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PRICE-BANDS: A TECHNICAL TOOL FOR STOCK TRADING

Jinwook Lee Joonhee Lee András Prékopa

Abstract. Given a stochastic process with known finite dimensional distributions, we construct lower and upper bounds within which future values of the stochastic process run, at a fixed probability level. For a financial trading business, such set of bounds are called "price-bands" or "trading-bands" that can be used as an indicator for successfully buying or short-selling shares of stock. In this paper, we present a mathematical model for the novel construction of price-bands using a stochastic programming formulation. Numerical examples using recent US stock market intraday data are presented.

Keywords. Price-bands (trading-bands); Technical Analysis; Gaussian Process; Binomial Moment Problem

1 Introduction

Stock trading can be approached in a multitude of ways, and the one fact practitioners agree upon is that there is no clear and easy way to outperform the market consistently. It is intuitively appealing that a company should have an "intrinsic value," and in the long run the stock price would converge to this value. This is the conventional wisdom of so-called "fundamental analysis," and the valuation of the company amounts to estimating this unknown "intrinsic value." This process is concerned mostly with the economic climate, interest rates, products, earnings, management, etc. There is a large number of valuation studies, and many of them are actually being used in financial practice.

On the other hand, technical (or quantitative) analysis is an evaluation process of securities based entirely on charting patterns, statistical approaches, and/or mathematical formulae. The technical analysis approach is particularly suited for short-term investing. In a certain sense, this amounts to an analysis of crowd psychology and behavior as well as investor philosophy, since it is believed that short-term patterns and trends result primarily from decisions by human investors. The tacit assumption underlying technical analysis is that a future price can be predicted by quantitative analysis of the past price movement. However, the efficient-market hypothesis (e.g., see Fama (1970), Damodaran (2012), etc.) asserts that no one can consistently outperform the market, since the market incorporates all information instantaneously. On the other hand, behavioral economists criticize the efficiency of the market for many reasons, such as irrationality of investors, information asymmetry, etc. For more details, we refer the reader to the literature, e.g., see Kahneman and Tversky (1979), Shleifer (2000), etc.

In this paper we suppose that stock prices are at least *partially* predictable based on recent market trends. Price patterns can be elusive, and the difficulty is amplified by the sheer complexity of the financial market, as well as market participants' philosophical and psychological states. We believe that price movement discloses investors' expectations in light of these (and many more) factors, and in this way accounts for them. There are some clear patterns in the stock market; for example, when a stock is in an up-trend with increasing volume, it is regarded as a sign of an up-market trend. Such patterns can be found by technical analysis, and this motivates the study of such methodologies. In what follows, we present a mathematical model for constructing new and reliable trading-bands (or price-bands) for price forecasting.

2 Construction of Price-bands

In the stock trading business, many different technical tools exist to guide traders through the swarm of information, e.g., trading bands, envelopes, channels, etc. Bollinger Bands is probably one of the most popular and successful models. Many traders use it daily as a tool for pattern recognition, augmenting technical trading strategies, and so on (see, e.g., Bollinger (2002), Grimes (2012), etc.). It is simple to see that Bollinger Bands is merely a confidence interval of future stock prices. Stock prices are expected to remain within the bands with a certain probability, depending on the width of the bands. Observation of stock price outside of the bands is considered a sign for "buying" or "(short) selling."

Table 1: The Bollinger Bands

Upper band	= n-day moving average$+k\sigma$
Middle band	= n-day moving average
Lower band	= n-day moving average$-k\sigma$

There are several ways to calculate moving average, e.g., simple moving average, front-weighted moving average, exponential moving average, and so on. The band width is determined by the multiplier k and standard deviation $\sigma = \sqrt{\frac{\sum(x_i-\mu)^2}{N}}$, *where* x_i *is the data point,* μ *the average, and N the number of points. The multiplier k can be chosen depending on the time periods n. Recommended (by Bollinger) width parameters with time periods are* $k = 1.9$ *if* $n = 10$, $k = 2.0$ *if* $n = 20$, $k = 2.1$ *if* $n = 50$, *etc.*

Let us consider the following simple example: Suppose we happen to observe a stock's price moving below the lower band. Then our expectation would be that the price will go up, moving back into the bands, *ceteris paribus.* If we make our investment decision solely on the basis of price-bands, then we would choose to "buy" while the price is below the lower band and subsequently "sell" within the bands. Consider, on the other hand, the case that a stock price is observed above the upper band. As one might expect, the proper action on the stock would be to "short sell" above the upper band and then "buy" within the bands.

For completeness we present the Bollinger Bands formulae in Table 1. Bollinger uses the mean and standard deviation to create price-bands as in Table 1, where the mean can be thought of as a central tendency and standard deviation as its volatility, thus determining the width of the bands. Like the Bollinger Bands, we assume stock price in a time period is normally distributed. However, rather than using simple mean and variance, we use conditional mean and conditional variance. By conditioning on a recent historical stock price data, we construct price-bands that are more sensitive to recent market information than Bollinger Bands.

Let $X(t)$ be the stock price at time t, and let $I_\tau = \{X(\tau_1), X(\tau_2), \ldots, X(\tau_m)\}$ be the history of past stock prices, available up to the present time, as described in Figure 1. Then given the information set I_τ, the probability p of the future stock prices running within $[a_1, b_1] \times \cdots \times [a_n, b_n]$ at time $t = 1, \ldots, n$, is the following:

$$p = P(a_i \leq X(t = i) \leq b_i, i = 1, \ldots, n \mid I_\tau). \tag{1}$$

If we use historical data of the past m days in order to predict the future n days of price changes, then the calculation of the future bounds in (1) has the following practical meaning. With a reasonable probability level (e.g., $p = 0.9, 0.95, 0.99$, etc.) for bounds of future stock prices, we can expect that the future stock price is likely to run within the upper and lower bounds. These bounds are paramount in such a trading strategy, and in this paper we present an efficient method of computing these values.

Let us assume that each random variable $X(t), t = 1, \ldots, n,$ is normally distributed. Then the stochastic process $\{X(t), t \geq 0\}$ is a Gaussian process, and $X = (X(1), \ldots, X(n))$ has a multivariate normal distribution. Let $m = (m_1, \ldots, m_n)$ denote the mean of a Gaussian random variable $X =$

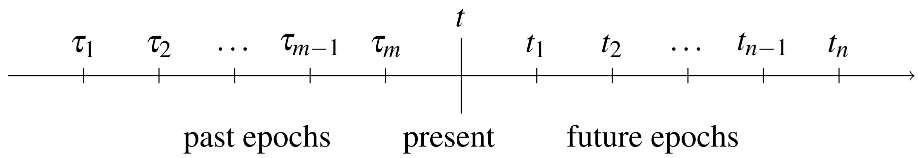


Figure 1: Past, present, future timeline

$(X(1),\ldots,X(n))$. Then the covariance matrix $\Sigma = (\Sigma_{ij})$ is defined by

$$\Sigma_{ij} = E((X(i)-m_i)(X(j)-m_j)). \tag{2}$$

A Gaussian process can be characterized by expectation and covariance of random variables, i.e., a multivariate normal distribution. We assume that stock prices follow a Gaussian process.

Let ξ_i denote the daily random stock prices on day $i,\ i=1,\ldots,n$, and let P and F denote "Past" and "Future." Out of the components $\xi_1,\ldots,\xi_{n+N}$, we form two random vectors:

$$\xi^{\mathbf{P}} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \xi^{\mathbf{F}} = \begin{pmatrix} \xi_{n+1} \\ \vdots \\ \xi_{n+N} \end{pmatrix}, \tag{3}$$

which have expectations:

$$e^P = E(\xi^{\mathbf{P}}) = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}, e^F = E(\xi^{\mathbf{F}}) = \begin{pmatrix} e_{n+1} \\ \vdots \\ e_{n+N} \end{pmatrix}. \tag{4}$$

Using (3) and (4), let us define components of covariance matrix of the random variables $\xi_1,\ldots,\xi_{n+N}$ by

$$\begin{aligned} S &= E[(\xi^F - e^F)(\xi^F - e^F)^T] \\ U &= E[(\xi^F - e^F)(\xi^P - e^P)^T] \\ T &= E[(\xi^P - e^P)(\xi^P - e^P)^T]. \end{aligned} \tag{5}$$

Then the covariance matrix C can be written as

$$C = \begin{pmatrix} S & U \\ U^T & T \end{pmatrix}. \tag{6}$$

It is well known that, given $\xi^P = x^P$, $\xi^{\mathbf{F}}$ has a normal distribution with "conditional" expectation vector

$$e^C = e^F + UT^{-1}(x^P - e^P), \tag{7}$$

and covariance matrix

$$S - UT^{-1}U^T. \tag{8}$$

The conditional probability density of $\xi^{\mathbf{F}}$, given $\xi^{\mathbf{P}} = \mathbf{x}^{\mathbf{P}}$, can be written up as

$$\begin{aligned} f(x^F \mid x^P) &= \left[\frac{\left|(S - UT^{-1}U^T)^{-1}\right|}{(2\pi)^N} \right]^{1/2} \\ &\quad \times \exp\left\{ -\tfrac{1}{2}(x^F - e^C)^T (S - UT^{-1}U^T)^{-1} (x^F - e^C) \right\}. \end{aligned} \tag{9}$$

We want to predict upper and lower bounds of a stock price for the next business day, based on historical data over a certain time period. Since we are interested in trading on the next day, ξ^F is a random variable, not a random vector. As S is the variance of ξ^F, it follows that S in the covariance matrix (6) is a number. Hence the conditional covariance matrix $S - UT^{-1}U^T$ is also a number. Given $\xi^{\mathbf{P}} = \mathbf{x}^{\mathbf{P}}$, let us denote the standard deviation of $\xi^{\mathbf{F}}$ by

$$\sigma^C = \sqrt{S - UT^{-1}U^T}. \tag{10}$$

If we use n intraday data points, the conditional expectation vector e^C in (7) has n components, denoting the expected value at corresponding time points. Then after the way of Bollinger, we can construct our trading bands with inputs (10) and (7).

Definition 1 (Prékopa-Lee Bands). *Based on n intraday data points from the past m days, we construct next-day trading bands by predicting volatility $\sigma^C = \sqrt{S - UT^{-1}U^T}$ and average price vector $e^C = e^F + UT^{-1}(x^P - e^P)$ as follows:*

$$\begin{aligned} &\textit{Upper band} = \overline{e^C} + k\sigma^C \\ &\textit{Middle band} = \overline{e^C} \\ &\textit{Lower band} = \overline{e^C} - k\sigma^C, \end{aligned} \tag{11}$$

where $\overline{e^C}$ is the average of components of vector e^C, and k is a multiplier that can be chosen based on investors' risk tolerance level, e.g., $k = 1.96$ for 95% confidence interval. Together with σ^C, k determines the width of the bands. The central tendency (i.e. $\overline{e^C}$) determines the moving direction of the bands.

We must find the values of $\sigma^C = \sqrt{S - UT^{-1}U^T}$ and $e^C = e^F + UT^{-1}(x^P - e^P)$. Since the stock price distribution of the next business day is unknown, there is no way to find the exact values of σ^C and e^C. However, given $\xi^{\mathbf{P}} = \mathbf{x}^{\mathbf{P}}$, the calculation of reasonable upper and lower bounds for the volatility σ^C is possible.

Among the terms in the expression for σ^C, $U = E[(\xi^F - e^F)(\xi^P - e^P)^T]$ represents the relationship between past and future stock prices. For this reason, $U = E[(\xi^F - e^F)(\xi^P - e^P)^T]$ is the key to measuring the upper and lower bounds of σ^C. If we use n intraday data points, measured at equally spaced time points, then we can consider them as elements in a set with 2^n distinct subsets. As n gets arbitrarily large, the complexity of bounding σ^C therefore increases exponentially. Thus, in order to efficiently obtain the upper and lower bounds of σ^C, we propose a stochastic programming formulation utilizing a binomial moment scheme. We have found that this set theoretical approach is fundamental in efficiently constructing the price-bands.

3 Binomial moment problem formulation

If we use n intraday data points (per day) with historical data points for the past m days (i.e. $x^P = \xi^P$), then $(\xi^F - e^F)$ will be an $1 \times n$ row vector and $(x^P - e^P)^T$ will be an $n \times m$ matrix given by:

$$\begin{aligned} (\xi^F - e^F) &= (\xi_1^F - e^F, \xi_2^F - e^F, \ldots, \xi_n^F - e^F) \\ (x^P - e^P)^T &= \begin{pmatrix} x_{11}^P - e_1^P & x_{21}^P - e_2^P & \ldots & x_{m1}^P - e_m^P \\ x_{12}^P - e_1^P & x_{22}^P - e_2^P & \ldots & x_{m2}^P - e_m^P \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n}^P - e_1^P & x_{2n}^P - e_2^P & \ldots & x_{mn}^P - e_m^P \end{pmatrix}. \end{aligned} \tag{12}$$

Since $T = E[(\xi^P - e^P)(\xi^P - e^P)^T] = (x^P - e^P)(x^P - e^P)^T$, T is an $m \times m$ matrix. U is a $1 \times m$ row vector, since $U = E[(\xi^F - e^F)(x^P - e^P)^T]$. Thus, the covariance matrix $S - UT^{-1}U^T$ of (8) is 1×1, i.e., a number.

The unit time period could be a day, a week, or a month, depending on the investment strategy in terms of realization of profit in a preferred length of time. Generally, price-bands are used for day trading, short-term investments, etc. If the unit time period is one day, then $x_{ji}^P - e_j^P$ could be said to be the difference between the stock price on the ith time data point for $i = 1, \ldots, n$ on day j and the average price of day j for $j = 1, \ldots, m$. Let us use the same variance for the next day as in the Bollinger bands:

$$S = \frac{\sum_{i=1}^N (x_i^P - \mu)^2}{N}, \tag{13}$$

where x_i^P's are all the given data points, μ their mean, and N the number of data points, i.e., $N = mn$.

Given $\xi^P = x^P$, we have

$$\begin{aligned} & S - UT^{-1}U^T \\ &= S - E\left[\sum_{i=1}^n (\xi^F - e^F)((x^P - e^P)^T)_i\right] \left[(x^P - e^P)(x^P - e^P)^T\right]^{-1} \left(E\left[\sum_{i=1}^n (\xi^F - e^F)((x^P - e^P)^T)_i\right]\right)^T, \end{aligned} \tag{14}$$

where S is defined by (13), and $((x^P - e^P)^T)_i$ is the ith row of matrix $(x^P - e^P)^T$ of (12). Let $\eta = \xi^F - e^F$. Then $\eta \sim N(0, S)$. Let Y^T denote the matrix $(\xi^P - e^P)^T$ of (12), and Y_i the ith row of matrix Y. Then we can write

$$\begin{aligned} U &= E\left[\sum_{i=1}^n (\xi^F - e^F)((x^P - e^P)^T)_i\right] \\ &= E\left[\sum_{i=1}^n \eta Y_i^T\right] \\ &= \frac{1}{n}\sum_{i=1}^n z_i Y_i^T, \end{aligned} \tag{15}$$

where z_i is the data point at time i of the next business day (i.e. realization of the random variable η at time i), n is the number of intraday data points, and U is a $1 \times m$ row vector.

By (14) and (15), the conditional variance $S - UT^{-1}U^T$ can be written as:

$$S - \frac{\sum_{i=1}^{n} z_i Y_i^T}{n} (YY^T)^{-1} \left(\frac{\sum_{i=1}^{n} z_i Y_i^T}{n} \right)^T, \tag{16}$$

where S and Y are defined as in (13), (15), respectively. Now, $z_i, i = 1, \ldots, n$, are the only unknowns. Since $\eta = \xi^F - e^F \sim N(0, S)$, it is reasonable to assume that $-4\sqrt{S} \leq z_i \leq 4\sqrt{S}$, $i = 1, \ldots, n$, where z_i's are realizations of the random variable η at time $i = 1, \ldots, n$.

For meaningful bounding values of (16), the future data points z_i, $i = 1, \ldots, n$, and their relations to all past data points at time i (i.e. $Y_i^T = \left((\xi^P - e^P)^T\right)_i$, $i = 1, \ldots, n$) are essential, and all possible cases must be examined. This is the fundamental motivation for implementing the set theoretical approach, and we formulate minimization and maximization problems for the calculation using a modified binomial moment method.

For details about the binomial moment scheme, we refer the reader to the literature, e.g., Prékopa (1988, 1995, 2003), etc. For completeness we present some basic definitions here. Let ν designate the number of events from $A_1, \ldots, A_n$ that occur. Let $v_i = P(\nu = i)$, $i = 1, \ldots, n$. Then

$$\sum_{i=0}^{n} \binom{i}{k} v_i = S_k, \ k = 0, 1, \ldots, n, \tag{17}$$

where, by definition, $S_k = E\left[\binom{\nu}{k}\right]$, $k = 0, \ldots, n$. Essentially, the binomial moment problem formulation is to optimize an objective function with a counting method leveraging the inclusion-exclusion principle.

Let us define the sets as follows:

$$A_j = \{t \mid t \leq z^{(j)}\}, \ j = 1, \ldots, n, \tag{18}$$

where $z^{(j)}$ is the jth largest among the next day's n intraday data points $z_1, z_2, \ldots, z_n$, where z_i is the realization of the random variable $\eta \sim N(0, S)$ at time i, $i = 1, \ldots, n$. Let us introduce the functions, for $k = 1, \ldots, n$:

$$\begin{aligned} S_k(z) &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(z^{(i_1)} \geq \eta, \ldots, z^{(i_k)} \geq \eta) \\ &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \ldots A_{i_k}), \end{aligned} \tag{19}$$

where A_{i_j}, $j = 1, \ldots, k$ are defined as in (18). Due to the shape of sets of (18), i.e., $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$, the probabilities v_i, $i = 1, \ldots, n$, can be expressed by

$$\begin{aligned} v_0 &= 1 - P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - P(A_1), \\ v_i &= P\left(A_i \cap \left(\bigcup_{k=i+1}^{n} A_k\right)\right) = P(A_i \backslash A_{i+1}), i = 1, \ldots, n, \end{aligned} \tag{20}$$

where $v_i = P(\nu = i),\ i = 1,\dots,n$, and ν designates the number of events out of $A_1,\dots,A_n$ that occur.

Equivalently, if the random variable η has a normal p.d.f. f, then we can write:

$$\begin{aligned} v_0 &= \int_{z^{(1)}}^{\infty} f(t)dt, \\ v_i &= \int_{z^{(i+1)}}^{z^{(i)}} f(t)dt,\ i = 1,\dots,n-1, \\ v_n &= \int_{-\infty}^{z^{(n)}} f(t)dt, \end{aligned} \tag{21}$$

where $z^{(n)} \le z^{(n-1)} \le \cdots \le z^{(1)}$ such that $\sum_{i=0}^{n} v_i = 1$. Probabilities $v_0, v_1, \dots, v_n$ are also described in Figure 2.

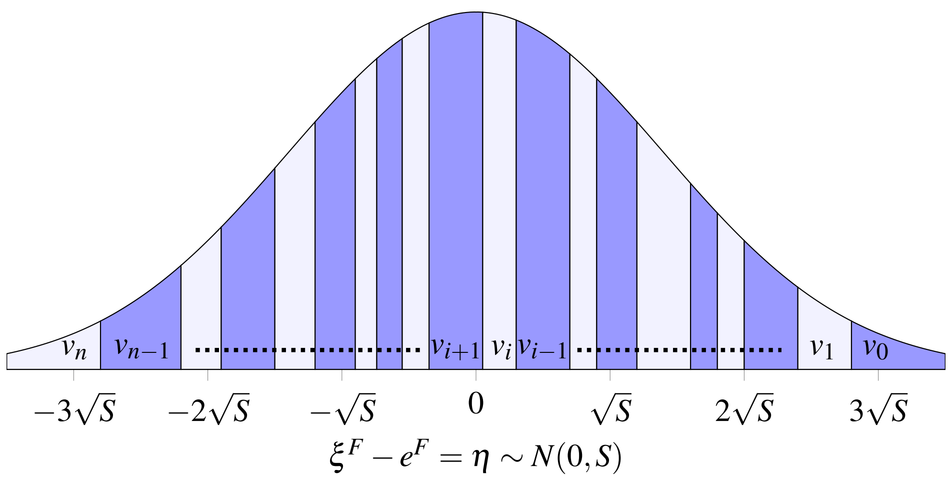


Figure 2: Description of data points and their corresponding probabilities $v_0 = \int_{z^{(1)}}^{\infty} f(t)dt,\ v_i = \int_{z^{(i+1)}}^{z^{(i)}} f(t)dt, i = 1,\dots,n-1$ and $v_n = \int_{-\infty}^{z^{(n)}} f(t)dt$ where f is normal p.d.f. of $\xi^F - e^F$ which is assumed to have mean zero and variance S.

We are ready to write up the formulation:

$$\begin{aligned} &\min(\max)\ S - \frac{\sum_{i=1}^{n} z_i Y_i^T}{n} (YY^T)^{-1} \left(\frac{\sum_{i=1}^{n} z_i Y_i^T}{n} \right)^T \\ &\text{subject to} \\ &\sum_{i=0}^{n} \binom{i}{k} v_i = S_k(z),\ k = 0,1,\dots,m \le n \\ &v_i \ge p_i, i = 0,1,\dots,n \\ &-4\sqrt{S} \le z_i \le 4\sqrt{S},\ i = 1,\dots,n, \end{aligned} \tag{22}$$

where $0 \leq p_i \ll 1,\ i = 1,\ldots,n$ are some fixed (very small) probabilities that can be chosen in various ways (e.g., $p_i = 0.001,\ i = 1,\ldots,n.$). Constraints on probabilities $v_i,\ i = 0,\ldots,n$, are to ensure reasonable placement of future data points at optimality of the problem, as described in Figure 2.

As a result of the special structure of the sets of (18) (i.e., $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$), the calculations of $S_k,\ k = 1,\ldots,n$, are manageable (in general, very expensive computationally). This is due to the fact that, for the calculations of $S_k,\ k = 1,\ldots,n$, in particular, we need to find only the minimum value among k data points over all cases of $\binom{n}{k},\ k = 1,\ldots,n$. We then calculate their CDF values, followed by summing up the values of all $\binom{n}{k},\ k = 1,\ldots,n$, cases to calculate the binomial functions $S_k,\ k = 0,\ldots,n$ as follows:

$$\begin{array}{lcl}
S_0(z) & \equiv & 1 \\
S_1(z) & = & P(\eta \leq z_1) + P(\eta \leq z_2) + \cdots + P(\eta \leq z_n) \\
S_2(z) & = & P(\eta \leq z_1, \eta \leq z_2) + P(\eta \leq z_1, \eta \leq z_3) + \cdots + P(\eta \leq z_{n-1}, \eta \leq z_n) \\
 & = & P(\eta \leq \min\{z_1, z_2\}) + P(\eta \leq \min\{z_1, z_3\}) + \cdots + P(\eta \leq \min\{z_{n-1}, z_n\}) \\
S_3(z) & = & P(\eta \leq z_1, \eta \leq z_2, \eta \leq z_3) + P(\eta \leq z_1, \eta \leq z_2, \eta \leq z_4) + \cdots + P(\eta \leq z_{n-2}, \eta \leq z_{n-1}, \eta \leq z_n) \\
 & = & P(\eta \leq \min\{z_1, z_2, z_3\}) + P(\eta \leq \min\{z_1, z_2, z_4\}) + \cdots + P(\eta \leq \min\{z_{n-2}, z_{n-1}, z_n\}) \\
S_4(z) & = & P(\eta \leq \min\{z_1, z_2, z_3, z_4\}) + P(\eta \leq \min\{z_1, z_2, z_3, z_5\}) + \cdots + P(\eta \leq \min\{z_{n-3}, z_{n-2}, z_{n-1}, z_n\}) \\
 & \vdots & \\
S_{n-1}(z) & = & P(\eta \leq \min\{z_1, \ldots, z_{n-1}\}) + P(\eta \leq \min\{z_1, \ldots, z_{n-2}, z_n\}) + \cdots + P(\eta \leq \min\{z_2, \ldots, z_n\}) \\
S_n(z) & = & P(\eta \leq \min\{z_1, \ldots, z_n\}),
\end{array} \tag{23}$$

where the binomial functions S_k have $\binom{n}{k}$ number of terms for all $k = 0,\ldots,n$. We note that the calculations of $S_k,\ k = 0,\ldots,n$ are computationally expensive when n is large, despite the fact that calculation of each term for the addition is relatively easy as a result of the special shape of sets in (18). In (23), the random variable η is normally distributed (setting $\eta = \xi^F - e^F$ as in (14) and (15), $\eta \sim N(0,S)$, where S is the next day's estimated variance, defined by (13).). These detailed binomial functions are equivalent to (19), and z is an n-tuple vector (vector of n intraday-data points); z_i designates the data point at time i on the next trading day for $i = 1,\ldots,n$ (i.e. the realization of random variable $\eta \sim N(0,S)$ at time $i = 1,\ldots,n$).

Like Bollinger bands, we also use the next day's estimated variance defined by $S = \frac{\sum_{i=1}^{N}(x_i^P - \mu)^2}{N}$, where x_i^P's are all the given data points from the m past days, μ their mean and N the number of data points, i.e., $N = mn$. Then, with the binomial functions of (23) in the RHS of the constraints and the pre-calcuated value S in the objective function, we can write the more detailed formulation:

$$
\begin{aligned}
&\min(\max)\ S-\left(\frac{\sum_{i=1}^{n} z_i Y_i^T}{n-1}\right)(YY^T)^{-1}\left(\frac{\sum_{i=1}^{n} z_i Y_i^T}{n-1}\right)^T \\
&\text{subject to}
\end{aligned}
$$

$$
\begin{array}{rcl}
\binom{0}{0}v_0+\binom{1}{0}v_1+\binom{2}{0}v_2+\cdots+\binom{n-1}{0}v_{n-1}+\binom{n}{0}v_n & = & S_0\equiv 1 \\
\binom{1}{1}v_1+\binom{2}{1}v_2+\cdots+\binom{n-1}{1}v_{n-1}+\binom{n}{1}v_n & = & S_1(z) \\
\binom{2}{2}v_2+\cdots+\binom{n-1}{2}v_{n-1}+\binom{n}{2}v_n & = & S_2(z) \\
\ddots & & \vdots \\
\binom{n-1}{n-1}v_{n-1}+\binom{n}{n}v_n & = & S_{n-1}(z) \\
\binom{n}{n}v_n & = & S_n(z) \\
v_i & \geq & p_i, i=0,1,\ldots,n \\
-4\sqrt{S}\leq z_i & \leq & 4\sqrt{S},\ i=1,\ldots,n,
\end{array}
\tag{24}
$$

where n is the number of intraday data points; $S_k(z), k=1,\ldots,n$ are defined by (23); $p_i \geq 0.001$, $i=1,\ldots,n$, or any small positive constant; Y^T denotes the matrix $(x^P-e^P)^T$ of (12), and Y_i^T the ith row of that matrix.

The variables of the above formulation (24) are $v_i,\ i=0,\ldots,n$, and $z_i,\ i=1,\ldots,n$. The optimal objective function value (i.e. the conditional variance at optimality), will be used for the calculation of σ^C to determine the width of the bands as in Definition 1. The optimal solution $z_i,\ i=1,\ldots,n$, are used for the calculation of the conditional mean, e^C, since with those optimal solutions we can find the value of $U=\frac{\sum_{i=1}^{n} z_i Y_i^T}{n}$ as in (15). Thus, the conditional mean $e^C=e^F+UT^{-1}(x^P-e^P)$ can be calculated with the estimated mean e^F (moving average, front-weighted, or exponential average, etc.) as in Bollinger Bands (see Table 1.).

U is a $1\times m$ row vector, T^{-1} is an $m\times m$ matrix, and (x^P-e^P) is an $m\times n$ matrix. It follows that e^C is a $1\times n$ vector whose components denote the conditional mean at corresponding time points. So we calculate the average of the components of vector e^C to find the "Middle band" in Table 2, since we look for the central tendency of a stock price in a time period.

As we solve both minimization and maximization problems, two different bands are formed as in Table 2. The two different bands can be interpreted as follows. The optimal objective function value (i.e. the conditional variance) from the minimization problem is the lowest volatility level that we can expect under a reasonable setting (i.e. the constraints of the problem (24)). The minimization problem provides us with the tightest bands. Thus, investors with a higher risk tolerance (i.e. more aggressive trader) may want to use the tightest bands constructed by solving the minimization problem. The bands from the maximization problem can be considered analogous.

The constraints of (24) can be customized to incorporate investment preference among various

Table 2: The Prékopa-Lee Bands

By the minimization problem of (24)	By the maximization problem of (24)
Upper bands $= \overline{e^C_{\min}} + k\sigma^C_{\min}$ Middle bands $= \overline{e^C_{\min}}$ Lower bands $= \overline{e^C_{\min}} - k\sigma^C_{\min}$	Upper bands $= \overline{e^C_{\max}} + k\sigma^C_{\max}$ Middle bands $= \overline{e^C_{\max}}$ Lower bands $= \overline{e^C_{\max}} + k\sigma^C_{\max}$
the lowest volatility level at a given setting	*the highest volatility level at a given setting*

$\sigma^C_{\min}$ can be calculated by the minimization problem of (24), and a vector $e^C_{\min}$ is determined by its solution. $\overline{e^C_{\min}}$ is the average value of components of vector $e^C_{\min}$. Similarly, $\sigma^C_{\max}$ is from the maximization problem of (24), and by its solution $\overline{e^C_{\max}}$ can be found. The multiplier k can be chosen depending on the time periods n. We recommend width parameters with time periods are $k = 2$ if $n = 10$, $k = 2.1$ if $n = 20$, $k = 2.2$ if $n = 50$, etc.

risk tolerance levels For example, we can modify the lower bounds of v_i $i = 0,\dots,n$, or the upper and lower bounds of $z_i,\ i = 1,\dots,n$. The lower bounds of v_0 and v_n limit the next day's highest and lowest stock price, respectively. This is because the lower bounds of v_0 and v_n determine the positions of "the upper bound of the highest" and "the lower bound of the lowest" price of stock for the optimization problem in the following way:

$$\begin{aligned} v_0 &= \int_{z^{(1)}}^{\infty} f(t)dt \\ v_n &= \int_{-\infty}^{z^{(n)}} f(t)dt, \end{aligned} \tag{25}$$

where $z^{(1)}$ and $z^{(n)}$ are the largest and the smallest stock price of the next business day. The optimization problem (24) looks for the optimal solutions $z_1,\dots,z_n$ and $v_0,\dots,v_n$. Thus the order of future data points (i.e. $z^{(1)},\dots,z^{(n)}$) is unknown which will be found by the the stochastic programming problem (24).

The proper usage of our bands is as follows. More aggressive investment strategy pairs with the bands constructed by solving the minimization problem, and conversely for the bands from the maximization problem. If the indicated actions agree, the result of price-bands analysis lends confidence to a potential investment decision. On the other hand, if price-bands analysis indicates a different action (e.g. "take no action"), this may suggest further verification of the potential investment decision-making process before execution.

4 Numerical examples and discussion

We construct the price-bands of Apple Inc. (NASDAQ: APPL) and Google Inc. (NASDAQ: GOOG) over the same time period from May 14 to July 17 in 2013, by solving the binomial moment problem (24). We do not only consider the price data for the bands construction, but we also take into account that "trading volume" plays a meaningful role in the stock price movement.

Trading volume can be thought as a measure of investors' interest in the stock. There is intrinsic duality between buying and selling (shares cannot be bought unless they are sold), and so the calculation of daily average price can be formulated by

$$e_j^P = \frac{\sum_{i=1}^n P_i^{(j)} V_i^{(j)}}{\sum_{i=1}^n V_i^{(j)}}, \tag{26}$$

where $P_i^{(j)}$ and $V_i^{(j)}$ denote the stock price and its volume at time i on day j, $i = 1,\dots,n$ and $j = 1,\dots,m$. This is called "volume weighted average price" (VWAP).

For both NASDAQ:APPL and NASDAQ:GOOG stocks, we use 30-minute intraday data points, i.e., the number of data points per day is $n = 12$. Using the historical stock prices data of the past 10 days, we calculate S and e^F. For S, let us use the same variance as in the Bollinger Bands for the next day:

$$S = \frac{\sum_{i=1}^N (x_i^P - \mu)^2}{N}, \tag{27}$$

where x_i^P's are all the given data points, μ their mean, and N the number of data points, i.e., $N = 12 \times 10 = 120$.

It is reasonable and widely accepted that recent time periods influence price movements more than earlier periods, and several measures are widely used in practice, e.g. m-day moving average, exponential average and front-weighted average. Here and in what follows we will use the front-weighted average. We calculate e^F as a front-weighted average given by

$$e^F = \frac{\sum_{i=1}^{m} i e_i^P}{\sum_{i=1}^{m} i}, \tag{28}$$

where $m = 10$ since we looked back over past 10 business days, and $e_i^P, i = 1,\dots,10$ are calculated by (26).

Then, with the next business day variance S from (27), we solve the following minimization and maximization problems:

$$\begin{aligned} &\min(\max)\ S - \frac{\sum_{i=1}^n z_i Y_i^T}{n} (YY^T)^{-1} \left(\frac{\sum_{i=1}^n z_i Y_i^T}{n} \right)^T \\ &\text{subject to} \\ &\sum_{i=0}^{n} \binom{i}{k} v_i = S_k(z),\ k = 0,1,\dots,12 \\ &v_i \geq 0.001, i = 0,1,\dots,12 \\ &-4\sqrt{S} \leq z_i \leq 4\sqrt{S},\ i = 1,\dots,12, \end{aligned} \tag{29}$$

where $S_k(z), k = 1, \ldots, 12$ are defined by

$$S_k(z) \quad = \quad \sum_{1 \le i_1 < \cdots < i_k \le n} P(z^{(i_1)} \ge \eta, \ldots, z^{(i_k)} \ge \eta), \tag{30}$$

where $\eta \sim N(0, S)$.

Using the optimal solutions to (29), together with e^F from (28), we find the conditional mean vector

$$e^C = e^F + UT^{-1}(x^P - e^P). \tag{31}$$

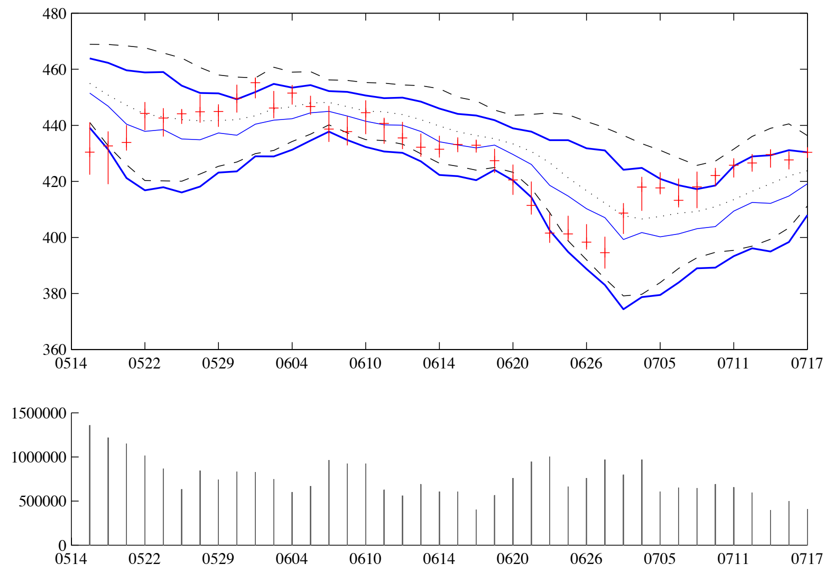
As explained in the previous section, for the central tendency of the next day, we take average of the components of vector e^C. In order to determine the "Middle band" in Table 2, the average values of the components of vector e^C from the minimization and maximization problems are designated $\overline{e^C_{\text{min}}}$ and $\overline{e^C_{\text{max}}}$, respectively.

By the results of (29) and (31), Prékopa-Lee Bands are constructed by the formulae in Table 2, and depicted in Figures 3 and 4. As the bar chart is widely used and easy to follow, we utilize it in Figures 3 and 4. The thin vertical line segments (red) are drawn to the high and low of the day, and the intersecting horizontal lines (red) represent closing prices. Let us refer to the results described in Figures 3 and 4, and the summary of numerical results presented in Table 3. The blue piecewise linear bands are the Prékopa-Lee Bands, while the dashed and dotted black bands are the Bollinger Bands. The lower figure of each subfigure represents the total trading volume of each day. Volume represents the amount of trading activity, and is a main indicator of investors' interest.

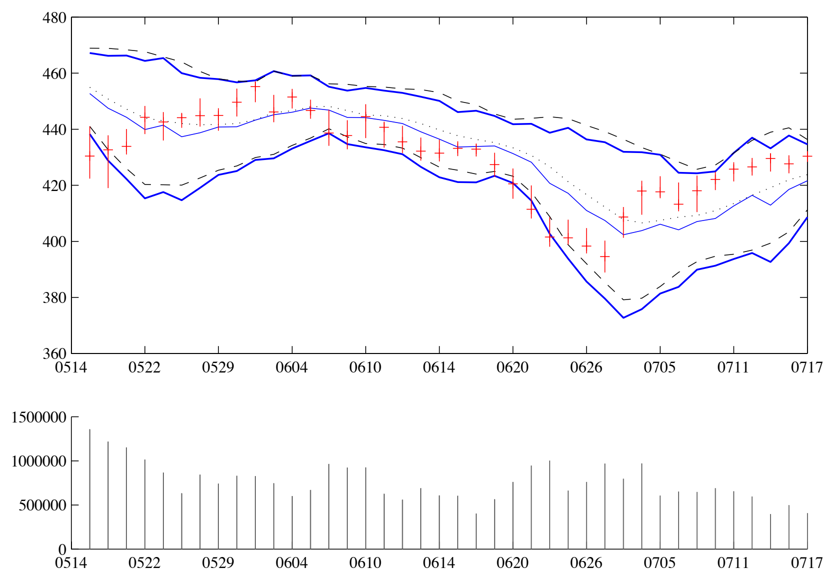
We are especially interested in finding opportunities for short-term profit realization—simply put, "buy low sell high." Given that stock prices are out of the bands, we can expect that the stock price will be back in the range of the bands in the same time period. If the red vertical line (daily price range of stocks) is fully in the bands, then this price-band strategy suggests taking no action. Otherwise, there exists the opportunity to make a profitable decision: Above the upper bands is indication to sell short and then buy as the price descends into the bands. Similarly, we can buy a stock if it's below the lower bands and sell it at a higher price upon entering the bands.

For the case of Apple Inc. (NASDAQ: AAPL) from May 14 to July 17 in 2013, Prékopa-Lee Bands provide more opportunities for making profit than Bollinger Bands. As we observe from Figure 3, the Bands from the minimization problem (29) look appealing—20 red vertical lines intersect either upper or lower bands, indicating good opportunities on 20 days out of 40 days. Another Prékopa-Lee Bands construction (by solving the maximization problem (29)) is depicted below in Figure 3 and yielded with profitable on 10 days out of 40 days, while the Bollinger Bands gives useful information from 9 days over the same timespan. For this example, the Prékopa-Lee Bands constructed from the min problem (29) perform much better than the other two. In other words, a more aggressive trading strategy works better in case of NASDAQ: APPL in that time period. We note that there is a performance gap between two different Prekopa-Lee Bands.

For Google Inc. (NASDAQ: GOOG) over the same time frame, they all performed at a similar level of effectiveness, although both Prékopa-Lee Bands marginally outperformed the Bollinger Bands in this case as well. However, in case of the bands from the minimization problem, there are four cases where lines completely out of the bands remain out in the next period, while those lines

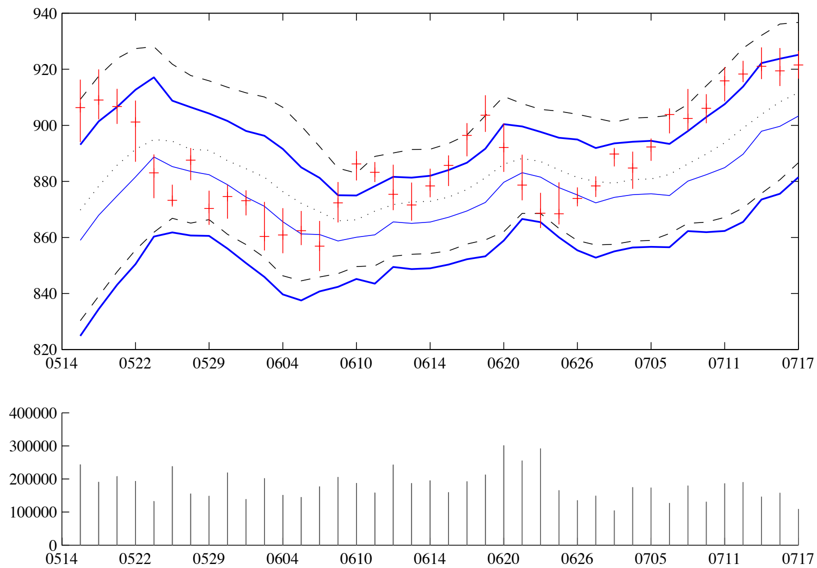


(a) From the min problem; the Prékopa-Lee bands are blue; black dotted are the Bollinger-bands

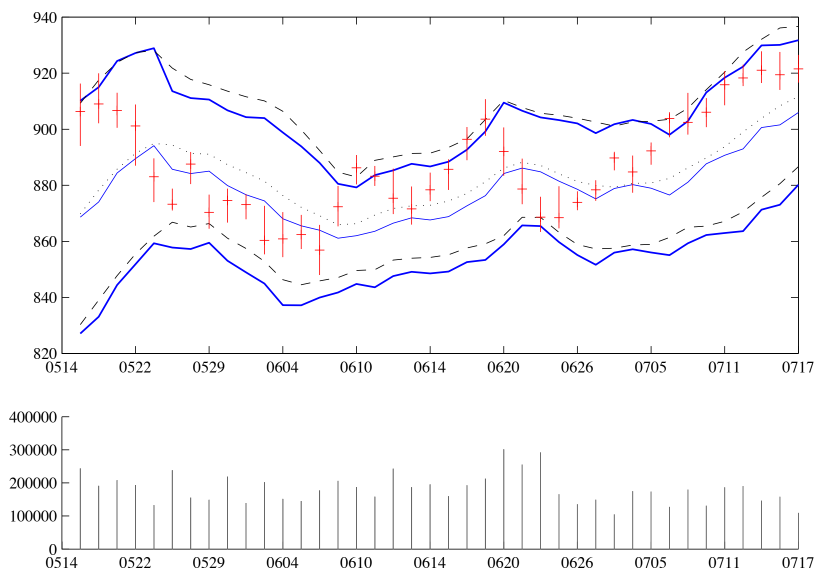


(b) From the max problem; the Prékopa-Lee bands are blue; black dotted are the Bollinger-bands

Figure 3: Prékopa-Lee bands on Apple Inc. (NASDAQ: AAPL) from May 14 to July 17 2013



(a) From the min problem; the Prékopa-Lee bands are blue; black dotted are the Bollinger-bands



(b) From the max problem; the Prékopa-Lee bands are blue; black dotted are the Bollinger-bands

Figure 4: Prékopa-Lee bands on Google Inc. (NASDAQ: GOOG) from May 14 to July 17 2013

Table 3: Comparison of the performances of Bollinger and Prékopa-Lee bands

	The Bollinger bands			The Prekopa-Lee bands			
	Number of opportunities for		total #	Number of opportunities for		total #	
	buying	short-selling		buying	short-selling		
NASDAQ: AAPL	9	0	9	8	12	20	(min)
				8	2	10	(max)
NASDAQ: GOOG	2	7	9	1	12	13	(min)
				1	11	12	(max)

For both stocks, APPL and GOOG, the Bollinger bands provide investors with 9 chances to make a profitable decision. On the other hand, for APPL, the Prékopa-Lee bands give 20 and 10 chances from the min and max problems, respectively; for GOOG, 13 and 12 chances from the min and max problems, respectively.

intersect or remain inside the Bollinger Bands. This is an undesirable situation, resulting in loss if the decision is made solely based on the bands from the minimization problem. On the other hand, the bands from the maximization problem catch two profitable cases of June 11 and July 11, while those daily vertical price lines are completely inside Bollinger Bands.

We do not insist that our model outperforms Bollinger Bands in more cases. What we hope for in our research is to provide practitioners with another useful tool of technical analysis for stock trading. Everyone loses money from trading. By spotting stock price trends with various functional tools, including Prékopa-Lee Bands, Bollinger Bands, and others, we believe that it would be possible to execute more winning trades than losing trades.

5 Concluding remark

In the current financial climate, low interest rates make stock investment more attainable, since low-cost borrowing is possible for most individuals (i.e. money is less expensive). However, successful investment remains elusive. Although it is simply determined by only four words: buy low, sell high, there is, ironically, no clear way for generating consistent profits from a stock trade, largely due to the mixture of the complexity and efficiency of the market and irrationality of its participants. Indeed, the emotional reactions of investors often (but not always) lead them to make poor real-time investment decisions. Price-bands are certainly helpful to deter investors from *entirely* following their feelings, and such tools have been widely used in practice, especially for short term investment, to help people validate their investment decisions. As one variant, we construct new price-bands via binomial moment problem formulation under the assumption that stock prices follow a Gaussian process. Usage of conditional probability distributions is the key attribute that differentiates our model. We hope that our model will pique the interest of many for both theoretical aspects and its applicability to stock trading businesses.

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