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Bottom-Up Approach to Measure Risk for Decision-Making on Corporate Mergers and Acquisitions (M&As)

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BOTTOM-UP APPROACH TO MEASURE RISK FOR DECISION-MAKING ON CORPORATE MERGERS AND ACQUISITIONS (M&AS)

Jinwook Lee   András Prékopa

Abstract. Corporate Mergers and Acquisitions (M&As) are notoriously complex, and risk management is one of the essential aspects of the analysis process for the decision-making on M&A deals. Empirically, we see that some M&A transactions are not successful in part because of the increased exposure to correlated sectors, suggesting that the merged entity possesses increased risk (strategic, operational, financial, etc.) in the market. This motivates our research on risk evaluation processes for corporate M&A deals, and in particular for a suitable, but not necessarily convex, risk measure for multiple correlated assets capturing their stochastic dependence structure. In this paper, such multivariate risk measure is introduced and surveyed. Numerical examples are presented.

Keywords. Corporate Mergers and Acquisitions; Risk Management; Multivariate Risk; Decision Analysis

1 Introduction

From a risk management perspective, comparison of the values of risk *before and after* corporate M&A deals may be useful for the decision-making process for the M&A transactions. This is because decision makers are interested in seeing if the M&A deals will result in a risk reduction. To calculate and compare the magnitudes of risks *before and after* M&As, a convex risk measure might not be suitable for the risk evaluation processes in M&A events due to its convexity, implying subadditivity, which is one of the axioms of coherence in the sense of Artzner et al. (1999).

The reason for this is that a risk measurement by a convex risk measure always indicates that the merged company will become less risky after the event because of its subadditivity property, no matter how badly the M&A fits the existing business models of the target and acquiring firms. Most research indicates that M&A transactions from 1995 to 2005 have an overall success rate of about 50%. This suggests that a convex risk measure will play a limited role, basically no better than a coin toss, in the risk evaluation processes for the decision-making on M&As.

Diversification of assets in portfolios is mainly sought to reduce risk. However, adding new assets to the portfolio will not always be beneficial even though it will become more diversified. This might lead to an unattractive risk-return profile of the portfolio, possibly due to a poor portfolio construction. In this respect, a suitable risk evaluation method using an appropriate risk measure for portfolio construction is useful. Examples of bad M&A deals and poorly constructed portfolios strongly motivate research of multivariate risk measures.

Value-at-Risk and Conditional (or Average) Value-at-Risk are widely accepted and used by both academics and practitioners. Value-at-Risk (VaR) has already existed in the statistical literature since the second half of the 19th century, under the name of quantile or percentile. The term Value-at-Risk was introduced at the beginning of the 1990s in the financial literature. For various topics of Value-at-Risk see Jorion (2006), Saita (2007), etc. Its multivariate counterpart turned up in the stochastic programming literature, primarily in the works of Prékopa (1990, 1995, etc.).

The term Conditional Value-at-Risk (CVaR) was introduced by Uryasev and Rockafellar (2000). The same notion was named Average Value at Risk (AVaR) by Föllmer and Schied (2002). This is also called *Expected Shortfall*, or *Tail Value at Risk*. However, it had already been presented in the earlier literature by Prékopa (1973) and Ben-Tal and Teboulle (1986). CVaR is a coherent risk measure in the sense of Artzner et al. (1999) while VaR is generally not (see, e.g., Pflug (2000).). For more about coherent (or convex) risk measures, the reader is referred to Acerbi and Tasche (2001), Szegö (2002), Frittelli and Gianin (2002), Jarrow and Purnanandam (2005), Föllmer and Penner (2006), Ben-Tal and Teboulle (2007), Pflug and Römisch (2007), Föllmer and Schied (2010), etc.

In the case of multiple correlated assets, however, it is not imperative to require convexity for a reasonable risk measure, and indeed, it is generally misguided. There are reasons for that, as we mentioned earlier in this section, such as bad corporate M&A deals and poorly constructed portfolios, which may have undesirable risk-return characteristics. For decision-making on M&As, especially from a risk management perspective, we believe that comparison of the risks before and after the M&A deals would be useful, since a positive decision can be made if the M&A is expected to reduce risk, i.e., the risk of a merged firm is less than the sum of the risks of the separate acquiring and target companies.

M&As are very complex since a number of things are involved and correlated to each other, and thus the use of a risk measure capable of handling a multidimensional situation may be a useful tool for the decision-making in M&As. Multidimensional settings can be managed in large part by capturing the dependence structure among key elements involved in the M&As. In order to deal with dependency structures among multiple correlated objects, copula approach has been developed and used for various practical applications, including finance, risk management, etc. For the theoretical and applicable aspects of copula, we refer the reader to Joe (1997), Embrechts et al. (1999), Embrechts et al. (2001), Ané and Kharoubi (2003), Luciano et al. (2004), Junker and May (2005), etc. Since copula functions can be used for a representation of association among random variables, a suitable application of copula might be useful for the risk management of M&As.

However, we are especially interested in the decision-making on M&As and a decision should be based on a suitable decision analysis process before taking action on the deal. We believe that, from a risk management perspective, comparisons of the expected losses from before and after M&As should be one of the decision criteria. In other words, we not only need to handle the dependence structure of components of the M&As, but also simultaneously quantify the expected losses from before and after the M&As in order to see if the M&As would result in a risk reduction. For this reason, application of copula would not be suitable for our goal, although it may play a role in the risk management in post-M&A processes.

In a Bottom-Up way of thinking, we came up with a new multivariate risk measure which quantifies the conditional expected loss of multiple correlated assets in some unfavorable situations, and also simultaneously incorporates stochastic dependency structures among the objects. In Section 4 we introduce such risk measure under the name of the worst-case Multivariate Individual Value-at-Risk (wMIVaR), which is developed through Sections 2 and 3, by a Bottom-Up approach. For decision-making on corporate M&As, especially from a risk management perspective, an appropriate methodology is presented in Section 5. Numerical examples about corporate M&As are presented and discussed in Section 6.

2 Favorable events for multidimensional situation, and taking a Bottom-Up approach

Let $X \in R$ be a random variable, interpreted as loss. Let F denote the probability distribution function of X: $F(z) = P(X \leq z), -\infty < z < \infty$. Let Q denote the p-quantile and equal to $\text{VaR}_p(X) = F^{-1}(p)$, where, by definition, $F^{-1}(p) = \min\{u \mid F(u) \geq p\}$. The event $X \leq Q$ (or $X \leq \text{VaR}_p(X)$) is preferred to the event $X > Q$ (or $X > \text{VaR}_p(X)$). In other words, the set $\{x \mid x \leq Q\}$ is favorable and its complement $\{x \mid x > Q\}$ is unfavorable. If X means profit, then this holds analogously. For a single asset, its Value-at-Risk can be represented as a single point on the real line as demonstrated in Figure 1.

If we are dealing with multiple assets jointly, then the level of loss of some of the assets can be expressed as a set rather than a single value. For a set of multiple assets, the level of *loss* is called Multivariate Value-at-Risk (MVaR) that has been known for some time as p-quantile or *p-Level Efficient Point* (pLEP), or briefly p-efficient point. The latter concept was introduced in Prékopa

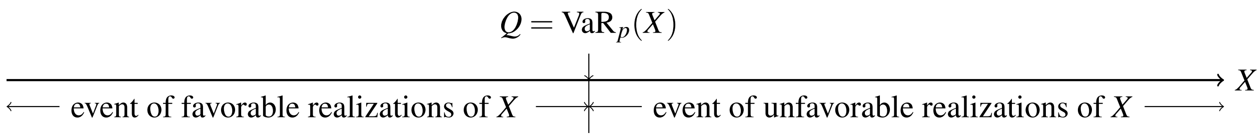


Figure 1: Favorable set and its complementary set in the case of a loss random variable

Favorable set and its complementary set of a loss random variable X.

(1990) and further studied in Prékopa et al. (1998), Prékopa (2012), Lee and Prékopa (2012), Boros et al. (2003). Multivariate Value-at-Risk (MVaR), the multivariate counterpart of VaR, is a set of points, rather than a single point as it is in the univariate case.

For the construction of portfolios, however, analysis of individual financial assets is essential, since every individual asset has its own attributes in various aspects – for example, categorization of stocks in the market can be done by business sector (healthcare, technology, services, etc.), capitalization (large, mid, or small Cap), style (growth or value) and many other different ways. Clearly, assets in different countries have different characteristics, even though the assets are of the same type, since there are various types of country risks, already reflected in the rating of assets. Furthermore, assets can also be handled in various ways by their classes; real estates, bonds, commodities, etc.

Examining the specific assets, followed by analysis of a set of these assets, can be called a Bottom-Up Approach for investment. From this point of view, the probability levels of single assets can be chosen individually based on the characteristics of each asset (though the probability levels of individual assets are not necessarily different). Then the probability level for a set of the multiple assets can be determined by (1) and (2) below.

The same reasoning applies to corporate M&A (Mergers and Acquisitions) activities. Before taking action for the M&A integration, a clear identification of the target company is essential. Furthermore, every relevant business sector should be closely examined for any potential effects of the event. In this respect, the detailed M&A plan can be made based on the Bottom-Up analysis approach to facilitate decision making. Note that M&A is a complex process so the risk measurement is only one of the key resources in a successful M&A integration.

Let us assume that we have a set of n different assets (or n different business sectors in a company). Let X_k be the loss random variable of asset k, $k = 1,\dots,n$ and p_k a given probability level for asset k, $k = 1,\dots,n$. Then asset k has its own quantile Q_k. With the probability distribution functions F_{X_k} the p_k-quantile points are $Q_k = F_{X_k}^{-1}(p_k) = \mathrm{VaR}_{p_k}(X_k)$, $k = 1,\dots,n$. Let Q be the p-quantile vector defined as:

$$Q = (Q_1,\dots,Q_n)^T = (F_{X_1}^{-1}(p_1),\dots,F_{X_n}^{-1}(p_n))^T = (\mathrm{VaR}_{p_1}(X_1),\dots,\mathrm{VaR}_{p_n}(X_n))^T, \qquad (1)$$

and define a corresponding probability level p as

$$p = P(X_1 \le Q_1,\dots,X_n \le Q_n) = F_X(Q), \qquad (2)$$

where F_X is the c.d.f. of the random vector $X \in R^n$.

Let B denote the most favorable outcome of the loss random vector $X \in R^n$, i.e.,

$$B = \{x \in R^n \mid x_1 \leq \mathrm{VaR}_{p_1}(X_1), x_2 \leq \mathrm{VaR}_{p_2}(X_2), \ldots, x_n \leq \mathrm{VaR}_{p_n}(X_n)\}. \tag{3}$$

Also let $A_j = \{X_j \leq \mathrm{VaR}_{p_j}(X_j)\}$, for $j = 1, \ldots, n$. Then the most favorable set is $B = \bigcap_{j=1}^n A_j$, and its complementary is an unfavorable set $B^c = \bigcup_{j=1}^n A_j^c$. Both are illustrated in Figure 2.

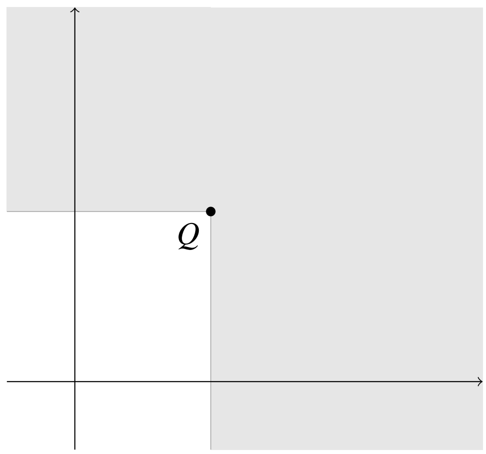


Figure 2: 2-D illustration of the favorable set and its complement.

The unshaded region represents the most favorable set B, and the shaded region describes its complement. Q is the p-quantile point in the sense that $p = P(X_1 \leq Q_1, \ldots, X_n \leq Q_n)$, $Q_k = VaR_{p_k}(X_k)$, $k = 1, \ldots, n$.

3 Multivariate Individual Value-at-Risk

In the definition of CVaR (or AVaR), we take the expectation of X given that $X > \mathrm{VaR}_p(X)$ (unfavorable outcome) if X means loss, i.e., $\mathrm{CVaR}_p(X) = E(X \mid X > \mathrm{VaR}_p(X))$. For the multivariate case (a set of multiple assets), we propose the following

Definition 1. *The Multivariate Individual Value-at-Risk, or MIVaR, of the loss random vector* $X \in R^n$ *is designated and defined as:*

$$MIVaR_p(X) = E(\psi(X) \mid X \notin B), \tag{4}$$

where ψ *is some n-variate function, B is the most favorable outcome and* $P(X \notin B) = 1 - P(X \in B) = 1 - p$, *where p is given by (2).*

MIVaR in (4) can also be rewritten as

$$\mathrm{MIVaR}_p(X) = E\left(\psi(X) \;\middle|\; X \in \bigcup_{j=1}^n A_j^c\right), \tag{5}$$

where $A_j^c = \{X_j > \text{VaR}_{p_j}(X_j)\}$ for $j = 1, \ldots, n$. Note that the event $\bigcup_{j=1}^n A_j^c$ allows for X the entire space excluding the single orthant $\{x \in R^n \mid x_j \le \text{VaR}_{p_j}(X_j),\ j = 1, \ldots, n\}$.

Let us define the function $\psi(u)$, $u = (u_1, \ldots, u_n)^T$ in the following way:

$$\psi(u) = \sum_{i=1}^{n} \lambda_i u_i, \tag{6}$$

where $\lambda_i,\ i = 1, \ldots, n$ can be chosen in a suitable way depending on how random variables are defined. For a risk evaluation process on corporate M&As, random variables may be defined as losses of business units of the companies involved in the M&As. In this case we may have $\lambda_i = 1,\ i = 1, \ldots, n$ in order to count each business unit once to quantify risk of the united firm after the M&As. For measuring risk on stock portfolios, random variables can be designated as losses of stocks. Then it would be suitable to have $\lambda_1, \ldots, \lambda_n$ to be integer-valued as their interpretation is the number of shares of corresponding stocks. We may also allow negative λ values, meaning short selling. If we want to assign weights on investment in Assets $1, \ldots, n$, then $\lambda_1, \ldots, \lambda_n$ are nonnegative constants satisfying $\sum_{i=1}^n \lambda_i = 1$. Depending on the meaning of random variables, a function $\psi(u),\ u \in R^n$ can be specialized in an appropriate way.

The calculation of MIVaR in (5) is not simple because it is defined in the space of the union of sets $A_1^c, \ldots, A_n^c$ (shaded region in Figure 2). To calculate, we can use the following equation:

$$E(\psi(X)) = E(\psi(X) \mid X \notin B) P(X \notin B) + E(\psi(X) \mid X \in B) P(X \in B), \tag{7}$$

from which we derive:

$$\begin{aligned} \text{MIVaR}_p(X) &= E(\psi(X) \mid X \notin B) \\ &= \frac{1}{P(X \notin B)} \Big(E(\psi(X)) - E(\psi(X) \mid X \in B) P(X \in B) \Big). \end{aligned} \tag{8}$$

Equation (8) can be written as:

$$\text{MIVaR}_p(X) = \frac{1}{1 - P(X \in B)} \left(\sum_{i=1}^{n} E(X_i) - \sum_{i=1}^{n} E(X_i \mid X \in B) P(X \in B) \right). \tag{9}$$

Since the set B is only a single-orthant in the n dimensional space (unshaded region in Figure 2), the formulation of (9) can be calculated. The set B represents the event of the best-case realizations of the random vector $X \in R^n$ as it is illustrated as an unshaded region in Figure 2. Multivariate Individual Value-at-Risk (MIVaR) gauges the expected loss amount in the set B^c, where the unfavorable events occur. B^c represents the whole space excluding the best-case scenarios and therefore B^c is the least risky of unfavorable events.

Let us now turn our attention to the worst-case event, the riskiest event among all possible outcomes under a set of individual probability levels. The expected loss amount in the worst-case event is clearly the largest, and so would cover any other risky situation. For this reason, it may be used for the calculation of minimum (but safe) required reserve for financial institutions, and this is our motivation in the next section.

4 Multivariate Individual Value-at-Risk in the worst-case event

The worst-case event should be considered in practice for various purposes: trading operations, asset management, or any business where a short-term catastrophe could result in complete collapse of the entity. The worst-case event will focus on a possible loss given that no favorable event occurs. Let W denote the worst-case event. If a random vector $X = (X_1, \ldots, X_n) \in R^n$ means losses, then the event W can be written as

$$\begin{aligned} W &= \left\{x \in R^n \mid x_i > \mathrm{VaR}_{p_i}(X_i), i = 1, \ldots, n\right\} \\ &= \left\{x \in R^n \mid x > Q\right\}, \text{ where } Q = \left(\mathrm{VaR}_{p_1}(X_1), \ldots, \mathrm{VaR}_{p_n}(X_n)\right)^T. \end{aligned} \tag{10}$$

Definition 2. *The worst-case MIVaR of the loss random vector $X \in R^n$ is designated and defined as:*

$$wMIVaR_p(X) = E(\psi(X) \mid X \in W), \tag{11}$$

where ψ is some n-variate function, W denotes the worst-case set as in (10), and the probability level p is

$$p = P(X \notin W) = 1 - P(X_1 > VaR_{p_1}(X_1), \ldots, X_n > VaR_{p_n}(X_n)). \tag{12}$$

Let us define the function $\psi(u)$ as in (6) with $\lambda_i = 1,\ i = 1, \ldots, n$. Note that the function ψ can be specialized in various ways, depending on the characteristics of the business, as mentioned earlier in Section 3.

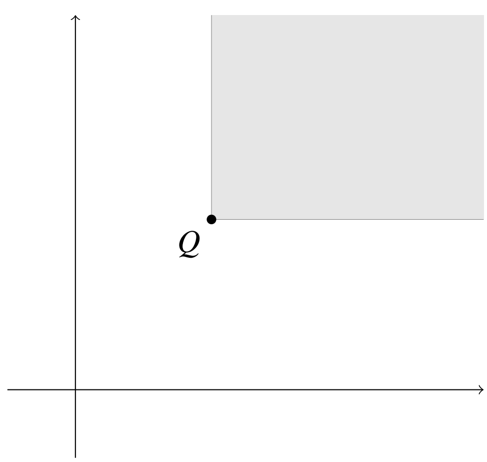


Figure 3: 2-D Illustration of the worst-case unfavorable set

2-D Illustration of the worst-case unfavorable set $W = \{x \in R^n \mid x > Q\}$ is the shaded region, where $Q = (VaR_{p_1}(X_1), \ldots, VaR_{p_n}(X_n))^T$.

The worst-case MIVaR (wMIVaR) can easily be calculated directly from the definition because the set W is a single-orthant in n-dimensional space, as described in Figure 3. If a random loss vector $X \in R^n$ has a continuous distribution, then the worst-case MIVaR (wMIVaR) can be formulated as:

$$\begin{aligned} \mathrm{wMIVaR}_p(X) &= E(\psi(X) \mid X \in W) = E(\psi(X) \mid X > Q) \\ &= E(X_1 + \cdots + X_n \mid X_1 > \mathrm{VaR}_{p_1}(X_1), \ldots, X_n > \mathrm{VaR}_{p_n}(X_n)) \\ &= \frac{\int_{\mathrm{VaR}_{p_n}(X_n)}^{\infty} \cdots \int_{\mathrm{VaR}_{p_1}(X_1)}^{\infty} (t_1 + \cdots + t_n) f_X(\mathbf{t})\, dt_1 \ldots dt_n}{\int_{\mathrm{VaR}_{p_n}(X_n)}^{\infty} \cdots \int_{\mathrm{VaR}_{p_1}(X_1)}^{\infty} f_X(\mathbf{t})\, dt_1 \ldots dt_n}, \end{aligned} \tag{13}$$

where f_X is the probability density function of a random vector $X \in R^n$ and the probability level p of $\mathrm{wMIVaR}_p(X)$ is given by

$$p = P(X \notin W) = 1 - \int_{\mathrm{VaR}_{p_n}(X_n)}^{\infty} \cdots \int_{\mathrm{VaR}_{p_1}(X_1)}^{\infty} f_X(\mathbf{t})\, dt_1 \ldots dt_n. \tag{14}$$

Theorem 1. *For a loss random vector $X = (X_1, \ldots, X_n)$ with independent components, we have the equation,*

$$wMIVaR_p(X) = CVaR_{p_1}(X_1) + \cdots + CVaR_{p_n}(X_n), \tag{15}$$

where we assume that $E(X_i)$, $i = 1, \ldots, n$ exist. The probability level p of $wMIVaR_p(X)$ is determined by the individual probability levels p_i of $CVaR_{p_i}(X_i)$, $i = 1, \ldots, n$ as in (12) or (14).

Proof. We have the equations

$$\begin{aligned} \mathrm{wMIVaR}_p(X) &= E\left(X_1 + \cdots + X_n \,\middle|\, \bigcap_{j=1}^n A_j^c\right) \\ &= \sum_{i=1}^n E\left(X_i \mid A_1^c \cdots A_n^c\right) \\ &= \sum_{i=1}^n E\left(X_i \mid A_i^c\right) \\ &= \sum_{i=1}^n E\left(X_i \mid X_i \geq \mathrm{VaR}_{p_i}(X_i)\right) \\ &= \sum_{i=1}^n \mathrm{CVaR}_{p_i}(X_i). \end{aligned} \tag{16}$$

□

Note that for a univariate random variable X, $\mathrm{wMIVaR}_p(X)$ is the same as $\mathrm{CVaR}_p(X)$.

Theorem 2. *Let $X, Y \in R^n$ be random vectors with finite expectations. Then the worst-case MIVaR (wMIVaR) exhibits the following properties:*

(1) $wMIVaR_p$ is translation-invariant: $wMIVaR_p(X + c) = wMIVaR_p(X) + c$.

(2) $wMIVaR_p$ is positively homogeneous: $wMIVaR_p(cX) = c \times wMIVaR_p(X), c \in R_+$.

(3) $wMIVaR_p$ is subadditive: $wMIVaR_p(X + Y) \leq wMIVaR_p(X) + wMIVaR_p(Y)$, when all $2n$ components in X and Y are independent.

(4) *$wMIVaR_p$ is monotonic with respect to the second order stochastic dominance:*

$$X \prec_{SD(2)} Y \textit{ implies } wMIVaR_p(X) \leq wMIVaR_p(Y),$$

when all $2n$ components of X and Y are independent.

(5) *$wMIVaR_p$ is additive in the sense that*

$$wMIVaR_p((X,Y)^T) = wMIVaR_\alpha(X) + wMIVaR_\beta(Y),$$

when all $n+m$ components in $X \in R^n$ and $Y \in R^m$ are independent. The probability level p is determined by the associated probability levels α and β as in (12).

Proof. The proofs of (1) and (2) are simple and therefore omitted.

(3) By Theorem 1, $\mathrm{wMIVaR}_p(X) = \sum_{i=1}^n \mathrm{CVaR}_{p_i}(X_i)$ and $\mathrm{wMIVaR}_p(Y) = \sum_{i=1}^n \mathrm{CVaR}_{p_i}(Y_i)$. Conditional (or Average) Value-at-Risk satisfies subadditivity, one of the coherence axioms of Artzner et al. (1999), which implies that $\sum_{i=1}^n \mathrm{CVaR}_{p_i}(X_i+Y_i) \leq \sum_{i=1}^n \mathrm{CVaR}_{p_i}(X_i) + \sum_{i=1}^n \mathrm{CVaR}_{p_i}(Y_i)$. Thus it can be written as follows.

$$\begin{array}{rcl} \mathrm{wMIVaR}_p(X+Y) & = & \sum_{i=1}^n \mathrm{CVaR}_{p_i}(X_i+Y_i) \\ & \leq & \sum_{i=1}^n \mathrm{CVaR}_{p_i}(X_i) + \sum_{i=1}^n \mathrm{CVaR}_{p_i}(Y_i) \\ & = & \mathrm{wMIVaR}_p(X) + \mathrm{wMIVaR}_p(Y). \end{array}$$

(4) About multivariate stochastic orders we refer the reader to the literature, e.g., Müller and Stoyan (2002). If X is second order stochastically dominated by Y, i.e., $X \prec_{\mathrm{SD}(2)} Y$, together with the independence assumption on the all components of X and Y, then $X_i \prec_{\mathrm{SD}(2)} Y_i,\ i = 1,\ldots,n$, which implies $\mathrm{CVaR}_{p_i}(X_i) \leq \mathrm{CVaR}_{p_i}(Y_i),\ i = 1,\ldots,n$ (see Pflug (2000).). By Theorem 1, we have $\mathrm{wMIVaR}_p(X) \leq \mathrm{wMIVaR}_p(Y)$ if $\mathrm{CVaR}_{p_i}(X_i) \leq \mathrm{CVaR}_{p_i}(Y_i),\ i = 1,\ldots,n.$

(5) It is a simple consequence of Theorem 1. When all components of the random vectors $X \in R^n$ and $Y \in R^m$ are independent of each other, we have

$$\begin{array}{rcl} \mathrm{wMIVaR}_p((X,Y)^T) & = & \mathrm{wMIVaR}_p((X_1,\ldots,X_n,Y_1,\ldots,Y_m)^T) \\ & = & \sum_{i=1}^n \mathrm{CVaR}_{p_i}(X_i) + \sum_{i=1}^m \mathrm{CVaR}_{p_{n+i}}(Y_i) \\ & = & \mathrm{wMIVaR}_\alpha((X_1,\ldots,X_n)^T) + \mathrm{wMIVaR}_\beta((Y_1,\ldots,Y_m)^T) \\ & = & \mathrm{wMIVaR}_\alpha(X) + \mathrm{wMIVaR}_\beta(Y), \end{array} \tag{17}$$

where

$$\begin{array}{rcl} \alpha & = & P(X_1 \geq \mathrm{VaR}_{p_1}(X_1),\ldots,X_n \geq \mathrm{VaR}_{p_n}(X_n)), \\ \beta & = & P(Y_1 \geq \mathrm{VaR}_{p_{n+1}}(Y_1),\ldots,Y_m \geq \mathrm{VaR}_{p_{n+m}}(Y_m)). \end{array} \tag{18}$$

□

Remark 1 (Geometric relationships among wMIVaR, VaR and CVaR). *For a random vector $X \in R^n$, if the worst-case set W in (10) is projected onto spaces of $X_j \in R,\ j = 1,\dots,n$, then the lower bound of the projection onto the space of $X_j \in R$ is $VaR_{p_j}(X_j),\ j = 1,\dots,n$. The projection of W is illustrated in Figure 4 for the 2-dimensional case. In Figure 4, the sets $\{z \mid z > VaR_{p_1}(X_1)\}$ and $\{z \mid z > VaR_{p_2}(X_2)\}$ are the projections of $W = \{x \in R^2 \mid x_1 > VaR_{p_1}(X_1), x_2 > VaR_{p_2}(X_2)\}$ onto the horizontal and vertical axes, respectively. If the random vector $X \in R^2$ has independent components X_1 and X_2, then, by Theorem 1, the following equation holds true: $wMIVaR_p(X) = CVaR_{p_1}(X_1) + CVaR_{p_2}(X_2)$. Figure 4 can be used to illustrate this.*

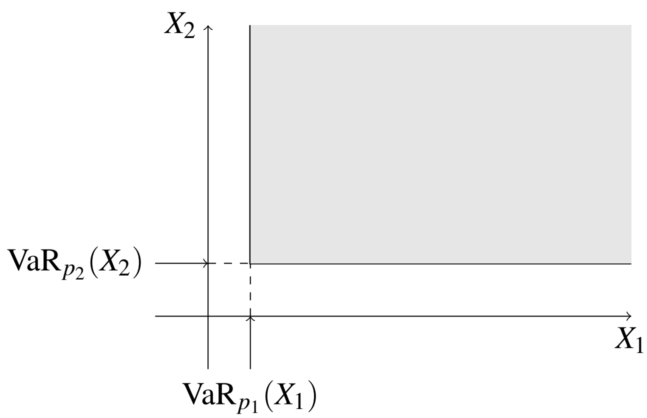


Figure 4: Description of projections of the set W

Projections of the set W, i.e., the worst case outcome as in (10) of the loss random vector $X = (X_1, X_2)^T$, from $\mathbf{R^2}$ onto the space of $\mathbf{X_1}$, $\mathbf{X_2} \in \mathbf{R}$. The points $VaR_{p_1}(X_1)$ and $VaR_{p_2}(X_2)$ are the lower bounds of the projection of the set $W \in R^2$, onto the space of X_1, $X_2 \in R$, respectively.

Remark 2 (Individual Probability levels of wMIVaR). *The probability level p for a set of multiple assets is determined by the probability levels for individual assets, as $p = 1 - P(X \in W)$, where $W = \{X \mid X_1 \geq VaR_{p_1}(X_1), \dots, X_n \geq VaR_{p_n}(X_n)\}$. If $P(X \in W)$ is too small (i.e., individual probability levels are large), then the wMIVaR associated with the corresponding p-level would not be useful in decision-making. This is because if the probability of the worst-case scenario is close to zero, then wMIVaR may become way too large. From a practical point of view, if the wMIVaR is used for the calculation for the reserve requirement of a financial institution, then it will result in an excessive reserve amount, and as a consequence, a negative effect on cash flow may occur. With this in mind, if we are dealing with a "high" dimensional case then "small" individual probability levels should be used to obtain a reasonable probability level p.*

5 Decision-making by measuring potential risk on corporate M&As (Mergers and Acquisitions) and demerger activities

Suppose that there is an ongoing analysis for a corporate M&A deal between Companies X and Y, and both companies run many different businesses. Assume that there are n and m business units in Companies X and Y, respectively. Let the random variables $X_i,\ i = 1,\dots,n$ and $Y_j,\ j = 1,\dots,m$ denote losses of the associated business sectors of Companies X and Y. Then the random vectors $X \in R^n$ and $Y \in R^m$ mean losses of the associated companies.

For a risk evaluation process on the corporate M&A deal for companies X and Y, a potential risk of the united firm can be calculated by

$$\text{wMIVaR}_p((X,Y)^T), \tag{19}$$

where the random vector $(X,Y)^T \in R^{n+m}$ means the losses of the new merged entity, and all components of the random vector $(X,Y)^T$ are loss random variables of corresponding business sectors of a new firm after the M&A deal of Company X and Y. For the M&A event of companies X and Y, it is reasonable to use a random vector $(X,Y)^T$, not a single variable $Z = \sum_{i=1}^n X_i + \sum_{j=1}^m Y_j$.

The reason behind this mathematical expression on the M&A risk evaluation as in (19) is that any two companies may not be the same in various aspects (e.g., quality of products and group of target customers, etc.). Furthermore, even if products of these companies were offering the same types of items or services before the M&A event, some differences between the products will continue to exist after the M&A event. More importantly, after an M&A deal the new company becomes a multi-plant firm and each business sector is still producing the same goods or services as it did for the former individual companies.

Successful integration of companies takes quite a long time, and the early phases of the integration process is sensitive and critical. We know that timing is crucial in a decision-making process. A risk measurement for a more critical period would be exceedingly valuable in setting up a suitable risk management plan. A restructuring project for a united firm can be initiated in the initial time period of the post-M&As while its business units are still operating in the same way as before the M&As. Thus, for risk evaluation for decision-making on M&As, it would be better to see all the business units individually as they were before M&As, but simultaneously look at all operating units as a whole, i.e., the analysis on the relationships among the business units of the united firm. This implies that the risk measure on a random vector $(X,Y)^T$ would be suitable for decision-making processes on M&As. This type of analysis is also in accord with our logical flow for this paper: a Bottom-Up approach.

In order to see the logic behind the mathematical risk assessment for decision-making on M&A deals, let us consider the simplest case: a univariate random variable corresponding to loss for each company, and a bivariate random vector meaning loss for a united firm. Let the random variables $X \in R$ and $Y \in R$ denote losses of Companies X and Y, respectively. Then $\text{CVaR}_p(X)$ and $\text{CVaR}_p(Y)$ indicate the amounts of losses of the respective Companies X and Y (i.e., the expected magnitude of losses beyond the $\text{VaR}_p(X)$, $\text{VaR}_p(Y)$, respectively). Suppose that M&A activity occurs between these companies and a new merged company is formed. Now let X and Y be random variables corresponding to losses of the business sectors X and Y of the new firm, respectively. If these business sectors are operating totally independently, then the risk associated with the new company should be $\mathrm{CVaR}_p(X) + \mathrm{CVaR}_p(Y)$: a simple sum of the risks.

However, if these are correlated, then the risk *after* the M&A deal is not the same, in general, as the simple sum of risks of the companies. In the case of correlated losses from the business sectors X and Y of the new company, risk measurement *before and after* the deal would be of either

$$\begin{array}{ll} \text{superadditivity:} & \mathrm{wMIVaR}_{p^*}\left((X,Y)^T\right) \geq \mathrm{CVaR}_p(X) + \mathrm{CVaR}_p(Y), \text{ or} \\ \text{subadditivity:} & \mathrm{wMIVaR}_{p^*}\left((X,Y)^T\right) \leq \mathrm{CVaR}_p(X) + \mathrm{CVaR}_p(Y), \end{array} \tag{20}$$

where p^* is determined by the individual probability level p as in (12). Between the two cases in (20), the case of subadditivity would be desirable for M&A deals, while the case of superadditivity is ideal for demerger (split-up) activities. This is because the LHS of (20) indicates a risk measurement of a united firm, and each term of the RHS of (20) represents that of a single company before an M&A event (or after a demerger activity). The same way of reasoning applies to an M&A event for multi-plant firms.

It is important to note that the risk evaluation process must be very detailed. For example, all business sectors that belong to the companies should be included and analyzed in the evaluation process, especially for their correlation structure. Furthermore, there are many types of risk evaluations useful for M&A deals, including strategic risk, compliance risk, operational risk, financial risk, reputation risk, etc., and each type of risk needs to be evaluated by a suitable risk measurement.

In the real world, the estimation of operational risk is complex and necessitates input from subject matter experts. Indeed, risk measurement is just one of the various key factors to analyze for in M&A decision-making processes. There are many other key factors to be considered as well, such as acculturation, human resource issues, post-marketing, and taxation to name just a few. For more about corporate M&A activities, we refer the reader to the literature (e.g., see Nahavandi and Malekzadeh (1988), Walsh (1988), Berger et al. (1998), Erickson (1998), Hagedoorn and Duysters (2002), Shimizu et al. (2004), Gregoriou and Renneboog (2007), Wulf and Singh (2011), etc.).

In section 6 numerical examples of corporate M&A deals with calculations of (19) for suitable comparisons as in (20) are presented and discussed.

6 Numerical examples and discussion

Adequate measure of potential risk in a corporate M&A (Mergers and Acquisitions) is essential. In this section we want to show, with illustrative examples of corporate M&A deals, how the worst-case Multivariate Individual Value-at-Risk (wMIVaR) plays a role in risk evaluation processes for various cases in terms of the stochastic dependence structure. Examples 1 and 2 present the cases of corporate M&A deals.

Example 1. *Company $\mathscr{A}$, a local newspaper company, wants to expand its business scope through a good M&A deal before the fourth quarter (Q4) of the year. Suppose that there are six candidates for the deal labeled by* $1,2,3,4,5$ *and 6, local business competitors of Company $\mathscr{A}$. The risk management arm of Company $\mathscr{A}$ wants to gauge and compare the risks to select the best case,* *in terms of risk, for the decision-making. The expected Q4 operational profits for the companies are: 4% for Company $\mathscr{A}$, -3% for Company 1, -4% for Company 2, 0.5% for Company 3, 2% for Company 4, 3% for Company 5 and 5% for Company 6.*

Let X denote the Q4 operational loss of Company $\mathscr{A}$, a normal random variable with mean -4% and variance 1, i.e., $X \sim N(-0.04, 1)$. Let $Y_i,\ i = 1,2,3,4,5,6$ be the normal random variables with

$$\begin{array}{lll} Y_1 \sim N(0.03, 0.7^2), & Y_2 \sim N(0.04, 1.7^2), & Y_3 \sim N(-0.005, 2^2), \\ Y_4 \sim N(-0.02, 1.6^2), & Y_5 \sim N(-0.03, 1.8^2), & Y_6 \sim N(-0.05, 1.5^2), \end{array} \tag{21}$$

meaning the Q4 operational losses of Companies $1,2,3,4,5$ and 6, respectively.

Then the random vectors $(X,Y_i)^T \in R^2,\ i = 1,2,3,4,5,6$ are assumed to have bivariate normal distributions, interpreted as Q4 operational losses of business units of the newly united firms. Suppose that the pairs $(X,Y_i),\ i = 1,2,3,4,5,6$ have correlation coefficients: $\rho = 0.9, -0.6, 0, 0.5, -0.3, 0.7$, respectively. Then we have

$$\begin{array}{lll} \mathrm{Cov}(X,Y_1) = (0.9)(0.7) = 0.63, & \mathrm{Cov}(X,Y_2) = (-0.6)(1.7) = -1.02, & \mathrm{Cov}(X,Y_3) = 0, \\ \mathrm{Cov}(X,Y_4) = (0.5)(1.6) = 0.8, & \mathrm{Cov}(X,Y_5) = (-0.3)(1.8) = -0.54, & \mathrm{Cov}(X,Y_6) = (0.7)(1.5) = 1.05. \end{array} \tag{22}$$

To make a decision regarding all the possible M&A events, especially in a risk evaluation process, two types of comparisons could be useful:

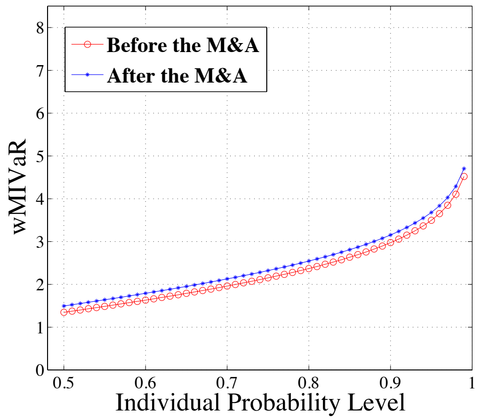
(a) Which M&A deal is ideal in terms of the risk?
We need to choose the smallest value among $\mathrm{wMIVaR}_p((X,Y_i)^T),\ i = 1,\dots,6$, i.e., compare the risk measurements *after* the M&A deals for all cases.

(b) Which M&A deals reduce risk?
We need to check which case will reduce the risk through the M&A deal, or equivalently, check if $\mathrm{wMIVaR}_p((X,Y_i)^T) < \mathrm{CVaR}_\alpha(X) + \mathrm{CVaR}_\beta(Y_i),\ i = 1,\dots,6$. This is the comparison of the risk measurements *before and after* for the M&A deal.

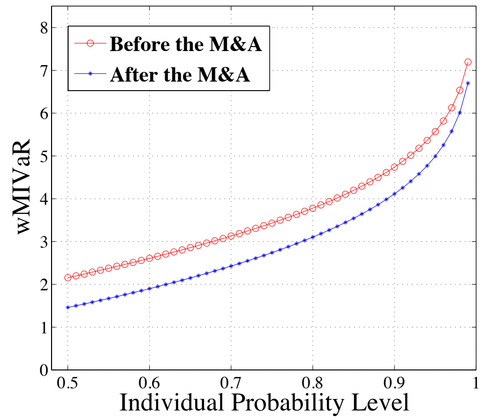
Let us use the same individual probability levels for each company. Then for each pair of merged companies the worst-case Multivariate Individual Value-at-Risk (wMIVaR) can be calculated by:

$$\begin{aligned} \mathrm{wMIVaR}_{p^*}((X,Y_i)^T) &= E(X + Y_i \mid X > \mathrm{VaR}_p(X), Y_i > \mathrm{VaR}_p(Y_i)) \\ &= \frac{\displaystyle\int_{\mathrm{VaR}_p(X)}^{\infty} \int_{\mathrm{VaR}_p(Y_i)}^{\infty} (x+y) f(x,y)\, dxdy}{\displaystyle\int_{\mathrm{VaR}_p(X)}^{\infty} \int_{\mathrm{VaR}_p(Y_i)}^{\infty} f(x,y)\, dxdy}, \end{aligned} \tag{23}$$

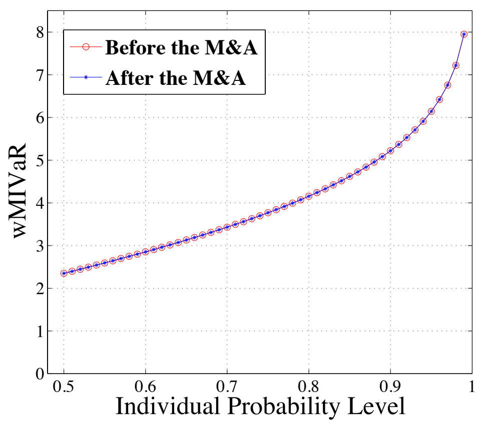
where $p^* = 1 - \int_{\mathrm{VaR}_p(X)}^{\infty} \int_{\mathrm{VaR}_p(Y_i)}^{\infty} f(x,y)\, dxdy$. In (23), f is the bivariate standard normal p.d.f. of $(X,Y_i),\ i = 1,\dots,6$ with expectations in (21) and covariances (22).



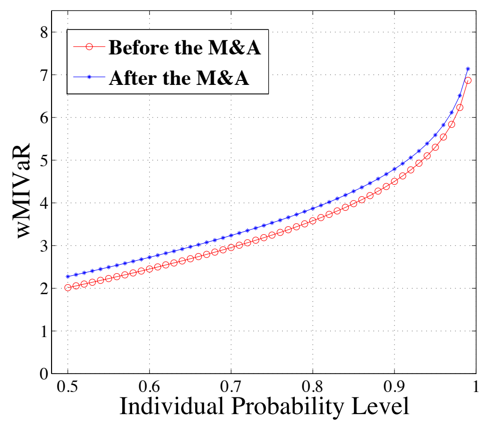
(a) Risk measurement of (X, Y_1)



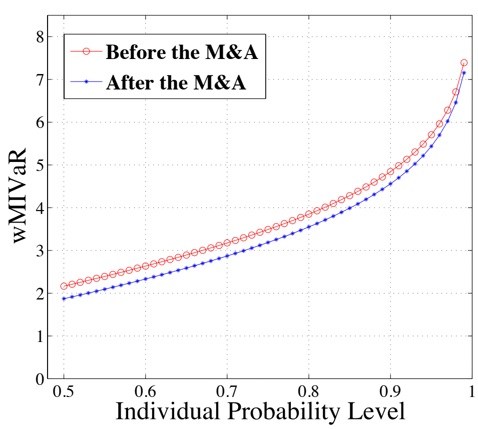
(b) Risk measurement of (X, Y_2)



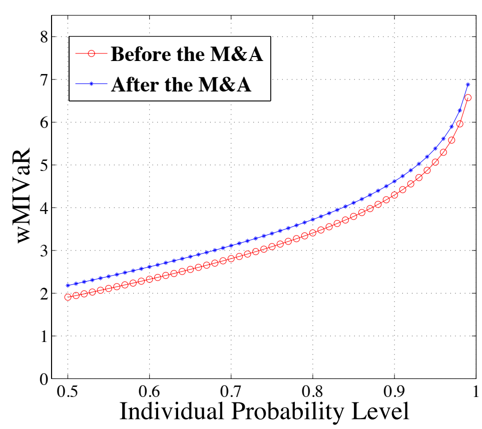
(c) Risk measurement of (X, Y_3)



(d) Risk measurement of (X, Y_4)



(e) Risk measurement of (X, Y_5)



(f) Risk measurement of (X, Y_6)

Figure 5: Comparison of risks among different M&A deals

We calculate wMIVaR at the same individual probabilities (two decimal places) from 0.50 to 0.99, and plot the results on the subfigures of Figure 5. Note that the wMIVaR is a function of the individual probability levels. For clarification, risk *after* M&As, i.e., wMIVaR at $p = 0.6, 0.7, 0.8, 0.9, 0.95$ are summarized in Table 1. In Table 2, risks *before* M&As, i.e., $\text{CVaR}_p(X) + \text{CVaR}_p(Y_i), i = 1, \ldots, 6$ are also presented.

We have obtained the following risk measurements for *before and after* the M&A deals between Companies $\mathscr{A}$ and its six target companies. For the M&As with Companies 1, 4 and 6, we have *superadditive* relationships:

$$\text{wMIVaR}_{p^*}((X, Y_i)^T) > \text{CVaR}_p(X) + \text{CVaR}_p(Y_i),\ i = 1, 4, 6, \tag{24}$$

for M&A activity with Company 3, we have additive relationship:

$$\text{wMIVaR}_{p^*}((X, Y_3)^T) = \text{CVaR}_p(X) + \text{CVaR}_p(Y_3), \tag{25}$$

and for the M&As with Companies 2 and 5, we have *subadditive* relationships:

$$\text{wMIVaR}_{p^*}((X, Y_i)^T) < \text{CVaR}_p(X) + \text{CVaR}_p(Y_i),\ i = 2, 5. \tag{26}$$

From a risk management perspective, cases of (26) are desirable since the mergers result in a risk reduction. These ideal cases can also be found from subfigures (b) and (e) of Figure 5. If decision will be made solely on the risk measurements, then (b) will be preferred to (e) since $\text{wMIVaR}_{p^*}(X, Y_2) < \text{wMIVaR}_{p^*}(X, Y_5)$. Note the case of the M&A between Companies $\mathscr{X}$ and 1 could also be beneficial, since $\text{wMIVaR}_{p^*}(X, Y_1)$ has the smallest value among all cases. In this respect, a thorough case study over the pairs of (X, Y_1) and (X, Y_2) will help decision makers to find a better deal and take action for the successful integration.

Table 1: Risk after M&A deals, i.e., risk of the new united firms

The worst-case MIVaR (wMIVaR), i.e., risk after M&As					
Individual p-levels:	$p = 0.60$	$p = 0.70$	$p = 0.80$	$p = 0.90$	$p = 0.95$
$\text{wMIVaR}_{p^*}(X, Y_1)$	1.79025356	2.12712366	2.54391028	3.15397777	3.67959583
$\text{wMIVaR}_{p^*}(X, Y_2)$	1.89906848	2.42685181	3.10301480	4.11400437	4.99177280
$\text{wMIVaR}_{p^*}(X, Y_3)$	2.85256302	3.43191854	4.15441809	5.21993051	6.14310303
$\text{wMIVaR}_{p^*}(X, Y_4)$	2.72411371	3.23586774	3.86782594	4.79169631	5.58741982
$\text{wMIVaR}_{p^*}(X, Y_5)$	2.33154399	2.87028090	3.54970143	4.55937126	5.43700554
$\text{wMIVaR}_{p^*}(X, Y_6)$	2.61648455	3.11191386	3.72329776	4.61625343	5.38458160

In this example, we can see a connection between wMIVaR and CVaR. From Tables 1 and 2 and equation (25), we observe that the values of wMIVaR in the case of $\rho = 0$ are just a sum of CVaR of each company (Zero correlation does not imply independence, in general.). By the additive relationship (25), Theorem 1 can be checked numerically. Table 4 summarizes probability levels p^* of wMIVaR in each case, and $1 - p^* = P(X \in W)$ means the probability of a worst-case outcome. In a setting of more than two variables, dependency structure would be more complex as we observe from Example 2.

Table 2: Sum of the risks of the companies before M&As

Risk of companies before the merger					
p-levels:	$p=0.60$	$p=0.7$	$p=0.8$	$p=0.9$	$p=0.95$
$\text{CVaR}_p(X)+\text{CVaR}_p(Y_1)$	1.63195576	1.96025814	2.36967632	2.97347163	3.49661176
$\text{CVaR}_p(X)+\text{CVaR}_p(Y_2)$	2.60781203	3.12923346	3.77948583	4.73845478	5.56932425
$\text{CVaR}_p(X)+\text{CVaR}_p(Y_3)$	2.85256302	3.43191853	4.15441808	5.21993051	6.14310303
$\text{CVaR}_p(X)+\text{CVaR}_p(Y_4)$	2.45122645	2.95333598	3.57950495	4.50295660	5.30305326
$\text{CVaR}_p(X)+\text{CVaR}_p(Y_5)$	3.11719163	3.75444932	4.54913547	5.72102543	6.73620487
$\text{CVaR}_p(X)+\text{CVaR}_p(Y_6)$	2.32464083	2.80743845	3.40952400	4.29745828	5.06678200

Table 3: Risk before M&As for each company

Risk of each company					
p-levels:	$p=0.60$	$p=0.7$	$p=0.8$	$p=0.9$	$p=0.95$
Risk of Company $\mathscr{A}$: $\text{CVaR}_p(X)$	0.92585633	1.11897538	1.35980960	1.71498331	2.02271280
Risk of Company 1: $\text{CVaR}_p(Y_1)$	0.70609943	0.84128276	1.00986672	1.25848832	1.47389896
Risk of Company 2: $\text{CVaR}_p(Y_2)$	1.68195571	2.01025808	2.41967623	3.02347147	3.54661145
Risk of Company 3: $\text{CVaR}_p(Y_3)$	1.92670669	2.31294315	2.79460848	3.50494720	4.12039023
Risk of Company 4: $\text{CVaR}_p(Y_4)$	1.52537012	1.83436060	2.21969535	2.78797329	3.28034046
Risk of Company 5: $\text{CVaR}_p(Y_5)$	2.19133530	2.635473941	3.18932587	4.00604212	4.71349207
Risk of Company 6: $\text{CVaR}_p(Y_6)$	1.39878450	1.68846307	2.04971440	2.58247497	3.04406920

Table 4: Probability levels of wMIVaR

probability levels of wMIVaR					
Individual p-levels:	$p=0.60$	$p=0.70$	$p=0.80$	$p=0.90$	$p=0.95$
	p^* as in (23)	p^* as in (23)	p^* as in (23)	p^* as in (23)	p^* as in (23)
Case of (X,Y_1)	0.66947743	0.76241307	0.85006754	0.93113505	0.96813223
Case of (X,Y_2)	0.93272725	0.97723374	0.99550175	0.99976102	0.99998881
Case of (X,Y_3)	0.84000011	0.91000008	0.96000005	0.99000002	0.99750001
Case of (X,Y_4)	0.76087275	0.84323267	0.91284943	0.96759847	0.98781057
Case of (X,Y_5)	0.88492698	0.94497326	0.98094402	0.99700071	0.99954134
Case of (X,Y_6)	0.72237245	0.80948141	0.88709824	0.95322102	0.98040069

Example 2. *Suppose that Company $\mathscr{A}$, the local newspaper company in the previous example, were able to grow throughout the M&A deal. This company is now considering acquiring the other media companies running other types of business, in order to become a media group. The target companies are as follows. Company* 1 *runs a magazine business, Company* 2 *is a radio broadcasting company, Company* 3 *has web-based technology and Company* 4 *does digital TV broadcasting service.*

Company $\mathscr{A}$ wants to acquire two of these companies before the second half of the year. The half-year business forecasts for Company $\mathscr{A}$ and Companies k, $k=1,2,3,4$ are 2% profit with standard deviation 0.8, -3% profit with standard deviation 0.6, 1.5% profit with standard deviation 1.2, 7% profit with standard deviation 2.3, and 9% profit with standard deviation 2.5, respectively. The risk management center of Company $\mathscr{A}$ wants to compare the risk measurements before and after *the all possible M&A activities, in order to use the result as one of the key factors of the decision-making on the event.*

Let $X,\ Z_k,\ k=1,2,3,4$ denote normally distributed random variables, meaning the half-year operational losses of Companies $\mathscr{A}$ and $k,\ k=1,2,3,4$, respectively:

$$\begin{gathered} X \sim N(-0.02, 0.8^2), \\ Z_1 \sim N(0.03, 0.6^2), Z_2 \sim N(-0.015, 1.2^2), Z_3 \sim N(-0.07, 2.3^2), Z_4 \sim N(-0.09, 1.9^2). \end{gathered} \tag{27}$$

Let us further assume for the correlation coefficients between the companies that $\rho_{X,Z_1} = -0.7, \rho_{X,Z_2} = 0.2, \rho_{X,Z_3} = 0.5, \rho_{X,Z_4} = -0.6, \rho_{Z_1,Z_2} = 0.3, \rho_{Z_1,Z_3} = -0.2, \rho_{Z_1,Z_4} = 0.6, \rho_{Z_2,Z_3} = -0.5, \rho_{Z_2,Z_4} = -0.2, \rho_{Z_3,Z_4} = -0.4$. Then the random vectors $(X, Z_i, Z_j)^T \in R^3,\ i \neq j,\ i,j = 1,2,3,4$ mean the set of operating losses from the corresponding business sectors of the united firm after the M&As.

To make a decision regarding all the possible M&A events, especially in a risk evaluation process, two different types of comparison may be useful:

(a) Which M&A deal is ideal in terms of the risk?
We need to choose the smallest value among $\text{wMIVaR}_p((X, Z_i, Z_j)^T),\ i \neq j,\ i = 1,2,3,4$, i.e., compare the risk measurements *after* the M&A deals for all cases.

(b) Which M&A deals reduce risk?
We need to check if $\text{wMIVaR}_p((X, Z_i, Z_j)^T) < \text{CVaR}_{p_1}(X) + \text{CVaR}_{p_2}(Z_i) + \text{CVaR}_{p_3}(Z_j),\ i \neq j,\ i,j = 1,2,3,4$. This is the comparison of the risk measurements *before and after* for the M&A deal.

Let us use the same probability levels for individual companies. Then for each case $(X, Z_i, Z_j),\ i \neq j,\ i = 1,2,3,4$, the worst-case Multivariate Individual Value-at-Risk (wMIVaR) can be calculated by:

$$\begin{aligned} \text{wMIVaR}_{p^*}((X, Z_i, Z_j)^T) &= E(X + Z_i + Z_j \mid X > \text{VaR}_p(X), Z_i > \text{VaR}_p(Z_i), Z_j > \text{VaR}_p(Z_j)) \\ &= \frac{\int\int\int_W (x+y+z) f(x,y,z)\, dxdydz}{\int\int\int_W f(x,y,z)\, dxdydz}, \end{aligned} \tag{28}$$

where $W = \{X > \text{VaR}_p(X), Z_i > \text{VaR}_p(Z_i), Z_j > \text{VaR}_p(Z_j)\}$ and $p^* = 1 - \int\int\int_W f(x,y,z)\, dxdydz$. In (28), f is the trivariate standard normal p.d.f. with expectation vector μ and covariance matrix

$C_k,\ k=1,\ldots,6$ in (29) according to the random vectors $(X,Z_i,Z_j),\ i\neq j,\ i,j=1,2,3,4$ as follows.

$$\text{Random vector } (X,Z_1,Z_2)^T \text{ has } \mu_1=\begin{pmatrix}-0.020\\0.030\\-0.015\end{pmatrix} \text{ and } C_1=\begin{pmatrix}0.640 & -0.336 & 0.192\\-0.336 & 0.360 & 0.216\\0.192 & 0.216 & 1.440\end{pmatrix},$$

$$\text{random vector } (X,Z_1,Z_3)^T \text{ has } \mu_2=\begin{pmatrix}-0.020\\0.030\\-0.070\end{pmatrix} \text{ and } C_2=\begin{pmatrix}0.640 & -0.336 & 0.920\\-0.336 & 0.360 & -0.276\\0.920 & -0.276 & 5.290\end{pmatrix},$$

$$\text{random vector } (X,Z_1,Z_4)^T \text{ has } \mu_3=\begin{pmatrix}-0.020\\0.030\\-0.090\end{pmatrix} \text{ and } C_3=\begin{pmatrix}0.640 & -0.336 & -1.200\\-0.336 & 0.360 & 0.900\\-1.200 & 0.900 & 3.610\end{pmatrix},$$

$$\text{random vector } (X,Z_2,Z_3)^T \text{ has } \mu_4=\begin{pmatrix}-0.020\\-0.015\\-0.070\end{pmatrix} \text{ and } C_4=\begin{pmatrix}0.640 & 0.192 & 0.920\\0.192 & 1.440 & -1.380\\0.920 & -1.380 & 5.290\end{pmatrix},$$

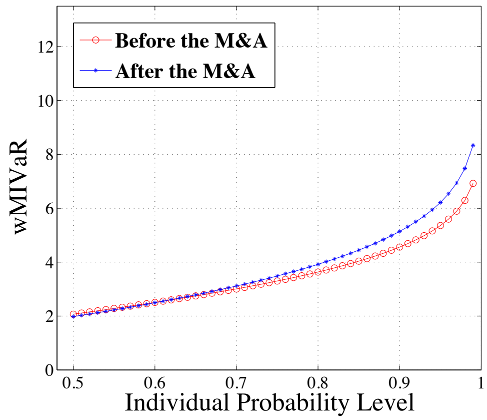
$$\text{random vector } (X,Z_2,Z_4)^T \text{ has } \mu_5=\begin{pmatrix}-0.020\\-0.015\\-0.090\end{pmatrix} \text{ and } C_5=\begin{pmatrix}0.640 & 0.192 & -1.200\\0.192 & 1.440 & -0.600\\-1.200 & -0.600 & 3.610\end{pmatrix},$$

$$\text{and random vector } (X,Z_3,Z_4)^T \text{ has } \mu_6=\begin{pmatrix}-0.020\\-0.070\\-0.090\end{pmatrix} \text{ and } C_6=\begin{pmatrix}0.640 & 0.920 & -1.200\\0.920 & 5.290 & -2.300\\-1.200 & -2.300 & 3.610\end{pmatrix}. \tag{29}$$

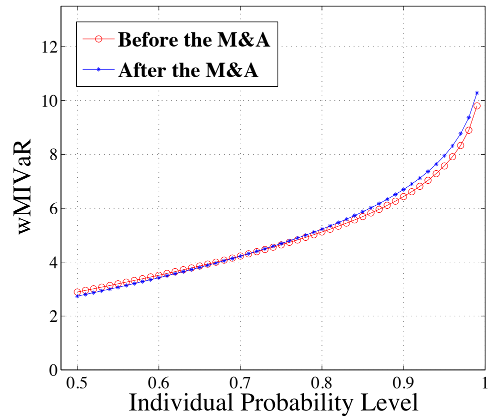
The risk measurements in the cases of different dependency structures as in (29) are calculated at the same individual probabilities (two decimal places) from 0.50 to 0.99. Due to the same individual p-levels they can be plotted in 2-D as in the subfigures of Figure 6. We observe some interesting results from Figure 6 – the cases in the subfigures (a), (b), (d) are neither subadditive nor superadditive. So we can expect that the stochastic dependence structure will be complicated in the risk evaluation processes in practice, as we observe that some cases are neither superadditive nor subadditive. Depending on stochastic dependence structures, these properties coexist and are separated by a certain level of probability.

Numerical results at individual probability levels $p=0.6,0.7,0.8,0.9,0.95$ are summarized in Tables 5, 6, 7 and 8. At these individual probability levels, we have relationships of superadditivity (30) and subadditivity (31) – comparisons of risks *before* and *after* the M&As. As illustrated in the subfigures (a),(b),(d) of Figure 6 and presented in Tables 5 and 6, at high individual probability levels we have *superadditive* relationships:

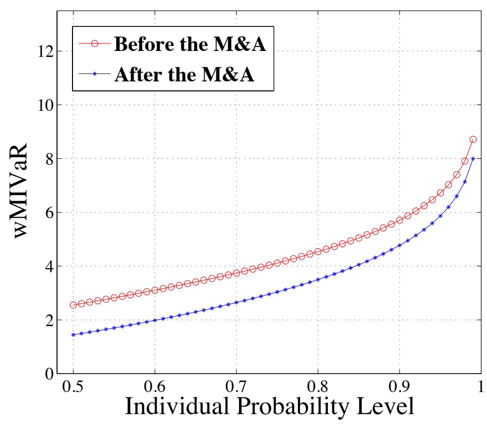
$$\begin{aligned}&\text{wMIVaR}_{p^*}((X,Z_1,Z_2)^T) > \text{CVaR}_p(X)+\text{CVaR}_p(Z_1)+\text{CVaR}_p(Z_2), p>0.7\\&\text{wMIVaR}_{p^*}((X,Z_1,Z_3)^T) > \text{CVaR}_p(X)+\text{CVaR}_p(Z_1)+\text{CVaR}_p(Z_3), p>0.8\\&\text{wMIVaR}_{p^*}((X,Z_2,Z_3)^T) > \text{CVaR}_p(X)+\text{CVaR}_p(Z_2)+\text{CVaR}_p(Z_3), p>0.95\end{aligned} \tag{30}$$



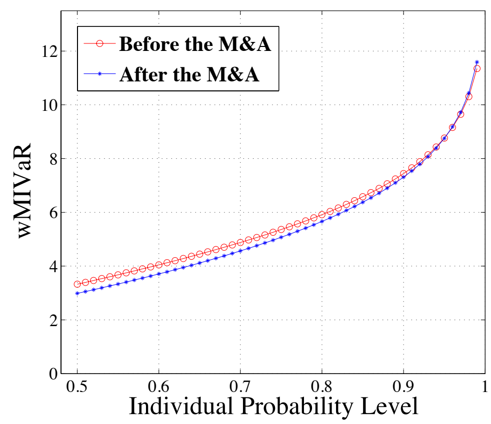
(a) Risk measurement of (X, Z_1, Z_2)



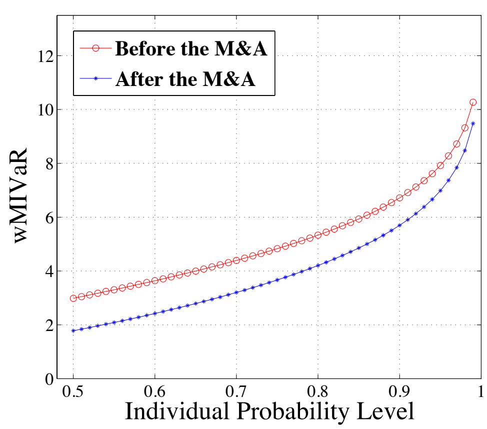
(b) Risk measurement of (X, Z_1, Z_3)



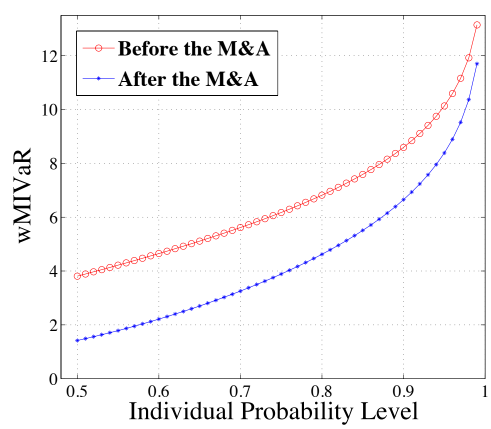
(c) Risk measurement of (X, Z_1, Z_4)



(d) Risk measurement of (X, Z_2, Z_3)



(e) Risk measurement of (X, Z_2, Z_4)



(f) Risk measurement of (X, Z_3, Z_4)

Figure 6: Comparison of risks among different M&A deals

We have *subadditive* relationships for all other cases:

$$\text{wMIVaR}_{p^*}((X,Z_i,Z_j)^T) < \text{CVaR}_p(X) + \text{CVaR}_p(Z_i) + \text{CVaR}_p(Z_j),\ i \neq j,\ i,j = 1,2,3,4. \quad (31)$$

Table 5: Risk after M&A deals, i.e., risk of the new united firms

The worst-case MIVaR (wMIVaR), i.e., risk after M&As					
Individual p-levels:	$p = 0.60$	$p = 0.70$	$p = 0.80$	$p = 0.90$	$p = 0.95$
$\text{wMIVaR}_{p^*}(X,Z_1,Z_2)$	2.49329287	3.11460792	3.91970243	5.14037649	6.21496657
$\text{wMIVaR}_{p^*}(X,Z_1,Z_3)$	3.42086954	4.22224287	5.22684633	6.69666789	7.94759881
$\text{wMIVaR}_{p^*}(X,Z_1,Z_4)$	1.98392938	2.64570195	3.49924961	4.77204115	5.87065544
$\text{wMIVaR}_{p^*}(X,Z_2,Z_3)$	3.70888752	4.56629510	5.66226146	7.30957330	8.75154331
$\text{wMIVaR}_{p^*}(X,Z_2,Z_4)$	2.42305069	3.20355632	4.20574281	5.69781483	6.98608444
$\text{wMIVaR}_{p^*}(X,Z_3,Z_4)$	2.21334102	3.25538668	4.62309213	6.65213752	8.38476756

Table 6: Sum of the risks of the companies before M&As

Risk of companies before the merger					
p-levels:	$p = 0.60$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$
$\text{CVaR}_p(X)$+$\text{CVaR}_p(Z_1)$+$\text{CVaR}_p(Z_2)$	2.50622646	3.00833599	3.63450496	4.55795663	5.35805322
$\text{CVaR}_p(X)$+$\text{CVaR}_p(Z_1)$+$\text{CVaR}_p(Z_3)$	3.51354385	4.22805169	5.11907625	6.43304858	7.57134263
$\text{CVaR}_p(X)$+$\text{CVaR}_p(Z_1)$+$\text{CVaR}_p(Z_4)$	3.10732475	3.74461726	4.53936957	5.71144107	6.72694516
$\text{CVaR}_p(X)$+$\text{CVaR}_p(Z_2)$+$\text{CVaR}_p(Z_3)$	4.04805773	4.87843701	5.91396195	7.44103858	8.76397032
$\text{CVaR}_p(X)$+$\text{CVaR}_p(Z_2)$+$\text{CVaR}_p(Z_4)$	3.64183854	4.39500261	5.33425538	6.71943107	7.91957284
$\text{CVaR}_p(X)$+$\text{CVaR}_p(Z_3)$+$\text{CVaR}_p(Z_4)$	4.64915592	5.61471818	6.81882662	8.59452302	10.13286217

Table 7: Risk before M&As for each company

Risk of each company					
p-levels:	$p = 0.60$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$
Risk of Company $\mathscr{A}$: $\text{CVaR}_p(X)$	0.75268507	0.90718030	1.09984768	1.38398666	1.63017025
Risk of Company 1: $\text{CVaR}_p(Z_1)$	0.60951380	0.72538523	0.86988576	1.08298999	1.26762768
Risk of Company 2: $\text{CVaR}_p(Z_2)$	1.14402760	1.37577046	1.66477152	2.09097998	2.46025537
Risk of Company 3: $\text{CVaR}_p(Z_3)$	2.15134501	2.59548615	3.14934281	3.96607194	4.67354471
Risk of Company 4: $\text{CVaR}_p(Z_4)$	1.74512586	2.11205173	2.56963612	3.24446443	3.82914722

From a risk management perspective, the subadditive cases of (31) are ideal since the M&As result in a risk reduction. As we mentioned in Remark 4, at "high" individual probability levels we will have unrealistic (way too large) magnitude of risk since probability level for the worst-case event will be too high. In Table 8 such situation can be confirmed – the probability levels p^* with individual probability levels $p = 0.6$ or 0.7 are only reasonable to be used. All the cases at lower individual p-levels are subadditive and so would be beneficial in terms of risk reduction. The last case depicted in subfigure (f) of Figure 6 would be the best case concerning efficiency of risk reduction throughout the M&As. However, a proper case study is necessary to find out

Table 8: Probability levels of wMIVaR

probability levels of wMIVaR					
Individual p-levels:	$p = 0.60$	$p = 0.70$	$p = 0.80$	$p = 0.90$	$p = 0.95$
	p^* as in (28)	p^* as in (28)	p^* as in (28)	p^* as in (28)	p^* as in (28)
Case of (X, Z_1, Z_2)	0.95860665	0.98867999	0.99845649	0.99996197	0.99999925
Case of (X, Z_1, Z_3)	0.96670203	0.99153104	0.99893355	0.99997599	0.99999955
Case of (X, Z_1, Z_4)	0.98374077	0.99775031	0.99989708	0.99999968	0.99999999
Case of (X, Z_2, Z_3)	0.92911062	0.97178483	0.99268452	0.99934013	0.99994519
Case of (X, Z_2, Z_4)	0.98899152	0.99868821	0.99995276	0.99999991	0.99999999
Case of (X, Z_3, Z_4)	0.99886459	0.99998574	0.99999999	0.99999999	0.99999999

logical and reasonable descriptions of why and how this results in risk reduction, in order to make an appropriate final decision on the deal.

Remark 3. *Note that M&As and demerger activities are like two sides of the same coin. Thus, these examples are also considered as the cases about corporate demerger events – corporate restructuring processes. For general corporate demerger activities, there could be several candidates in terms of how to split business sectors of the company. For example, if there are m different ways of restructuring, then we may need to choose the least risky case. By comparison of risks among different cases, we can see what would be the most desirable case in a demerger from a risk management perspective, and eventually it will be used to make a decision on the demerger event. For demerger events in the real world, analyses are required on each pertinent business sector and on the correlation among all the sectors. Then this should be followed by some suitable partition options for the inputs of the risk evaluation processes. Furthermore, a decision should not be made based only on risk measurement since demerger is a very complex process with various reasons behind it. For successful M&As and demergers, detailed post-event plans should be set up before taking action on the deals.*

7 Concluding remarks

Risk, in practice, may not be equal to a real valued random variable, rather, it is frequently represented by a finite collection of random variables, i.e., a random vector. A company typically has many different assets, portfolios, business sectors, exposed to different kinds of randomness, influencing the overall behavior of the company. In order to characterize it, from the point of view of risk exposure, we need to work with the joint probability distribution of the random variables involved. Multivariate risk measures have already been introduced in the literature and the starting point of our investigation is the paper by Prékopa (2012), where the concepts of Multivariate-Value-at-Risk (MVaR) and Multivariate-Conditional-Value-at-Risk (MCVaR) are introduced and explored. See also Lee and Prékopa (2012), where new methods for numerical calculations of these concepts are presented.

In this paper we construct a further multivariate risk measure: the worst case Multivariate-Individual-Value-at-Risk (wMIVaR), where only one orthant of the space represent unfavorable set and its vertex is at the vector with components equal to the individual VaR's. Properties of this risk measure are derived but they fail to satisfy the convexity inequality required by one of the risk measure axioms of Artzner et al. (1999). In our opinion a risk measure should signal the advantages or disadvantages of corporate M&As and demergers rather than to always satisfy an axiom, however, attractive it is from a purely mathematical point of view; wMIVaR is constructed to serve this objective. In Section 5 we show in what way it can be used in practice. In the numerical examples we look at one company which considers M&As with one or two of a few target candidates, we calculate which M&A deal is ideal in terms of risk and which M&A deals reduce risk. To do the above analysis we have introduced vector operations, where we put together risk vectors to create new risk vector with increased number of components, to describe M&As, and split a risk vector into parts, to describe demergers.

Comparison of risks before and after M&As or demergers activities will be useful for decision-making before taking action on the deals. It is the job of the decision makers to find out reasonable justification for an increase or decrease in risks throughout the M&A deals, in order to make the best decision. We hope that these aspects of wMIVaR will attract interest for future research.

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