u, v, and w are mutually non-adjacent in G^2. This means that $\{u, v, w\}$ induce a triangle in $\overline{G^2}$, and so we have a contradiction to the fact that G and $\overline{G^2}$ are isomorphic where G is triangle-free. This proves the desired result. □

The following simple observation (that will be used in Section 5) shows that the independence number of every squco graph is bounded from below by its maximum degree plus one.

Proposition 4.10. *Every squco graph G satisfies $\alpha(G) > \Delta(G)$.*

Proof. Let $v \in V(G)$ be a vertex in G of maximum degree. Then $S := N_G[v]$ induces a complete graph in G^2. Consequently, S is an independent set in $\overline{G^2}$. The conclusion follows since $\overline{G^2} \cong G$ and the independence number of a graph is preserved under isomorphism. □

This is not the first time that the independence number has been studied in the context of graph squares, see, e.g., [16].

We now show that in order to obtain arbitrarily large squco graphs, the maximum degree must also be unbounded.

Proposition 4.11. *If G is a squco graph, then*

$$n(G) \leq \Delta(G)(\delta(G) + 1) + 1 \tag{1}$$

with equality only if the following hold for each vertex v of minimum degree in G.

(i) *All neighbors of v have maximum degree, and*

(ii) *the vertex v is not contained in any triangles or quadrangles.*

Proof. Let G denote a squco graph, and let v denote a vertex of G of minimum degree. Now, G contains at most $\delta(G)(\Delta(G) - 1)$ $(N(v, G), N_2(v, G))$-edges, and, since each vertex of $N_2(v, G)$ has at least one neighbor in $N(v, G)$, it follows that $N_2(v, G)$ contains at most $\delta(G)(\Delta(G) - 1)$ vertices. Thus, we have proved that $V(G) \setminus B_2(v, G)$ contains at least

$$n(G) - \delta(G)(\Delta(G) - 1) - \delta(G) - 1$$

vertices. Since, in $\overline{G^2}$, v is adjacent to every vertex of $V(G) \setminus B_2(v)$, we have

$$\deg(v, \overline{G^2}) \geq n(G) - \delta(G)(\Delta(G) - 1) - \delta(G) - 1 \tag{2}$$

Since $\Delta(G) = \Delta(\overline{G^2}) \geq \deg(v, \overline{G^2})$, the inequality (1) now follows directly from (2). By the argument above it easily follows that (i) and (ii) of the proposition must hold if equality is attained in (1). □

Finally, we observe that, given a squco graph G, no other squco graphs can be obtained from it by either adding or deleting some edges.

Proposition 4.12. *If G is a squco graph, then no proper spanning subgraph of G is a squco graph.*

Proof. Suppose that G and H are squco graphs, and H is a proper spanning subgraph of G. Then $m(G^2) \geq m(H^2)$, and so, since $n(G) = n(H)$, $m(\overline{G^2}) \leq m(\overline{H^2})$. This along with the fact that both G and H are squco graphs implies

$$m(G) = m(\overline{G^2}) \leq m(\overline{H^2}) = m(H)$$

which contradicts the fact that H has strictly fewer edges than G. This contradiction establishes the desired result. □

The simple idea of the proof of Proposition 4.12 has some interesting implications.

Proposition 4.13. *Let G and H be two squco graphs.*
If $n(G) = n(H)$ and $m(G) > m(H)$, then $m(G^2) < m(H^2)$.
If $n(G) = n(H)$ and $m(G) = m(H)$, then $m(G^2) = m(H^2)$.

Proof. Suppose that G and H are squco graphs with $n(G) = n(H)$ and $m(G) > m(H)$ but $m(G^2) \geq m(H^2)$. Then $m(\overline{G^2}) \leq m(\overline{H^2})$. This along with the fact that both G and H are squco graphs implies

$$m(G) = m(\overline{G^2}) \leq m(\overline{H^2}) = m(H)$$

which contradicts the fact that $m(G) > m(H)$. This contradiction establishes the first part of the proposition.

To show the second part, suppose that G and H are squco graphs with $n(G) = n(H)$ and $m(G) = m(H)$. By symmetry, it suffices to consider the case $m(G^2) \geq m(H^2)$. Then, since $n(G) = n(H)$, $m(\overline{G^2}) \leq m(\overline{H^2})$. This along with the fact that both G and H are squco graphs implies

$$m(G) = m(\overline{G^2}) \leq m(\overline{H^2}) = m(H).$$

Since $m(G) = m(H)$ we have that $m(\overline{G^2}) = m(\overline{H^2})$ and the desired equality holds. □

Define $G := C_7[1, 2, 1, 2, 2, 1, 1]$ and $H := C_7[4, 1, 1, 1, 1, 1, 1]$. Then, by Proposition 3.2, both G and H are squco graphs of order 10. Moreover, $m(G) = 14$, while $m(H) = 13$. Now, by Proposition 4.13, we should have $m(G^2) < m(H^2)$ and this is indeed the case. We have $m(G^2) = 31$ and $m(H^2) = 32$.

We conclude this section by observing that there exist non-isomorphic squco graphs G and H with $n(G) = n(H)$ and $m(G) = m(H)$. For example, we may take G to be the graph depicted in Figure 3, and H to be the graph depicted on Figure 4. Indeed, $n(G) = n(H) = 13$ and $m(G) = m(H) = 21$ while the two graphs are not isomorphic (notice, for example, that the 3 vertices of degree 4 have a common neighbor in G but not in H).

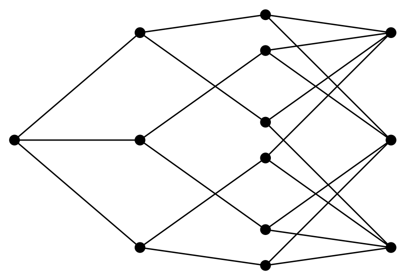

Figure 4: A bipartite squco graph on 13 vertiecs.

5 Characterizations of squco graphs in particular graph classes

In this section, we give several characterizations of squco graphs in particular graph classes. We consider several subclasses of perfect graphs, graphs of independence number at most three, graphs of maximum degree at most three, and graphs on at most 11 vertices.

Theorem 5.1. *Let G be a tree, a cograph, a split graph, an interval graph or a block graph. Then, G is squco if and only if $G = K_1$.*

Proof. For trees, the result follows immediately from Theorem 4.4. For cographs, the result follows directly from Proposition 4.2 and the fact that cographs are P_4-free ([6]). For split graphs, the result follows directly from Proposition 4.3, since every connected split graph is of radius at most 2 (every vertex in the clique part of G is of eccentricity at most 2). Now, let G be a squco interval graph. Then G^2 is also interval ([10, 25]). Hence, since G is squco, the graph $\overline{G} \cong G^2$ is an interval graph. Therefore, G is interval and co-interval, hence also chordal and co-chordal and thus split. The conclusion now follows from the result for split graphs. Finally, if G is a squco block graph then Theorem 4.4 implies that G is complete. In particular, G has radius less than 3, and hence the conclusion follows from Proposition 4.3. □

In the following proposition, we characterize squco graphs with maximum degree at most 2.

Proposition 5.2. *The only nontrivial squco graph with maximum degree at most 2 is C_7.*

Proof. Let G be a nontrivial squco graph of maximum degree at most 2. Since G is connected, it is a path or a cycle. By Theorem 5.1, G must be a cycle, say C_k. By Proposition 4.7, we see that $k \leq 7$. By Proposition 4.10, we have $k \in \{6, 7\}$. By Proposition 4.8, we have $k = 7$, and the result follows. □

An immediate consequence is the characterization of squco graphs with independence number at most 3.

Corollary 5.3. *Let G be a graph with $\alpha(G) \leq 3$. Then, G is squco if and only if $G = K_1$ or $G = C_7$.*

Proof. Let G denote a squco graph other than K_1 and C_7. Then, by Proposition 5.2, $\Delta(G) \geq 3$, and so, by Proposition 4.10, $\alpha(G) \geq 4$. □

We now turn our attention to small graphs. As announced in the introduction, the 7-cycle is the unique smallest nontrivial squco graph.

Observation 5.4. *The only squco graph on at most six vertices is the singleton K_1.*

Proof. Suppose G is a squco graph on at most six vertices. If $\Delta(G) \leq 2$, then, by Proposition 5.2, G is the singleton K_1. Thus we may assume $\Delta(G) \geq 3$. Let v denote a vertex of maximum degree in G. By Proposition 4.2, there is a vertex u at distance at least 3 from v in G. Hence $n(G) = 1 + |N_1(v, G)| + |N_2(v, G)| + |N_{\geq 3}(v, G)| \geq 1 + \Delta(G) + 1 + 1 \geq 6$, and so, since $n(G) \leq 6$, we must have $|N_2(v, G)| = |N_{\geq 3}(v, G)| = 1$. Let u denote a vertex of $N_1(v, G)$ adjacent to the unique vertex of $N_2(v, G)$. Then every vertex of G is within distance at most 2 from u; a contradiction to Proposition 4.2. □

Together with Corollary 3.3, Observation 5.4 implies a complete characterization of natural numbers n for which there exists a square-complementary graph of order n:

Proposition 5.5. *There exists a squco graph on n vertices if and only if $n = 1$ or $n \geq 7$.*

The remainder of this section is devoted to proving the following characterization of squco graphs of maximum degree at most 3 (also called *subcubic* graphs).

Theorem 5.6. *Let G be a graph of maximum degree at most 3. Then, G is squco if and only if $G = K_1$, $G = C_7$, $G = C_7[2, 1, 1, 1, 1, 1, 1]$, or G is the Franklin graph.*

The proof of Theorem 5.6 is obtained via a rather lengthy case analysis organized in a sequence of propositions. For the sake of readability, we only present in this section the first two propositions, dealing with graphs of order at least 12. The remaining cases can be settled similarly; proofs can be found in B. The validity of Theorem 5.6 has also been checked with computer search, using the program `geng` ([19]) (see also [18]) and SAGE ([29]).

By Proposition 4.11, every subcubic squco graph has at most 13 vertices. We first show that this bound can be improved.

Proposition 5.7. *Every subcubic squco graph has at most 12 vertices.*

Proof. Let G denote a subcubic squco graph. By Proposition 4.11, G has at most 13 vertices. Suppose that $|V(G)| = 13$. Then, since $\Delta(G^2) \leq 9$ and the number of odd-degree vertices in G^2 is even, at least one vertex, say v, has degree at most 8 in G^2. Consequently, the degree of v in $\overline{G^2}$ is at least $13 - 1 - 8 = 4$, which is a contradiction with the assumption that G is a squco subcubic graph. □

Proposition 5.8. *The only subcubic squco graph on 12 vertices is the Franklin graph.*

Proof. Let G denote a subcubic squco graph on 12 vertices. By Proposition 5.2, $\Delta(G) = 3$, and, by Proposition 4.1, both G and $\overline{G}$ are connected. Let v denote an arbitrary vertex of G. We have $\overline{G^2} \cong G$, and so, since G is subcubic, $N_{\geq 3}(v, G)$ contains at most three vertices. If some vertex v in G has degree at most 2, then $B_2(v)$ contains at most seven vertices, and so $N_{\geq 3}(v)$ contains at least 5 vertices, a contradiction. Hence G is cubic.

Similarly, if some vertex v lies on a triangle of G, then $N_{\geq 3}(v, G)$ contains at least 4 vertices, a contradiction. Hence G is triangle-free.

If some vertex v of G does not lie on a 4-cycle, then $N_{\geq 3}(v, G)$ contains two vertices, a contradiction to the fact that G is cubic. Hence every vertex of G lies on a 4-cycle.

If two 4-cycles intersect, then, since G is cubic, their intersection forms a path with one or two edges. If two 4-cycles share a path with two edges, then for each endpoint u of this path, the set $N_{\geq 3}(u, G)$ contains at least four vertices, a contradiction. If two 4-cycles share a common edge, say uv, then again $N_{\geq 3}(u, G)$ contains at least four vertices, a contradiction. Hence, every vertex of G lies on a unique 4-cycle.

Thus, the vertex set of G can be partitioned into three vertex-disjoint induced 4-cycles, say $abcd$, $efgh$ and $ijkl$. These three cycles have 12 edges in total, hence, since G is cubic, there are six edges connecting vertices from different cycles. For every two of these three cycles, there are at most two edges between them, since otherwise another 4-cycle would arise. Consequently, every two of these three cycles are connected to each other with precisely two edges. Without loss of generality, we now fix some of the remaining edges, for example let $ae \in E(G)$ and $fj \in E(G)$.

Suppose that $bg \in E(G)$. Then, to avoid another 4-cycle between vertices of $abcd$ and those of $ijkl$, we may assume without loss of generality that $ci, dk \in E(G)$. Consequently, vertices h and l must also be adjacent. In this case the graph G contains a 5-cycle $abgfe$. On the other hand, the graph $\overline{G^2}$ is isomorphic to the Franklin graph (see Fig. 5), and hence bipartite.

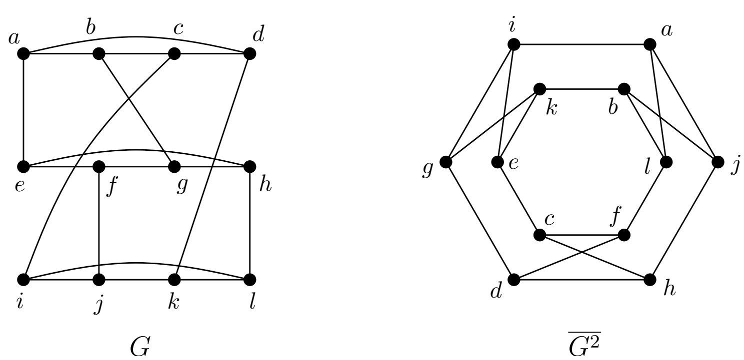

Figure 5: Graphs G and $\overline{G^2}$ in the case $bg \in E(G)$

This implies that $G \not\cong \overline{G^2}$, a contradiction which shows that $bg \notin E(G)$. By symmetry, we conclude that $dg \notin E(G)$, $hi \notin E(G)$ and $hk \notin E(G)$.

Consequently, the missing edge between cycles $abcd$ and $efgh$ is incident with c, and the missing edge between cycles *efgh* and *ijkl* is incident with l.

Suppose that $ch \in E(G)$. Then $gl \in E(G)$. Without loss of generality, the remaining edges can be fixed so that $bi \in E(G)$ and $dk \in E(G)$. Again, the graph G contains a 5-cycle *fglkj*, while the complement of its square is isomorphic to the (bipartite) Franklin graph (see Fig. 6), a contradiction.

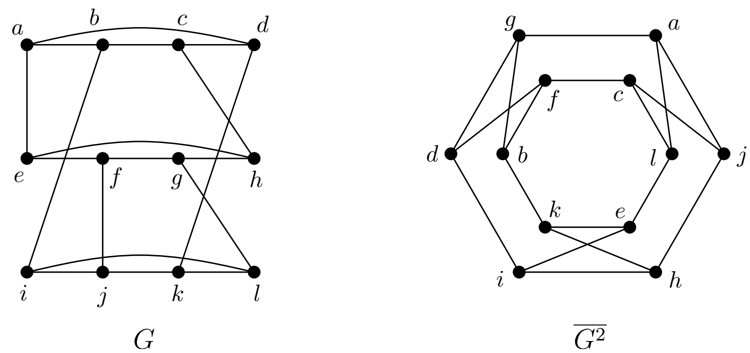

Figure 6: Graphs G and $\overline{G^2}$ in the case $ch \in E(G)$

This implies that $ch \notin E(G)$. Since the missing edge between cycles *abcd* and *efgh* is incident with c, we must have $cg \in E(G)$ and consequently $hl \in E(G)$. Again, no generality is lost in fixing the remaining edges so that $bi \in E(G)$ and $dk \in E(G)$. In this case, as Figure 7 shows, G is isomorphic to the Franklin graph, which, by Observation 1.2, is square-complementary.

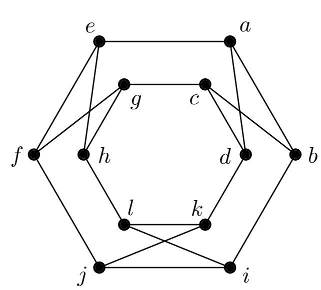

Figure 7: A drawing of G showing an isomorphism between G and the Franklin graph.

This completes the proof. □

We conclude this section with a report on some computational results. Using the programs `geng` ([19]) and SAGE ([29]), we did an extensive search for small squco graphs, in particular, we were able to determine all the squco graphs with at most 11 vertices

The results are summarized in the following theorem and in Table 2.

order	all graphs	max degree at most 4	max degree at most 3
1	K_1	K_1	K_1
2	none		
3	none		
4	none		
5	none		
6	none		
7	C_7	C_7	C_7
8	$C_7[2,1,1,1,1,1,1]$	$C_7[2,1,1,1,1,1,1]$	$C_7[2,1,1,1,1,1,1]$
9	$C_7[3,1,1,1,1,1,1]$	$C_7[3,1,1,1,1,1,1]$	none
10	$C_7[4,1,1,1,1,1,1]$ & $C_7[1,2,1,2,2,1,1]$	$C_7[4,1,1,1,1,1,1]$ & $C_7[1,2,1,2,2,1,1]$	none
11	$C_7[5,1,1,1,1,1,1]$ & $C_7[2,1,1,2,1,2,2]$	$C_7[2,1,1,2,1,2,2]$	none
12	unknown	the Franklin graph	the Franklin graph

Table 2: Small squco graphs

Theorem 5.9. *Let G be a graph with at most 11 vertices. Then, G is squco if and only if G is one of the following 7 graphs:*

$$K_1, \quad C_7, \quad C_7[2,1,1,1,1,1,1], \quad C_7[3,1,1,1,1,1,1], \quad C_7[4,1,1,1,1,1,1]\,,$$

$$C_7[1,2,1,2,2,1,1], \quad C_7[5,1,1,1,1,1,1], \quad C_7[2,1,1,2,1,2,2]\,.$$

The encodings in the g6 format of all small squco graphs and of all small bipartite squco graphs is available at the homepage of one of the authors ([23]).

6 Bipartite squco graphs

The Franklin graph (defined in Figure 1) is an example of a bipartite squco graph. In this section, we give a characterization of bipartite squco graphs, and exemplify it with a construction (based on bipartite bi-Cayley graphs over cyclic groups) of an infinite family of bipartite vertex-transitive squco graphs generalizing the Franklin graph.

Assume that G is a bipartite squco graph. Then G does not have triangles; in particular, by Observation 4.9, G does not contain three vertices with mutually disjoint closed neighborhoods. It is also clear that if all neighborhoods intersect then the diameter is at most 2 and, by Proposition 4.3, G is not squco. Hence, the maximum cardinality of a set of vertices with mutually disjoint closed neighborhoods is 2.

We start with a preliminary result.

Lemma 6.1. *Let G be a bipartite squco graph with bipartition $\{U, W\}$. Then, the closed neighborhoods of any two vertices in U intersect nontrivially.*

Proof. Assume to the contrary that there exist $u, v \in U$ with disjoint closed neighborhoods; that is, u and v have no common neighbor. We claim that the neighborhood of every vertex in $U \setminus \{u, v\}$ contains $N(u)$ or $N(v)$. In fact, if $x \in U$, $u' \in N(u)$ and $v' \in N(v)$ are such that x is adjacent neither to u' nor to v' then (u, v', x, u', v) is a pentagon in $\overline{G^2}$, contradicting the assumption that G is a bipartite squco graph.

Due to the triangle-freeness of $\overline{G^2}$, every vertex in W is adjacent to either u or v (but not both due to the assumption on u and v). On the other hand, no vertex x of U can be adjacent to all vertices in W since otherwise $N_2(x) = G$, again contradicting Proposition 4.3. This implies that for every $x \in U$ we have that either $N(u) \subseteq N(x)$ or $N(v) \subseteq N(x)$ but not both. This naturally partitions U in the sets $U_u := \{x \in U \mid N(u) \subseteq N(x)\}$ and $U_v := \{x \in U \mid N(v) \subseteq N(x)\}$.

Since G is connected, we may assume without loss of generality that there is an edge between a vertex $x \in U_u$ and a vertex $z \in N(v)$. Then the neighborhood of x in $\overline{G^2}$ is properly contained in $N(v)$ and the neighborhood of z in $\overline{G^2}$ is properly contained in U_u. Note that the neighborhood of any vertex of G contains $U_u, U_v, N(u)$ or $N(v)$, implying that the cardinality of either U_v or of $N(u)$ is strictly smaller than the minimum of the cardinalities of U_u and $N(v)$. A similar argument would imply that if there is also an edge between a vertex in U_v and a vertex in $N(u)$ then the cardinality of either U_u or of $N(v)$ is strictly smaller than the minimum of the cardinalities of U_v and $N(u)$, a contradiction. Hence $U_v \cup N(u)$ is an independent set.

Now we determine the bipartition in $\overline{G^2}$. Note that u is adjacent in $\overline{G^2}$ to every vertex in $U_v \cup N(v)$, and therefore $U_v \cup N(v)$ is contained in one of the parts X of $\overline{G^2}$. Similarly, v is adjacent in $\overline{G^2}$ to every vertex in $N(u)$ implying that $N(u)$ is contained in the part $X' \neq X$ of $\overline{G^2}$. Finally, for every vertex $y \in U_u$, the neighborhood of y in $\overline{G^2}$ is contained in $U_v \cup N(v)$ and therefore U_u is also contained in X'. Hence the parts of $\overline{G^2}$ are $U_u \cup N(u)$ and $U_v \cup N(v)$, but then u is adjacent in $\overline{G^2}$ to every vertex in the other part, contradicting once more Proposition 4.3. This finishes the proof. □

The *bipartite complement* of a bipartite graph G with bipartition $\{U, W\}$ is the graph obtained from G by replacing $E(G)$ by $UV \setminus E(G)$ (where UV denotes the set of all pairs consisting of a vertex in U and a vertex in W). A bipartite graph G with bipartition $\{U, W\}$ is said to be *bipartite self-complementary* if it is isomorphic to its bipartite complement ([1, 12]).

The previous discussion and Lemma 6.1 imply that if G is a bipartite squco graph with bipartition $\{U, W\}$ then every two vertices in U have a common neighbor, and every two vertices in W have a common neighbor. Consequently, $U \subseteq N_2(u)$ for every $u \in U$, and $W \subseteq N_2(v)$ for every $v \in W$ and the next theorem with its subsequent corollary follow.

Theorem 6.2. *A bipartite graph G is a squco graph if and only if it satisfies the following conditions:*

- *G is a connected bipartite self-complementary graph.*
- *Every two vertices in the same part have a common neighbor.*

Corollary 6.3. *Every nontrivial bipartite squco graph is of diameter* 3.

6.1 Bipartite bi-Cayley graphs

If Γ is a group and $A_1, \ldots, A_n$ are subsets of Γ, then let $A_1 \ldots A_n = \{a_1 \ldots a_n \mid a_i \in A_i\}$ and $A_i^{-1} = \{a_i^{-1} \mid a_i \in A_i\}$. If $A_i = \{a_i\}$ is a singleton, we will often abuse notation and simply write a_i instead of $\{a_i\}$. Finally, we write $\overline{A_1}$ for $\{g \in \Gamma \mid g \notin A_1\}$ (Γ should be clear from the context).

Let Γ be a finite group and let $A \subseteq \Gamma$ be a subset of Γ. The *bipartite bi-Cayley graph* $\mathrm{BBC}(\Gamma, A)$ is the graph with vertex-set $\Gamma \times \{0,1\}$ and two vertices $(x,0)$ and $(y,1)$ are adjacent if and only if $x^{-1}y \in A$ (and no other adjacencies). (See, e.g., [17].) There is a natural bipartition of G given by $\{\Gamma \times \{0\}, \Gamma \times \{1\}\}$. Bipartite bi-Cayley graphs always admit some very natural automorphisms, as we see in Lemma 6.4.

Lemma 6.4. *Let $G = BBC(\Gamma, A)$. Then the automorphism group of G acts transitively on each of the parts of the natural bipartition of G. Moreover, if A is preserved by conjugation in Γ (in other words, if $g^{-1}ag \in A$ for every $a \in A$ and every $g \in \Gamma$), then G is vertex-transitive.*

Proof. For an element $g \in \Gamma$, let σ_g be the permutation of $V(G)$ which maps every vertex (x,i) to the vertex (gx, i). It is simply a matter of routine to check that σ_g is an automorphism of G. Clearly, the group $\langle \sigma_g \mid g \in \Gamma \rangle$ acts transitively on each of the parts of the natural bipartition of G, proving our first claim.

Now, suppose that A is preserved by conjugation in G and let $\varphi : V(G) \to V(G)$ mapping (x,i) to $(x^{-1}, 1-i)$. We show that φ is an automorphism of G. Indeed, $\varphi((x,0)) = (x^{-1}, 1)$ and $\varphi((y,1)) = (y^{-1}, 0)$ are adjacent if and only if $(y^{-1})^{-1}x^{-1} = yx^{-1} \in A$ if and only if $x^{-1}y \in x^{-1}Ax = A$ if and only if $(x,0)$ and $(y,1)$ are adjacent. This shows that φ is an automorphism. Clearly, φ exchanges the parts of the natural bipartition of G and hence G is vertex-transitive. □

It follows from Lemma 6.4 that a bipartite bi-Cayley graph on an abelian group is always vertex-transitive. We now give a sufficient condition on (Γ, A) for $\mathrm{BBC}(\Gamma, A)$ to satisfy the conditions of Theorem 6.2.

Proposition 6.5. *Let Γ be a group and let A be a subset of Γ which satisfies the two following conditions:*

- *$AA^{-1} = \Gamma = A^{-1}A$, and*
- *there exists an automorphism λ of Γ and $\tau, \mu \in \Gamma$ such that $\tau\lambda(A)\mu = \overline{A}$.*

Then $BBC(\Gamma, A)$ is a bipartite squco graph.

Proof. Let $G = \mathrm{BBC}(\Gamma, A)$. We will verify that the conditions of Theorem 6.2 are satisfied, with respect to the natural bipartition of G. First, let us show that every two vertices in one part of the bipartition have a common neighbor. Let $x, y \in \Gamma$ be two different group elements. By the first condition in the hypothesis, we can write $x^{-1}y$ as ab^{-1} for some $a, b \in A$. This implies that $(xa, 1) = (yb, 1)$ is a common neighbor of the vertices $(x,0)$ and $(y,0)$. Similarly, we can write $x^{-1}y$ as $c^{-1}d$ for some $c, d \in A$ which implies that $(xc^{-1}, 0) = (yd^{-1}, 0)$ is a common neighbor of the vertices $(x, 1)$ and $(y, 1)$.

Note that $|\tau\lambda(A)\mu| = |A|$ and hence the second condition in the hypothesis implies that $|A| = |\overline{A}|$. In particular, this implies that $|\Gamma| \geq 2$, and consequently, since every two vertices in one part of the bipartition have a common neighbor, G is a connected graph. Let $G^* = \mathrm{BBC}(\Gamma, \overline{A})$ denote the bipartite complement of G. Since $|A| = |\overline{A}|$, it follows that $|E(G)| = |E(G^*)|$. We will use this fact to show that an isomorphism between G and its bipartite complement is given by the mapping $\varphi : V(G) \to V(G)$ defined as follows:

$$\varphi((x, 0)) = (\lambda(x)\tau^{-1}, 0), \quad \varphi((x, 1)) = (\lambda(x)\mu, 1).$$

There is a natural permutation of $E(G) \cup E(G^*)$ induced by φ. We will show that $\varphi(E(G)) \subseteq E(G^*)$. Indeed, if $(x, 0)$ and $(y, 1)$ are adjacent then $x^{-1}y \in A$, which implies that $\tau\lambda(x^{-1}y)\mu \in \overline{A}$. Since λ is an automorphism of Γ, it follows that $(\lambda(x)\tau^{-1})^{-1}(\lambda(y)\mu) = \tau\lambda(x)^{-1}\lambda(y)\mu \in \overline{A}$ and hence $\varphi((x, 0))$ and $\varphi((y, 1))$ are not adjacent in G.

We have just shown that $\varphi(E(G)) \subseteq E(G^*)$. Together with the fact that $|E(G)| = |E(G^*)|$, this implies that φ interchanges $E(G)$ and $E(G^*)$. This shows that φ is an isomorphism between G and its bipartite complement and completes the proof. □

Note that, if (Γ, A) is a pair which satisfies the second condition in Proposition 6.5, then $|A| = |\Gamma|/2$. Given a group of even order Γ and a subset $A \subseteq \Gamma$ with $|A| = |\Gamma|/2$, call an element $g \in \Gamma$ *bad with respect to* A if $g \notin AA^{-1}$ or $g \notin A^{-1}A$. With this terminology, the first condition in Proposition 6.5 can be restated as saying that Γ contains no bad element with respect to A. Here are a few simple facts about bad elements.

Lemma 6.6. *Let Γ be a group of even order, let A be a subset of Γ with $|A| = |\Gamma|/2$ and let $g \in \Gamma$ be bad with respect to A. Then either ($gA = \overline{A}$ and $g\overline{A} = A$) or ($Ag = \overline{A}$ and $\overline{A}g = A$). Moreover, g has even order and every odd power of g is bad with respect to A.*

Proof. Suppose that $g \notin AA^{-1}$. Then $gA \cap A = \emptyset$ and, since $|A| = |gA| = |\overline{A}|$, it follows that $gA = \overline{A}$. Left multiplication by g is a permutation of Γ. Since $|A| = |\Gamma|/2$, this shows that $g\overline{A} = A$ and $g^2A = A$. It follows that g has even order and odd powers of g are also bad. The proof is analogous in the case $g \notin A^{-1}A$. □

6.2 Cyclic groups

We now specialise to the case when Γ is a cyclic group. To emphasize this, we switch to the additive notation for the group operation. Throughout this section, let $n \geq 2$ be even, let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ denote the cyclic group of order n and let $A \subseteq \mathbb{Z}_n$ such that $|A| = n/2$.

For a subset B of $\mathbb{Z}_n$, let $\chi(B)$ denote its characteristic string, that is, $\chi(B)$ is the string of length n, indexed over elements of $\mathbb{Z}_n$, with a "1" in the ith position if $i \in B$ and a "0" otherwise. Given strings S and T, let S^{-1} denote the *inverse* of S (that is, S in reverse order), let $S \circ T$ denote the concatenation of S and T and, for a positive integer i, let S^i denote the string obtained by concatenating i copies of S. In fact, all our strings will be binary, that is, with entries in $\{0,1\}$. Given a binary string S, let $\overline{S}$ denote its *complement* (that is, every 0 is replaced by a 1 and vice-versa). Note that $\overline{S^{-1}} = \overline{S}^{-1}$ for every string S, and that for any $\mu \in \mathbb{Z}_n$ and $A \subseteq \mathbb{Z}_n$, $\overline{A+\mu} = \overline{A} + \mu$.

Lemma 6.7. *Let $i \in \{1,\ldots,n-1\}$ be bad with respect to A and let $d = \gcd(i,n)$. Then, for every $\mu \in \mathbb{Z}_n$, there exists a binary string S of length d such that $\chi(A+\mu) = (S \circ \overline{S})^{\frac{n}{2d}}$.*

Proof. Note that the order of i is n/d, which by Lemma 6.6 is even. Write $d = xi + yn$ with x, y integers. Let 2^a be the highest power of 2 dividing d. Since n/d is even, it follows that 2^{a+1} divides n. This shows that 2^a is the highest power of 2 dividing xi, but d divides i and hence so does 2^a. This implies that x is odd and, by Lemma 6.6, that xi is bad and hence so is d. Let S be the starting substring of $\chi(A+\mu)$ of length d. Since d is bad, $A + d = \overline{A}$ and $\overline{A} + d = A$. It follows that the substring of $\chi(A+\mu)$ of length d starting at position d is $\overline{S}$. This is followed by S and so on. Since d has the same order as i, the order of d is even and hence $2d$ divides n. This completes the proof. □

A binary string T is called *troubled* if $T = T^{-1}$ or there exists an integer $i \geq 1$ and a string U such that $U^{-1} = U$ and $T = (U \circ \overline{U})^i$.

Theorem 6.8. *Suppose that there exist $\mu \in \mathbb{Z}_n$ and a binary string T such that T is not troubled and $\chi(A+\mu) = T \circ \overline{T}^{-1}$. Then BBC$(\mathbb{Z}_n, A)$ is a vertex-transitive bipartite squco graph.*

Proof. We will show that the conditions of Proposition 6.5 are satisfied. Note that if B is a subset of $\mathbb{Z}_n$ such that $\chi(B) = T \circ \overline{T}^{-1}$ for some binary string T, then $-B-1 = \overline{B}$. The hypothesis that $\chi(A+\mu) = T \circ \overline{T}^{-1}$ thus implies that $-(A+\mu)-1 = \overline{A}+\mu$. It follows that $-A-2\mu-1 = \overline{A}$ and the second condition of Proposition 6.5 is satisfied (recall that, in an abelian group, multiplying by -1 is an automorphism).

Suppose that the first condition of Proposition 6.5 is not satisfied. In other words, there exists a bad element with respect to A and, by Lemma 6.7, we obtain that $\chi(A+\mu) = (S \circ \overline{S})^{\frac{n}{2d}}$ for some binary string S and integer d dividing $n/2$. By hypothesis, we have $\chi(A+\mu) = T \circ \overline{T}^{-1}$ and hence $(S \circ \overline{S})^{\frac{n}{2d}} = T \circ \overline{T}^{-1}$.

Suppose first that $\frac{n}{2d}$ is odd. Then $T = (S \circ \overline{S})^{\lfloor \frac{n}{4d} \rfloor} \circ S$ and $\overline{T}^{-1} = \overline{S} \circ (S \circ \overline{S})^{\lfloor \frac{n}{4d} \rfloor}$. On the other hand, $\overline{T}^{-1} = \overline{S}^{-1} \circ (S^{-1} \circ \overline{S}^{-1})^{\lfloor \frac{n}{4d} \rfloor}$. This shows that $S = S^{-1}$ and hence $T = T^{-1}$, contradicting the fact that T is not troubled.

Now, suppose that $\frac{n}{2d}$ is even. It follows that $T = (S \circ \overline{S})^{\frac{n}{4d}}$ and $\overline{T}^{-1} = (S \circ \overline{S})^{\frac{n}{4d}}$. On the other hand, $\overline{T}^{-1} = (S^{-1} \circ \overline{S}^{-1})^{\frac{n}{4d}}$ and hence $S = S^{-1}$, again contradicting the fact that T is not troubled.

We have obtained a contradiction in both cases, which shows that the first condition of Proposition 6.5 is satisfied and hence BBC$(\mathbb{Z}_n, A)$ is a bipartite squco graph. Finally, it follows from Lemma 6.4 that BBC$(\mathbb{Z}_n, A)$ is vertex-transitive. □

Theorem 6.8 gives a relatively easy way to construct a large number of vertex-transitive bipartite connected squco graphs. Indeed, start with a binary string T which is not troubled, let $n = 2|T|$ and let A be the unique subset of $\mathbb{Z}_n$ such that $\chi(A) = T \circ \overline{T}^{-1}$. Then $\mathrm{BBC}(\mathbb{Z}_n, A)$ will be as required. We now give a very simple lemma which helps in determining whether a given string is troubled.

Lemma 6.9. *Let T be a binary string such that there exists an integer $i \geq 1$ and a string U such that $U^{-1} = U$ and $T = (U \circ \overline{U})^i$. Then there exists an integer $a \geq 0$ and a string V such that $V^{-1} = V$ and $T = (V \circ \overline{V})^{2^a}$.*

Proof. Write $i = 2^a j$ with j odd and let $V = (U \circ \overline{U})^{\frac{j-1}{2}} \circ U$. Since $U^{-1} = U$, it follows that $V^{-1} = V$. Moreover, $\overline{V} = (\overline{U} \circ U)^{\frac{j-1}{2}} \circ \overline{U}$, hence $V \circ \overline{V} = (U \circ \overline{U})^j$ and $T = (V \circ \overline{V})^{2^a}$. □

Lemma 6.9 makes it rather straightforward to determine if a given string T is troubled. For example, if $|T|$ is odd, then it follows from Lemma 6.9 that T is troubled if and only if $T = T^{-1}$. The shortest binary string which is not troubled is $(0, 0, 1)$ and it is unique, up to inversion and taking complements. Applying the construction which follows Theorem 6.8 to $(0, 0, 1)$ yields our old friend, the Franklin graph.

7 Concluding remarks and open problems

We initiated the study of square-complementary graphs — which we also call squco graphs for short. We gave several examples of squco graphs, including infinite families of bipartite squco graphs, proved several necessary conditions for a graph to be square-complementary, gave ways of building new squco graphs from existing ones, and characterized squco graphs in certain graph classes.

Regarding algorithmic and complexity issues, it is worth mentioning that, while the computational complexity of recognizing squco graphs is unknown, the squco graphs recognition problem is not more difficult than the graph isomorphism problem. Indeed, it is not hard to see that whenever $\mathcal{C}$ is a polynomially recognizable graph class such that the isomorphism problem is polynomial for graphs in $\mathcal{C}$, recognizing whether a graph G is a squco graph can be done in polynomial time whenever $G \in \mathcal{C}$, $G^2 \in \mathcal{C}$ or $\overline{G} \in \mathcal{C}$. Examples of such classes $\mathcal{C}$ include: circulant graphs ([22]) (which are closed under complements and squares), permutation graphs ([5, 28]), planar graphs ([13, 14]) (and more generally, graphs of bounded genus ([8, 20, 21])) and graphs of bounded tree-width ([2, 3]). Moreover, Proposition 4.11 implies that there are only finitely many squco graphs of bounded maximum degree, so for graph classes of bounded degree, the problem of recognizing squco graphs can be solved in $O(1)$ time.

Results obtained in this paper motivate a further study of squco graphs. Since a complete characterization of squco graphs seems perhaps too ambitious, we pose the following partial questions and open problems:

- Is there a squco graph of diameter 4?
- Is there a squco graph of radius 4?

- Is there a squco graph of girth 5? Of girth 6?
- Is there a nontrivial chordal squco graph? A negative answer to this question would follow from a negative answer to any of the following two questions.
- Is there a nontrivial squco graph containing a simplicial vertex (that is, a vertex whose neighborhood forms a clique)?
- Is there a squco graph containing a separating clique (that is, a clique whose removal disconnects the graph)?
- Is there a graph H such that every squco graph G is H-free? In particular:
  - Can squco graphs contain arbitrarily large cliques?
    (We do not know of any squco graph containing a K_4. For an example of a squco graph containing a triangle, consider the circulant $C_{41}(\{4, 5, 8, 10\})$.)
  - Can squco graphs contain arbitrarily long induced paths?
- Is the chromatic number of squco graphs bounded by a constant?
- What is the computational complexity of the problem of recognizing squco graphs? In particular, is this a GI-complete problem?

To conclude we mention a natural generalization of self- and square-complementary graphs. For a positive integer k, we say that a graph G is *k-th-power-complementary* if $G^k \cong \overline{G}$. For $k = 1$ and $k = 2$, we obtain the classes of s.c. and squco graphs, respectively. It turns out that, except for the singleton K_1, all these graph classes are disjoint.

Proposition 7.1. *If G is k-th-power-complementary and ℓ-th-power-complementary with $k \neq \ell$, then $G \cong K_1$.*

Proof. Let G be k-th-power-complementary and ℓ-th-power-complementary with $k \neq \ell$, where $k < \ell$. A slight modification of the first part of the proof of Proposition 4.1 shows that G is connected. Let us now show that G^k is complete. Suppose for a contradiction that G^k is not complete. Then there exist two vertices $x, y \in V(G)$ such that $d_G(x, y) > k$. Notice that $d_G(x, y)$ is finite since G is connected, therefore there exists a vertex z in G such that $d_G(x, z) = k + 1$. Then, x and z are not adjacent in G^k, however since $\ell \geq k + 1$, they are adjacent in G^ℓ. Since G^k is a subgraph of G^ℓ, it follows that G^ℓ has strictly more edges than G^k. So G^k and G^ℓ cannot be isomorphic. Hence, $\overline{G} \cong G^k \ncong G^\ell \cong \overline{G}$, a contradiction.

Since $G^k \cong \overline{G}$ and G^k is complete, G is edgeless. The only edgeless connected graph is K_1. □

Several of the results obtained in this paper for squco graphs (for example Observation 1.1 and Propositions 4.1,4.2 and 4.3) admit natural generalizations to k-th power-complementary graphs for values of k other than 2. A more systematic investigation of k-th power complementary graphs for $k > 2$ seems to deserve further attention.

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A Some squco circulants

In this section, we argue why the circulants mentioned in Section 3 are squco. Suppose that $G = C_n(D)$ and $G' = C_n(D')$ are two circulants to be tested for isomorphism. Let $\mathbb{Z}_n^*$ be the set of all residues modulo n which are coprime to n. If $D = mD'$ for some $m \in \mathbb{Z}_n^*$, then the permutation $x \mapsto mx$, with $x \in \mathbb{Z}_n$, is an isomorphism between G and G' ([22]).

The following list gives the examples of squco circulants mentioned in Section 3 and for each of them provides an argument why the graph is square-complementary:

(1) $G = C_7(\{1\})$, the 7-cycle.

The square G^2 is the circulant graph $C_7(\{1, 2\})$. The complement of the square $\overline{G^2}$ is the circulant graph $C_7(\{3\})$. An isomorphism between G and $\overline{G^2}$ is obtained by multiplying each distance $d \in \{1\}$ by 3.

(2) $G = C_{17}(\{1, 4\})$.

The square G^2 is the circulant graph $C_{17}(\{1, 2, 3, 4, 5, 8\})$. The complement of the square $\overline{G^2}$ is the circulant graph $C_{17}(\{6, 7\})$. An isomorphism between G and $\overline{G^2}$ is obtained by multiplying each distance $d \in \{1, 4\}$ by 6.

(3) $G = C_{24}(\{1, 3, 11\})$.

The square G^2 is the circulant graph $C_{24}(\{1, 2, 3, 4, 6, 8, 10, 11, 12\})$. The complement of the square $\overline{G^2}$ is the circulant graph $C_{24}(\{5, 7, 9\})$. An isomorphism between G and $\overline{G^2}$ is obtained by multiplying each distance $d \in \{1, 3, 11\}$ by 5.

(4) $G = C_{40}(\{1, 3, 5, 9, 17\})$.

The square G^2 is the circulant graph $C_{40}(\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 16, 17, 18, 20\})$. The complement of the square $\overline{G^2}$ is the circulant graph $C_{40}(\{7, 11, 13, 15, 19\})$. An isomorphism between G and $\overline{G^2}$ is obtained by multiplying each distance $d \in \{1, 3, 5, 9, 17\}$ by 11.

(5) $G = C_{40}(\{1, 3, 5, 13, 19\})$.

The square G^2 is the circulant graph $C_{40}(\{1, 2, 3, 4, 5, 6, 8, 10, 12, 13, 14, 16, 18, 19, 20\})$. The complement of the square $\overline{G^2}$ is the circulant graph $C_{40}(\{7, 9, 11, 15, 17\})$. An isomorphism between G and $\overline{G^2}$ is obtained by multiplying each distance $d \in \{1, 3, 5, 13, 19\}$ by 11.

(6) $G = C_{40}(\{1,3,5,17,19\})$.

The square G^2 is the circulant graph $C_{40}(\{1,2,3,4,5,6,8,10,12,14,16,17,18,19,20\})$. The complement of the square $\overline{G^2}$ is the circulant graph $C_{40}(\{7,9,11,13,15\})$. An isomorphism between G and $\overline{G^2}$ is obtained by multiplying each distance $d \in \{1,3,5,17,19\}$ by 11.

(7) $G = C_{40}(\{1,5,9,11,19\})$.

The square G^2 is the circulant graph $C_{40}(\{1,2,4,5,6,8,9,10,11,12,14,16,18,19,20\})$. The complement of the square $\overline{G^2}$ is the circulant graph $C_{40}(\{3,7,13,15,17\})$. An isomorphism between G and $\overline{G^2}$ is obtained by multiplying each distance $d \in \{1,5,9,11,19\}$ by 3.

(8) $G = C_{41}(\{4,5,8,10\})$.

The square G^2 is the circulant graph $C_{41}(\{1,2,3,4,5,6,8,9,10,12,13,14,15,16,18,20\})$. The complement of the square $\overline{G^2}$ is the circulant graph $C_{41}(\{7,11,17,19\})$. An isomorphism between G and $\overline{G^2}$ is obtained by multiplying each distance $d \in \{4,5,8,10\}$ by 6.

B Subcubic square-complementary graphs of order 7 to 11

Here we provide the rest of the proof of Theorem 5.6, handling separately the subcubic squco graphs having exactly k vertices, for $k \in \{7,8,9,10,11\}$. The case of at most 6 vertices has been considered in Observation 5.4.

B.1 The unique subcubic square-complementary graph of order 7

Observation B.1. *If G is a subcubic squco graph on seven vertices and G is not the 7-cycle, then $\alpha(G) = 4$.*

Proof. Suppose G denote a subcubic squco graph on seven vertices which is not the 7-cycle. By Corollary 5.3, $\alpha(G) \geq 4$. By Proposition 4.1, G is connected, and so, in particular, $\alpha(G) \leq 6$. If $\alpha(G) = 6$, then G must be the star $K_{1,6}$; however, by Theorem 5.1, $K_{1,6}$ is not a squco graph. Hence we must have $\alpha(G) \leq 5$.

Suppose $\alpha(G) = 5$, and let S denote a maximum independent set of G. Define $T := V(G) \setminus S$, and let t_1 and t_2 denote the two vertices of T. Suppose t_1 and t_2 are adjacent. Now, each vertex in S has degree at least 1, since G is connected, and so each neighbor of a vertex in S must be in T, it follows that G contains at least 5 (S,T)-edges. On the other hand, since t_1 and t_2 are adjacent and $\Delta(G) = 3$, there can be at most 4 (S,T)-edges, a contradiction. Hence we may assume that t_1 and t_2 are non-adjacent. This means that G is a bipartite graph with bipartition (S,T). By Theorem 5.1, G cannot be a tree and so G must contain a cycle C. Since G is bipartite with bipartition (S,T), this cycle C must go between S and T. This means that C must be a 4-cycle containing t_1 and t_2. Now the three vertices $S \setminus V(C)$ must all be adjacent to at least one vertex in T, and so at least one of the vertices t_1 and t_2 is of degree at least 4, a contradiction. This contradiction shows that G cannot have independence number 5, and the proof is complete. □

Observation B.2. *If G is a subcubic squco graph on seven vertices, then G contains no 5-cycles.*

Proof. Suppose G is a subcubic squco graph on seven vertices. Of course, if $G \cong C_7$, then G contains no 5-cycles, so suppose $G \ncong C_7$. Then, by Observation B.1, $\alpha(G) = 4$. However, if G contains a 5-cycle C, then C^2 is a 5-clique, and so $\alpha(G) = \alpha(\overline{G^2}) \geq \alpha(\overline{C^2}) = 5$, a contradiction. □

Observation B.3. *If G is a squco graph on seven vertices and G is not the 7-cycle, then G contains no 7-cycles.*

Proof. Suppose that G is a squco graph on seven vertices and G is not the 7-cycle but G contains a 7-cycle C. By Observation 1.1, C is a squco graph, and so, since C is also a spanning proper subgraph of G, we have a contradiction to Proposition 4.12. □

Corollary B.4. *If G is a subcubic squco graph on seven vertices and G is not the 7-cycle, then the only possible odd cycles are triangles.*

Proposition B.5. *The 7-cycle is the only squco graph on seven vertices.*

Proof. Let G denote a squco graph on 7 vertices. If $\Delta(G) \leq 2$, then the desired result follows from Proposition 5.2. Hence we may assume that $\Delta(G) \geq 3$. Recall that $\delta(G) \geq 2$ because of Theorem 4.4.

Suppose first that $\Delta(G) \geq 4$ and let v denote a vertex of maximum degree in G. Then, the degree of v in $\overline{G^2}$ is at most 2. By Theorem 4.4, $\overline{G^2} \cong G$ has no cut vertices, and consequently the degree of v in $\overline{G^2}$ is exactly 2. By Proposition 4.3, we get that $|N_2(v, G)| = |N_3(v, G)| = 1$, which implies that the unique vertex in $N_2(v, G)$ is a cut vertex, contradicting Theorem 4.4. This contradiction shows that $\Delta(G) = 3$.

Let v denote a vertex of degree 3 in G. By Proposition 4.2, $N_3(v, G)$ is non-empty. Suppose that $N_4(v, G)$ is non-empty. Then, since $n(G) = 7$, we must have $|N_2(v, G)| = |N_3(v, G)| = |N_4(v, G)| = 1$. Now the vertex, say u, of $N_4(v, G)$ is adjacent to every vertex of $N_G[v]$ in $\overline{G^2}$, and so, in particular, u has degree 4 in $\overline{G^2}$, a contradiction. Thus, $N_4(v, G)$ must be empty. This leaves us with two cases to consider: (i) $|N_2(v, G)| = 1$ and $|N_3(v, G)| = 2$, and (ii) $|N_2(v, G)| = 2$ and $|N_3(v, G)| = 1$. Case (i) is impossible as it would imply that the unique vertex in $N_2(v, G)$ is a cut vertex, contradicting Theorem 4.4.Thus, we may assume that (ii) holds. Let the unique vertex of $N_3(v, G)$ be denoted q. Let t be a neighbor of v at distance 2 from q. Then, t has degree at least 5 in G^2. This means that t is of degree exactly 1 in $\overline{G^2}$. However, this implies that $\overline{G^2}$ contains a cut vertex, and hence by Theorem 4.4 cannot be isomorphic to G. We have reached a contradiction, which completes the proof. □

B.2 The unique subcubic square-complementary graphs of order 8

First, we collect some properties of subcubic squco graphs on eight vertices.

Observation B.6. *If G is a subcubic squco graph on eight vertices, then*

(i) $\delta(G) \geq 2$,

(ii) $\alpha(G) = 4$,

(iii) *G does not contain C_5 as a subgraph,*

(iv) *G does not contain $K_{2,3}$ as a subgraph,*

(v) *G does not contain K_4^- as a subgraph,*

(vi) *G does not contain C_4 as a subgraph unless $G \cong C_7[2,1,1,1,1,1,1]$,*

(vii) *G does not contain a triangle.*

Proof. Suppose G is a subcubic squco graph on eight vertices.

(i) The fact that G must have minimum degree at least 2 follows immediately from Proposition 4.11.

(ii) By Corollary 5.3, $\alpha(G) \geq 4$. By Proposition 4.1, G is connected and so $\alpha(G) \leq 7$. If $\alpha(G) \in \{5,6,7\}$, then we obtain a contradiction to the fact that G has minimum degree 2 and maximum degree 3. Hence we must have $\alpha(G) = 4$.

(iii) If G contains C_5 as a subgraph, then $V(C)$ would induce a 5-clique in G^2, and so $\alpha(\overline{G^2}) \geq 5$, which contradicts (ii) and the fact that $\overline{G^2} \cong G$.

(iv) If G contains $K_{2,3}$ as a subgraph, then $V(K_{2,3})$ would induce a 5-clique in G^2, and so $\alpha(\overline{G^2}) \geq 5$, which contradicts (ii) and the fact that $\overline{G^2} \cong G$.

(v) Suppose G contains K_4^- as a subgraph with vertices labelled p, q, r, and s such that q and s are nonadjacent in K_4^-. Since G is connected, subcubic and of order more than 4, q and s must be non-adjacent and at least one of q and s must be adjacent to a vertex of $V(G) \setminus V(K_4^-)$. If just on of the vertices q and s, say q, is adjacent to a vertex of $V(G) \setminus V(K_4^-)$, then s has degree 4 in $\overline{G^2}$, a contradiction. Let t and u denote the neighbors of q and s in $V(G) \setminus V(K_4^-)$, respectively. Note that t and u must be distinct, since otherwise $V(K_4^-) \cup \{t\}$ would contain a $K_{2,3}$ which contradicts (iv). Also, by (iii), t and u must be non-adjacent. Let the two vertices of $V(G) \setminus (V(K_4^-) \cup \{t,u\})$ be denoted by v and x.

If one of the vertices t and u, say t, is adjacent to both v and x, then t has degree 1 in $\overline{G^2}$ which contradicts (i) and the fact that $\overline{G^2}$ and G are isomorphic. On the other hand, since $3 \geq \Delta(G) \geq \delta(G) \geq 2$, we may, by symmetry, assume that t and u are adjacent to v and x, respectively. Again, since $\delta(G) \geq 2$, it follows by the above that v and x must be adjacent. Now the graph G has been completely determined, in particular, we have found that G contains four vertices of degree 2 and four vertices of degree 3. However, it is now also easy to see that $\overline{G^2}$ contains six vertices of degree 2 and two vertices of degree 3, and so we have a contradiction to the fact that G and $\overline{G^2}$ are isomorphic. This shows that G cannot contain K_4^- as a subgraph.

(vi) Suppose that G contains a 4-cycle $C : p, q, r, s$. Since G is subcubic, connected and has order 8, not both diagonals of C can be presented in G, and so, by (v), no diagonal of C is present in G.

Similarly, not all vertices of C can have degree 2 in G. If three vertices of C has degree 2 in G, then it is easy to seee that $\Delta(\overline{G^2}) \geq 4$, a contradiction. Hence at most two of the vertices of C have degree 2 in G. Suppose that C has two adjacent vertices of degree 2. We may assume, w.l.o.g., that $\deg(p, G) = \deg(q, G) = 2$. By (iii), G does not contain a 5-cycle, and so s and r have two distinct neighbors, s' and r' respectively, in $V(G) \setminus V(C)$. Let t and u denote the two vertices of $V(G) \setminus (V(C) \cup \{r', s'\})$. Since G is connected and subcubic, we may, w.l.o.g., assume that t is adjacent to s'. Now the 2-ball $B_2[s]$ contains all vertices of G except u, and so the vertex s has degree 1 in $\overline{G^2}$, a contradiction. This contradiction implies that no two adjacent vertices of C both have degree 2 in G. This implies that C contains two non-adjacent vertices both of degree 3. By symmetry, we may assume that both q and s have degree 3 in G. Let t and u denote the neighbors of q and s, respectively, in $V(G) \setminus V(C)$. If $t = u$, then G contains $K_{2,3}$ as a subgraph; a contradiction to (iv). Hence $t \neq u$. Let v and x denote the two vertices of $V(G) \setminus (V(C) \cup \{t, u\})$. Since G contains no 5-cycle, it follows that t and u can be adjacent to no other vertices of $V(C)$ except q and s, respectively. Similarly, t and u are not adjacent. If one of the vertices t and u are adjacent to both v and x, then $\delta(\overline{G^2}) \leq 1$, a contradiction. Since $\delta(G) \geq 2$, it now follows that we may assume that t and u are adjacent to v and x, respectively.

If one of the vertices p and r are adjacent to v or x, then $\delta(\overline{G^2}) \leq 1$, a contradiction. This implies that both p and r has degree 2 in G. Now in order for both v and x to have degree at least 2 in G, it must be the case that v and x are adjacent. It follows from the above that no further edges can be present in G, and so G is isomorphic to $C_7[2, 1, 1, 1, 1, 1, 1]$. This completes the argument.

(vii) Suppose G contains a triangle $T : pqr$. By (iii) and (vi) we may assume that G contains neither a 4-cycle nor a 5-cycle. By (i), $\delta(G) \geq 2$. Of course, since G is connected, at least one of the vertices of T has degree 3 in G.

If T contains just one vertex of degree 3 in G, then each of the vertices of T of degree 2 in G has degree 4 in $\overline{G^2}$, a contradiction.

Suppose that exactly one of the vertices of T has degree 2 in G, say p has degree 2 in G. Let q' and r' denote the neighours of q and r, respectively, in $V(G) \setminus V(T)$. Since G

contains no 4-cycles, q' and r' are distinct vertices, and, since G contains no 5-cycle, q' and r' are non-adjacent. Since $\delta(G) \geq 2$, it follows that q' and r' both have neighbors in $V(G) \setminus (V(T) \cup \{q', r'\})$. Let q'' and r'' denote neighbors in $V(G) \setminus (V(T) \cup \{q', r'\})$ of q' and r', respectively. Since G contains no 5-cycle, q'' and r'' are distinct. Let x denote the unique vertex of $V(G) \setminus (V(T) \cup \{q', q'', r', r''\})$. If x is adjacent to q' or r', then we find that G^2 contains a vertex of degree 6 and so $\delta(\overline{G^2}) \leq 1$ which contradicts (i) and the fact that G and $\overline{G^2}$ are isomorphic. Since $\delta(G) \geq 2$, it must be the case that x is adjacent to both q'' and r''. The fact that G^2 contains no vertex of degree 6 or higher can be used to deduce that G contains no additional edges. Now G is the 8-cycle with one chord which is contained in a triangle, in particular, G contains six vertices of degree 2 and two vertices of degree 3 while $\overline{G^2}$ contains four vertices of degree 2 and four vertices of degree 3. This contradicts the fact that G and $\overline{G^2}$ are isomorphic.

Finally, suppose all three vertices of T have degree 3 in G. Let p', q', and r' denote the neighours of p, q, and r, respectively, in $V(G) \setminus V(T)$. Since G is connected and subcubic, G must contains an edge joining one of the vertices p', q', or r' to a vertex in $V(G) \setminus (V(T) \cup \{p', q', r'\})$. By symmetry we may assume that p' is adjacent to $V(G) \setminus (V(T) \cup \{p', q', r'\})$. This, however, implies that p has degree at least 6 in G^2 and so we obtain a contradiction as above. Thus, the argument for statement (vii) is complete.

□

Theorem B.7. *The only subcubic squco graph on eight vertices is* $C_7[2, 1, 1, 1, 1, 1, 1]$.

Proof. It is straightforward to verify that $C_7[2, 1, 1, 1, 1, 1, 1]$ is a subcubic squco graph. Let G denote a subcubic squco graph on eight vertices other than $C_7[2, 1, 1, 1, 1, 1, 1]$. By Observation B.6, G has girth at least 6 and $\delta(G) \geq 2$.

Suppose that G has girth 6, and let C denote a 6-cycle in G. Since G is connected, we may assume that some vertex $p \in V(C)$ is adjacent to some vertex $q \in V(G) \setminus V(C)$ in G. Now the vertex q has degree at least 2 in G. If q is adjacent to some vertex of $V(C) \setminus \{p\}$, then G has girth at most 5, a contradiction. Hence q is adjacent to the unique vertex of $V(G) \setminus (V(C) \cup \{q\})$ which we denote r. Now the 2-ball $B_2[p]$ contains all vertex of G except perhaps one vertex and so $\delta(\overline{G^2}) \leq 1$, a contradiction.

Suppose that G has girth 7. Let C denote a 7-cycle in G. Since $\delta(G) \geq 2$, the unique vertex of $V(G) \setminus V(C)$ must be adjacent to at least two vertices in $V(C)$. This, however, implies that G contains a cycle of length at most 5, a contradiction.

If G is of girth 8, then G is isomorphic to the 8-cycle, but the 8-cycle is not a squco graph, a contradiction.

If G is of girth larger than 8, then G is a tree, and so we have a contradiction to Theorem 5.1.

This completes the proof of the theorem. □

B.3 No subcubic square-complementary graphs has order 9

Observation B.8. *If G is a subcubic squco graph on nine vertices, then*

(i) $\Delta(G) = 3$,

(ii) $\delta(G) = 2$,

(iii) *G does not contain a 3-path uvw where* $\deg(u, G) = \deg(v, G) = \deg(w, G) = 2$,

(iv) *no 2-vertex of G is on a triangle,*

(v) *no 2-vertex of G is on a $K_{2,3}$,*

(vi) *no 4-cycle contains two adjacent 2-vertices,*

(vii) $\alpha(G) = 4$

(viii) *G contains no 5-cycles,*

(ix) *no 2-vertex of G is on a 4-cycle,*

(x) *G contains no $K_{2,3}$,*

(xi) *G contains no induced 4-cycle, and*

(xii) *G contains no 4-cycle.*

Proof. Suppose G is a subcubic squco graph on nine vertices. By Proposition 4.1, G is connected.

(i) It follows from Proposition 5.2 that G must have maximum degree 3.

(ii) It follows from Proposition 4.11 that the minimum degree of G is at least 2. Since G has maximum degree 3 and the number of vertices of odd degree in any graph is even, it follows that G contains at least one vertex of degree 2.

(iii) If G contains a 3-path uvw where $\deg(u, G) = \deg(v, G) = \deg(w, G) = 2$, then the vertex v has degree at least 4 in $\overline{G^2}$, a contradiction.

(iv) If a 2-vertex u of G is on a triangle uvw of G, then u has degree at least 4 in $\overline{G^2}$, a contradiction.

(v) Proved by an argument similar to the one given for (iv).

(vi) Proved by an argument similar to the one given for (iv).

(vii) It follows from (i) and Proposition 4.10, that the independence number of G is at least 4. Since G is connected and of maximum degree 3, it easily follows that the independence number of G is at most 6. Let S denote an independent set of G of cardinality $\alpha(G)$, and let T denote the set $V(G) \setminus S$.

Suppose $\alpha(G) = 6$. Then, since $|S| = 6$, $\delta(G) = 2$ and S is an independent set, it follows that G contains at least 12 (S,T)-edges. On the other hand, since $|T| = 3$ and $\Delta(G) = 3$, it follows that G contains at most 9 (S,T)-edges, a contradiction.

Suppose $\alpha(G) = 5$. By an argument as above, it can be shown that G contains at least 10 and at most 12 (S,T)-edges.

(1) Suppose that G contains exactly 10 (S,T)-edges. Then, since $|S| = 5$ and $\delta(G) = 2$, it follows that each vertex in S have degree 2. Since $\Delta(G) = 3$ and the number of 3-vertices in G must be even, it follows that T contains either two or four 3-vertices. If T contains exactly two 3-vertices, then in order for G to contain exactly 10 (S,T)-edges, it must be the case that T is an independent set, in particular, each of the 2-vertices of T have their neighbors among the vertices of S. Now, since the vertices of S are all 2-vertices, we have a contradiction to (iii). Thus, all four vertices of T must be 3-vertices. Since G contains exactly 10 (S,T)-edges, it must be the case that two vertices of T are incident to three (S,T)-edges each while the remaining two vertices, say u and v, of T are incident to two (S,T)-edges each. Let x and y denote the two neighbors of u in S. Then, in particular x and y are 2-vertices. By (iv), $N_G(u)$ is an independent set. Let p and q denote the two neighbors of v in $V(G) \setminus N_G[u]$. Then p and q are both in S, in particular, both p and q are 2-vertices.

Since $\{x, y, p, q\} \subseteq S$, neither x nor y is adjacent to p or q. To avoid that u would have degree at most 1 in $\overline{G^2}$, we conclude that x and y have a common neighbor in $T \setminus \{u\}$, say w. Similarly, p and q have a common neighbor in $T \setminus \{v\}$, say z. Vertices u, v, w, z are pairwise distinct, and the only way to complete the graph is to connect the remaining vertex, say t, (which belongs to S and is of degree 2) to w and z. This graph is not squco since only three vertices in it (namely u, v and t) have degree 2 in $\overline{G^2}$, a contradiction to the assumption that G and $\overline{G^2}$ are isomorphic.

This completes case (1).

(2) Suppose that G contains exactly 11 (S,T)-edges. Then S contains exactly four 2-vertices and one 3-vertex. Since the number of 3-vertices in G is even, it follows that T contains either one or three 3-vertices. If T contains just one 3-vertex, then G contains at most 9 (S,T)-edges, a contradiction. Hence T contains exactly three 3-vertices. Now, since G contains exactly 11 (S,T)-edges, it follows that the vertices of T must be non-adjacent. Let z denote the unique 2-vertex of T, and let x and y denote the neighbors of z. Then both x and y are in S. It follows from (iii) that at least one of the vertices x and y is a 3-vertex. We may, w.l.o.g.,

assume that x has degree 3 in G. Then y has degree 2 in G. Let p and q denote the neighbors of x distinct from z.

Let r denote the unique neighbor of y distinct from z. It follows from (vi) that r is distinct from p and q.

Now z is in T while x and y are in S. Since G is a connected bipartite graph with bipartition (S, T), it follows that p, q and r must be in T. Since $|T| = 4$, it follows that the vertices of $S' := V(G) \setminus \{p, q, r, x, y, z\}$ must be in S. Since S is an independent set, each vertex of S' must be adjacent to a vertex in $\{p, q, r\}$, in fact, each vertex of S' must have two neighbors in $\{p, q, r\}$, and so each vertex of S' must have at least one neighbor in $\{p, q\}$. This shows that x is adjacent to each vertex of the set $S' \cup \{p, q, y, z\}$ in G^2, in particular, x has degree at most 1 in $\overline{G^2}$, a contradiction. This completes case (2).

(3) Suppose that G contains exactly 12 (S, T)-edges. Then, since $|T| = 4$, it follows that the vertices of T are also mutually non-adjacent and have degree 3 in G. Since $|S| = 5$ and S is an independent set of 2-vertices and 3-vertices, it follows that S consists of exactly three 2-vertices and two 3-vertices. Let p denote one of the 3-vertices of S, and let q, r, and s denote the neighbors of p. Define $T' := \{q, r, s\}$. Since q, r and s are all in T, they are all mutually non-adjacent 3-vertices. Let S' denote the set of vertices in $S \setminus \{p\}$ which are adjacent to at least one vertex in T'. There are exactly 6 (S', T')-edges in G.

Since each vertex in T' has two neighbors in S' and G is connected with maximum degree 3, it follows that S' must contain at least three vertices. If S' contains more than three vertices, then p has degree at most 1 in $\overline{G^2}$, a contradiction. Hence S' consists of exactly three vertices. The set S' consists of 2-vertices and perhaps one 3-vertex. For G to be connected, at least one vertex of S' is adjacent to a vertex, say t, in $V(G) \setminus (\{p\} \cup S' \cup T'\})$, and so S' must contain a 3-vertex. Now, it is easy to see that the unique vertex in $V(G) \setminus (\{p, t\} \cup S' \cup T'\})$ can only be adjacent to t. This, however, contradicts the fact that G has minimum degree 2. This completes case (3).

(viii) Suppose that G contains a 5-cycle C. Then $V(C)$ induces a 5-clique in G^2, and so $V(C)$ is an independent set of size 5 in $\overline{G^2}$, a contradiction to (vii).

(ix) Suppose that some 4-cycle $C : abcd$ contains a 2-vertex, say a is a 2-vertex. Then, by (iv) and (vi), both b and d are two non-adjacent 3-vertices. Let e and f denote the neighbor of b and d, respectively, in $V(G) \setminus V(C)$. By (v), e and f are distinct vertices, and, by (viii), $\{c, e, f\}$ is an independent set. Note that $N_2(a)$ is the set $\{c, e, f\}$.

(1) Suppose that c is a 2-vertex. Let g, h, and i denote the three vertices of $V(G) \setminus B_2(a)$. The set $\{a\} \cup N_2(a)$ is an independent set of G, and so, since $\alpha(G) = 4$, each vertex of $\{g, h, i\}$ has at least one neighbor in $N_2(a)$. Since c is a 2-vertex, at least one of the vertices e and f must have at least two neighbors in the set $\{g, h, i\}$. We

may, w.l.o.g., assume that e is adjacent to g and h. Thus, $N(e) = \{b, g, h\}$, and so i must be adjacent to f (or else, $\{a, c, e, f, i\}$ is an independent set with 5 vertices, contradicting (vii)). If neither g nor h is adjacent to f, then $\{a, c, f, g, h\}$ is an independent set of G of size 5, a contradiction. Thus, we may, w.l.o.g., assume that h is adjacent to f. Now, both g and i must have at least one additional neighbor; if g and i are not adjacent, then they must both be adjacent to h which implies that h had degree at least 4, a contradiction. Hence g and i are adjacent, but then *egifh* is a 5-cycle in G, which contradicts (viii). This final contradiction implies that c cannot have degree 2.

(2) Suppose that c is a 3-vertex. Let g denote the unique neighbor of c in $V(G) \setminus B_2(a, G)$. Let h and i denote the two vertices of $V(G) \setminus (B_2(a, G) \cup \{g\})$.

Now, $\{a, c, e, f\}$ is an independent set of G of size 4, and so, since $\alpha(G) = 4$, both h and i must be adjacent to some vertex of $\{e, f\}$. (They cannot be adjacent to c, since $N(c, G) = \{b, d, g\}$.) If h and i have a common neighbor in $\{e, f\}$, say e, then $\deg(b, \overline{G^2}) \leq 1$, a contradiction. Thus, by symmetry, we may assume that e and f are the unique neighbors of h and i, respectively, in $N_2(a)$. Now, by (viii), g is adjacent to neither i nor h, since otherwise G would contain a 5-cycle. Since $\deg(g, G) \geq 2$, we may, w.l.o.g., assume that g is adjacent to e. Note that i must be adjacent to h since otherwise $\{b, d, g, h, i\}$ would be an independent set with 5 vertices. By (viii), f and g must be non-adjacent (for otherwise *ehifg* would be 5-cycle in G). Vertex i is a 2-vertex in G, and by (iv), vertices f and h are non-adjacent. Now, the edges of the graph G have been determined completely, and we find that $\{h, i, f\}$ induce a 3-path in G and $\deg(h, G) = \deg(i, G) = \deg(f, G) = 2$ which contradicts (iii). This contradiction completes case (2).

We have proved that no 4-cycle in G contains a vertex of degree 2 in G.

(x) Suppose that G contains a $K_{2,3}$ as a subgraph. Then $V(K_{2,3})$ induces a clique in G^2, and so $V(K_{2,3})$ is an independent set of size 5 in $\overline{G^2}$, a contradiction to (vii) and the assumption that G is isomorphic to $\overline{G^2}$.

(xi) Suppose that G contains an induced 4-cycle $C : v_1v_2v_3v_4$. By (ix), no vertex in $V(C)$ has degree 2 in G, that is, they all have degree 3. For each vertex $v_i \in V(C)$, let v_i' denote the unique neighbor of v_i in G not in $V(C)$. By (viii) and (x), G contains neither a 5-cycle nor $K_{2,3}$, and so the vertices v_1', v_2', v_3', and v_4' are all distinct. Let x denote the unique vertex of $V(G) \setminus \{v_i, v_i' \mid i \in [4]\}$. The neighbors of x must be in the set $\{v_i' \mid i \in [4]\}$. If x is adjacent to v_i' and v_{i+1}' for $i \in [3]$ or to v_1' and v_4', then it is easy to see that G contains a 5-cycle, a contradiction. Thus, since also $\delta(G) \geq 2$, we may, w.l.o.g., assume that $N(x)$ is equal to $\{v_1', v_3'\}$. Now it is easy to see that $\{v_1, v_3, v_2', v_4', x\}$ is an independent set of G of size 5, and so we have a contradiction to (vii). This completes the argument for (xi).

(xii) Suppose that G contains a 4-cycle C : *abcd*. Since G is connected, subcubic and has more than four vertices, it follows that $V(C)$ does not induce a 4-clique in G. Thus, we may assume that a and c are non-adjacent in G. By (xi), G contains no induced 4-cycle, and so it must be the case that b and d are adjacent. By (ix), both a and c must have degree 3 in G. Let a' and c' denote the unique neighbors of a and c, respectively, in $V(G) \setminus V(C)$. If a' and c' were to be identical, then the subgraph of G induced by $V(C) \cup \{a'\}$ would contain a copy of $K_{2,3}$ contradicting (x). Thus, a' are c' are distinct vertices. By Theorem 4.4, a' and c' must have neighbors a'' and c'' in $V(G) \setminus (V(C) \cup \{a', c'\})$. We may assume (again, by Theorem 4.4) that $a'' \neq c''$.

Let x denote the unique vertex of $V(G) \setminus (V(C) \cup \{a', a'', c', c''\})$. The neighbors of x must all be contained in the set $\{a', a'', c', c''\}$.

If x is adjacent to neither a'' nor c'', then it is straightforward to deduce that the subgraph of G induced by the vertices of the set $\{a', a'', c', c'', x\}$ is a 5-cycle; this contradicts (viii). Thus, we may, by symmetry, assume that x is adjacent to a''.

If x is adjacent to both a' and c', then, in order for c'' to have degree at least 2, it must be the case that c'' is adjacent to a'' which implies that the subgraph of G induced by the vertices of the set $\{a', a'', c', c'', x\}$ contains a 5-cycle, a contradiction. Hence x is not adjacent to both a' and c'.

If x is adjacent to neither a' nor c', then a'' and c'' must be the neighbors of x and in this case it is easy to derive a contradiction to either (iii), (iv), or (ix). Hence x is adjacent to at least one of the vertices a' and c'.

Suppose x is adjacent to c''. Then we may, by symmetry, assume that x is adjacent to a'. By (iv), a'' must be adjacent to c' or c''. If a'' is adjacent to c', then the subgraph of G induced by the vertices of the set $\{a', a'', c', c'', x\}$ contains a 5-cycle, a contradiction. Hence a'' is adjacent to c''. We have now determined the edges of G completely, and we see that G has just one vertex of degree 2, namely c'. However, it is easy to see that $\overline{G^2}$ contains at least two vertices of degree 2 (for example a and a'), and so we have a contradiction to the assumption that G and $\overline{G^2}$ are isomorphic. This shows that x is not adjacent to c''.

Suppose a'' and c'' are adjacent. Then, by (iv), the fact that $\Delta(G) = 3$ and x is not adjacent to both a' and c', it follows that x must be adjacent to c'. Moreover, since $\Delta(G) = 3$, c' and a'' are non-adjacent, and so it the subgraph of G induced by $\{a'', c', c'', x\}$ is a 4-cycle; this contradicts (xi).

Hence a'' and c'' are not adjacent, and so, in order for c'' to have degree at least 2, it must be the case that c'' is adjacent to a'. Since $\Delta(G) = 3$, it follows that x is adjacent to c'. Now, it is easy to see that the subgraph of G induced by the vertices of the set $\{a', a'', c', c'', x\}$ contains a 5-cycle, a contradiction. This final contradiction implies that G contains no 4-cycle.

□

Proposition B.9. *There are no subcubic squco graph on nine vertices.*

Proof. Suppose G is a subcubic squco graph on nine vertices. By Proposition 4.1, G is connected. By Observation B.8, $\delta(G) = 2$ and $\Delta(G) = 3$. Let v_1 denote a vertex of degree 3 in G, and let the neighbors of v_1 be denoted v_2, v_3, and v_4.

Of course, each vertex of $N_2(v_1, G)$ has at least one neighbor in $N(v_1, G)$, and, since, by Observation B.8, G contains no 4-cycle, it follows that each vertex of $N_2(v_1, G)$ has exactly one neighbor in $N(v_1, G)$.

Since $\delta(G) = 2$ and $\Delta(G) = 3$, it must be the case that $N_{\geq 3}(v_1, G)$ contains two or three vertices. Consequently, $N_2(v_1, G)$ contains two or three vertices.

(i) Suppose $|N_2(v_1, G)| = 2$. If the two vertices of $N_2(v_1, G)$ have a common neighbor, say v_2, in $N(v_1, G)$, then it is easy to see that $N(v_1, G) \setminus \{v_2\}$ induces a triangle in G with two vertices of degree 2 in G; a contradiction to Observation B.8. Let v_5 and v_6 denote the two vertices of $N_2(v_1, G)$. We may assume that v_2 and v_3 are adjacent to v_5 and v_6, respectively. Now, in order for v_4 to have degree at least 2 in G it must be adjacent to at least one of the vertices v_2 and v_3. If v_4 is adjacent just adjacent to one of the vertices v_2 and v_3, then G contains a triangle with a 2-vertex, a contradiction to Observation B.8. Thus, v_4 is adjacent to both v_2 and v_3. This implies that the subgraph of G induced by $N[v_1, G]$ contains a 4-cycle; again, a contradiction to Observation B.8.

(ii) Suppose $|N_2(v_1, G)| = 3$. Then $|N_3(v_1, G)| = 2$. Let the vertices of $N_2(v_1, G)$ be denoted v_5, v_6, and v_7. As noted above, each vertex of $N_2(v_1, G)$ has exactly one neighbor in $N(v_1, G)$. Since $\Delta(G) = 3$, at most two vertices of $N_2(v_1, G)$ have a common neighbor in $N(v_1, G)$. Assume that two vertices of $N_2(v_1, G)$ have a common neighbor in $N(v_1, G)$; by symmetry, we may assume that v_5 and v_6 are both adjacent to v_2. By symmetry, we may assume that v_7 is adjacent to v_3. Since $\delta(G) = 2$, $\Delta(G) = 3$ and G contains no 4-cycle, it must be the case that v_4 is adjacent to v_3. This, however, implies that v_4 is a 2-vertex on a triangle in G, a contradiction to Observation B.8. Thus, we may assume that v_5 is adjacent to v_2, v_6 is adjacent to v_3, and v_7 is adjacent to v_4. Since G contains no 5-cycles, $N_2(v_1, G)$ is an independent set of G. Let v_8 and v_9 denote the two vertices of $N_3(v_1, G)$.

Suppose that one of the vertices of $N_3(v_1, G)$, say v_8, has three neighbors in $N_2(v_1, G)$. Then, since G contains no 4-cycles, v_9 is adjacent to just one vertex of $N_2(v_1, G)$. By symmetry, we may assume that v_9 is adjacent to v_7. However, v_9 cannot be adjacent to v_8, since then v_8 would have degree 4. Hence v_9 has degree 1 in G, a contradiction. Thus, both v_8 and v_9 have at most two neighbors in $N_2(v_1, G)$.

By symmetry, we may assume that v_5 and v_6 are adjacent to v_8, while v_7 is adjacent to v_9. Suppose that v_9 is adjacent to neither v_5 nor v_6. Then, v_9 must be adjacent to v_8. Moreover, to avoid that v_7 would be of degree 4 in $\overline{G^2}$, it must be the case that v_4 is adjacent to v_3 (say). Now, we see that v_8 is a vertex of degree 3 in G all the neighbors of which are of degree 2. On the other hand, each of the four vertices of degree 3 in $\overline{G^2}$ (namely, v_2, v_5, v_7, and v_9) is adjacent to some vertex of degree 3 (in $\overline{G^2}$). This contradicts the assumption that G is a squco graph.

Thus, by symmetry, we may assume that v_9 is adjacent to v_6. Neither of the edges v_2v_3 and v_3v_4 can be in G, since G contains no 5-cycles. We have now determined all adjacencies between vertices of G, except for v_2v_4 and v_8v_9.

If neither of the edges v_2v_4 and v_8v_9 is present in G, then vertex v_5 is of degree 4 in $\overline{G^2}$, a contradiction. By symmetry, we may assume that v_2 and v_4 are adjacent.

If v_8 and v_9 are adjacent, then $\overline{G^2}$ is a 2-regular graph, a contradiction. Hence v_8 and v_9 are non-adjacent. Similarly as above, we now reach a contradiction: vertex v_6 is a vertex of degree 3 in G all the neighbors of which are of degree 2. On the other hand, each of the four vertices of degree 3 in $\overline{G^2}$ (namely, v_5, v_7, v_8, and v_9) is adjacent to some vertex of degree 3 (in $\overline{G^2}$). This contradicts the assumption that G is a squco graph.

In both case we have derived a contradiction, and so there can be no subcubic squco graph on nine vertices. □

B.4 No subcubic square-complementary graphs of order 10

Given a graph G, we let $n_i(G)$ denote the number of i-vertices in G.

Proposition B.10. *There are no subcubic squco graphs on ten vertices.*

Proof. Suppose that G is a subcubic squco graph on ten vertices. By Proposition 4.1 and Proposition 5.2, G is connected and has maximum degree 3. By Proposition 4.11, G has minimum degree at least 2.

(i) Suppose $\delta(G) = 2$, and let v denote a vertex of degree 2 in G. Then, according to Proposition 4.11, the two neighbors of v both have degree 3 and v is neither on a triangle nor on a 4-cycle, and this implies that $V(G) \setminus B_2(v, G)$ consists of exactly three vertices, in particular, v has degree 3 in $\overline{G^2}$.

Thus, we have shown that $n_3(\overline{G^2}) \geq n_2(G)$. Since $\overline{G^2} \cong G$, $n(G) = 10$, and no graph contains an odd number of vertices of odd degree, it follows that $n_2(G)$ must be either 2 or 4.

(i.1) Assume first that $n_2(G) = 4$. Then there are six vertices with degree 3 and at least two of them must be adjacent.

We note first that G has girth at most 4. Assume that u, v are two adjacent vertices of degree 3, and neither u nor v belong to a triangle or a square. Then the neighbors of u other than v must have degree 2, since otherwise $N_3(u)$ contains at most one vertex. Similarly, the neighbors of v other than u have degree 2. Then the four vertices in $N_{\geq 2}(u, G) \cap N_{\geq 2}(v, G)$ must have degree 3, and form a square if $N_2(u, G) \cap N_2(v, G) = \emptyset$ or a triangle with a pendant vertex otherwise. In the first case, $\overline{G^2}$ contains six vertices with degree 2, a contradiction. If the vertices in

$N_{\geq 2}(u,G) \cap N_{\geq 2}(v,G)$ induce a triangle with a pendant vertex, then the unique vertex $w \in N_{\geq 3}(u,G) \cap N_{\geq 3}(v,G)$ is adjacent to u and v in $\overline{G^2}$ and has also degree 2 in $\overline{G^2}$ (just like u and v), but this is not possible. Hence we may assume that, for each pair u and v of adjacent vertices with degree 3 there is a square or a triangle containing at least one of u and v.

As noted above, all vertices on a triangle or square must have degree 3.

An edge uv of G cannot belong simultaneously to two triangles, since otherwise $B_2(u,G)$ contains at most 6 elements implying that G has a vertex of degree at least 4. Also an edge cannot belong simultaneously to a triangle and to a square, otherwise the five vertices involved have degree 3, and four of the remaining vertices have degree 2, which would force two of them to be adjacent.

If G has two disjoint triangles, then their six vertices have degree 3 and the remaining four vertices have degree 2. It is easy to see that this would force two vertices of degree 2 to be adjacent. Therefore, if G has a triangle then it has only one triangle and no squares.

Assume that G has only one triangle and no squares then the three vertices of the triangle have degree 3. Four of the remaining vertices have degree 2 and the remaining three vertices have also degree 3, but no two vertices of the same degree (out of these seven) can be adjacent. It is immediate that if the neighbors of all vertices in the triangle have degree 3, then two vertices of degree 2 must be adjacent. If all neighbors of two vertices of the triangle have degree 3, and the remaining one has a neighbor of degree 2 then the remaining vertex of degree 3 must be adjacent to another vertex of degree 3 (forcing another triangle or a square). If precisely two vertices of the triangle are adjacent to vertices with degree 2 then either two vertices of degree 2 are adjacent, or the two remaining vertices of degree 3 are adjacent. Finally, if all vertices of the triangle are adjacent to a vertex with degree 2 then there is no way to avoid edges between two of the three remaining vertices with degree 3. In all cases we arrive to a contradiction. Therefore G cannot have triangles and its girth is 4.

Suppose that there is a common neighbor of two opposite vertices of a square in G. Then this vertex must have degree 3, and there must be two adjacent vertices among the four vertices of degree 2 (there is only one vertex of degree 3 left, plus three free edges from the square and the common neighbor of two diagonal vertices), a contradiction. Hence the neighbors outside the square of the four vertices of the square are all distinct.

If all vertices of the square are adjacent to vertices of degree 2, the remaining two vertices of degree 3 must be adjacent, but none of them can belong to a triangle or a square. Hence there is one vertex w with degree 3 which does not belong to the square and is adjacent to a vertex of the square. Note that w is adjacent to the two vertices at distance 2 from the square. (If this were not the case, vertex w would be adjacent to another vertex not belonging to the square, say z, with a neighbor in the square. If the respective neighbors of w and z in the square, say w' and z', are adjacent, then two vertices of degree 2 in G are adjacent. If w' and z' are non-adjacent, then $V(G) \setminus B_2[w']$ consists of exactly one vertex. In either case, we reach a contradiction.) This implies that these two vertices have degree 2, and there is a vertex z with degree 3, other than w, adjacent to a vertex in the square, not belonging to the square. Then z and w must be adjacent to the two vertices at distance 2 from the square, forcing the remaining two vertices of degree 2 to be adjacent, which is impossible. This concludes the proof that a subcubic square-complementary graph of order 10 cannot have 4 vertices with degree 2.

(i.2) Let G be a square-complementary graph with ten vertices, eight of which have degree 3 while the remaining two have degree 2. At most four of the vertices with degree 3 are adjacent to a vertex with degree 2, and hence there are at least four vertices with degree 3 all whose neighbors have degree 3. Since there are eight vertices with degree 3 in G, there are at least two vertices x_1, x_2 with degree 3 all whose neighbors have also degree 3 and with the property that $1 + |N_1(x_i, G)| + |N_2(x_i, G)| = 7$ for $i = 1, 2$.

Let y_1, y_2, y_3 be the neighbors of x_1 and $N_2(x_1, G) =: \{z_1, z_2, z_3\}$. Then, for $i = 1, 2, 3$, $d_G(y_i) = 3$ and one of the edges incident to y_i is also incident to x_1. We shall consider two cases, depending on whether there is an edge between y_i and y_j for some $i \neq j$ or not.

Assume first that there is no edge between two vertices in $\{y_1, y_2, y_3\}$, that is, for $i = 1, 2, 3$ there are two edges incident to y_i that are also incident to vertices in $N_2(x_1, G)$. If all vertices in $N_2(x_1, G)$ are adjacent to precisely two vertices in $\{y_1, y_2, y_3\}$, then all vertices in $N_2(x_1, G)$ belong to a square and hence have degree 3. We note then that the number of vertices at distance at most 2 from y_i is 8, contradicting the fact that G is square-complementary with exactly two vertices with degree 2. (Here we used that no two vertices in $N_2(x_1, G)$ are adjacent, since otherwise the remaining one would be a cut vertex.) Therefore we may assume that not all vertices in $N_2(x_1, G)$ have two neighbors in $\{y_1, y_2, y_3\}$, which, without loss of generality, implies that z_1 is adjacent to the three vertices y_i, z_2 is adjacent to y_1 and y_2, and z_3 is adjacent to y_3. Then the subgraph G' of G induced by z_2, z_3 and the three vertices w_1, w_2, w_3 at distance at least 3 from x_1 has five vertices and five edges. Furthermore, z_2 must have degree 1 in G' since it belongs to a square of G. As a consequence, G' contains a square or a triangle not including z_2 as a vertex. Since every vertex of a square or a triangle must have degree 3, G must have at least 9 vertices with degree 3 ($x_1, y_1, y_2, y_3, z_1, z_2$ plus all vertices in the square or triangle), contradicting our hypothesis.

The previous analysis implies that there is an edge between y_1 and y_2 (say), forming a triangle with x_1.

Note that if two triangles share an edge uv then the vertex u cannot have at least seven vertices at distance at most 2, contradicting our hypothesis. This implies that y_3 is adjacent to two vertices in $\{z_1, z_2, z_3\}$. Assume without loss of generality that y_3 is adjacent to z_2 and z_3. Then z_1 is adjacent to y_1 (say), and y_2 must be adjacent to z_2 (say).

The vertex z_2 cannot be adjacent to z_1, since otherwise the vertex y_2 would have only six vertices at distance at most 2. On the other hand, if z_2 is adjacent to z_3 then the subgraph G^* induced by z_1, z_3 and the three vertices at distance at least 3 from x_1 has five vertices and five edges. Furthermore, z_3 must have degree 1 in G^* since otherwise z_1 would be a cut vertex of G. This implies that G^* has a triangle or a square and all the vertices in this cycle must have degree 3. But then G has at most 1 vertex with degree 2, contradicting our hypothesis. Hence we may assume that z_2 is not adjacent to z_1 and z_3, and since it belongs to a square of G, it must be adjacent to a vertex w_1 at distance 3 from x_1.

Now, the subgraph $\tilde{G}$ of G induced by z_1, z_3 and the three vertices $\{w_1, w_2, w_3\}$ at distance at least 3 from x_1 has five vertices and five edges. This implies that $\tilde{G}$ has either a triangle, a square or a pentagon, however, if there was a square in $\tilde{G}$ then all its vertices would have degree 3 in G implying that G has at most one vertex with degree 2.

If $\tilde{G}$ has a triangle, to avoid the two vertices with degree 2 to be adjacent or either of them to belong to the triangle, it must contain the vertices w_2 and w_3. Furthermore, the other vertex cannot be z_3 or w_1 since otherwise there would be nine vertices at distance at most 2 from y_3 or z_2 respectively, contradicting that G is a squco graph with exactly two vertices of degree 2. Hence, the remaining vertex of the triangle is z_1, and the two vertices of degree 2 in G are w_1 and z_3. We may assume without loss of generality that w_1 is adjacent to w_2, and z_3 to w_3. Note that w_2 and w_3 are two adjacent vertices in G, each of degree 3 with the property that both are adjacent to vertices with degree 2. The vertices with degree 2 in $\overline{G^2}$ are y_1 and z_1 with neighbors z_3, w_1 and y_3, z_2 respectively in $\overline{G^2}$, but no neighbor in $\overline{G^2}$ of y_1 is adjacent (in $\overline{G^2}$) to a neighbor in $\overline{G^2}$ of z_1. This contradicts the fact that G is a squco graph.

Finally, if $\tilde{G}$ is a pentagon, then w_2, w_3 have degree 2. Neither z_3 nor w_1 can be the common neighbor of w_2 or w_3, since otherwise the number of vertices at distance at most 2 from y_3 or z_2 would be 9, again a contradiction. It follows that the common neighbor of w_2 and w_3 is z_1. Now the number of vertices at distance at most 2 from y_1, y_3 and z_2 is 8, contradicting that G is a squco graph with only two vertices with degree 2.

(ii) Suppose $\delta(G) = 3$. Suppose, in addition, that G is bipartite. Let (A, B) denote a bipartition of G, and let p denote some vertex in A. Let q_1, q_2, and q_3 denote the three vertices of $N(p)$. Since $\overline{G^2}$ is cubic, it follows that $V(G) \setminus B_2(p, G)$ contains exactly three vertices. This, of course, means that $N_2(p, G)$ contains exactly three vertices. Let r_1, r_2, and r_3 denote the three vertices of $N_2(p, G)$, and let s_1, s_2, and s_3 denote the three vertices of $V(G)\setminus B_2(p,G)$. Since G is bipartite, cubic and the set $V(G)\setminus B_2(p,G)$ consists of just three vertices, it follows that the vertices of $V(G)\setminus B_2(p,G)$ must all be at distance 3 from p. This in turn implies that each of the three vertices of $N_3(p,G)$ has three neighbors in $N_2(p,G)$, which implies that no vertex of $N_2(p,G)$ can have a neighbor in $N_1(p,G)$, a contradiction.

Suppose that G contains an independent set S of size at least 5. Then, since G is cubic, it must be the case that $V(G)\setminus S$ and S have equal size and that G is bipartite, a contradiction. Hence, G has independence number at most 4, and so G contains no 5-cycles.

Suppose that G contains a K_4^-, that is, K_4 with one edge removed, say, $G[v_1,v_2,v_3,v_4]\cong K_4^-$ where v_1 and v_3 are non-adjacent in G. Now it is easy to see that $V(G)\setminus B_2(v_2,G)$ contains at least four vertices, and so v_2 has degree at least 4 in $\overline{G^2}$, a contradiction. This shows that G does not contain a K_4^-.

Suppose that G contains a triangle, say $G[v_1,v_2,v_3]\cong K_3$. Then, since G is connected, cubic, has order more than 4, and does not contain K_4^-, it follows that each vertex v_i with $i\in[3]$ has a unique neighbor u_i not in $\{v_1,v_2,v_3\}$, and that the vertices u_1, u_2, and u_3 are distinct. The set $V(G)\setminus B_2(v_1,G)$ must contain exactly three vertices, and so $N_2(v_1,G)$ must contain three vertices two of which are u_2 and u_3. This means that u_1 must be adjacent to u_2 or u_3 which implies that G contains a 5-cycle, a contradiction.

Thus G contains no triangle. Let u denote some vertex of G. Then, as noted above, we must have $|N_1(u,G)|=|N_2(u,G)|=|N_3(u,G)|=3$. Since G contains no triangles $N_1(u,G)$ is an independent set and $N_3(u,G)$ contains at least two non-adjacent vertices, say v and w. Now, $N_1(u,G)\cup\{v,w\}$ is an independent set of size 5 in G, a contradiction. This contradiction shows that G cannot be cubic.

□

B.5 No subcubic square-complementary graphs has order 11

Observation B.11. *There is no subcubic squco graph on 11 vertices.*

Proof. Suppose G is a subcubic squco graph on 11 vertices. By Proposition 4.1 and Proposition 5.2, G is a connected graph with maximum degree 3. Since no graph contains an odd number of vertices of odd degree, G contains at least one vertex, say v, of degree 2. Now, $\deg(v,\overline{G^2})=|N_{\geq 3}(v,G)|\geq 11-1-2-4=4$, which is impossible, since $\overline{G^2}\cong G$, where G is subcubic. This contradiction shows that there can be no subcubic squco graph on 11 vertices. □