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Generating Minimal Valid Inequalities

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Generating Minimal Valid Inequalities

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Abstract. We consider the problem of generating a lattice-free convex set to find a valid inequality that minimizes the sum of its coefficients for 2-row simplex cuts. Multi-row simplex cuts has been receiving considerable attention recently and we show that a pseudo-polytime generation of a lattice-free convex set is possible. We conclude with a short numerical study.

Keywords: integer programming; lattice-free convex sets; valid inequalities.

1 Basic Definitions and Notation

Let us consider an integer programming problem and its set of feasible solutions

$$P_I \;=\; \operatorname{conv}\{x \in \mathbb{Z}^n \mid Ax = b, x \geq 0\} \tag{1}$$

where $A \in \mathbb{Q}^{m\times n}$, and $b \in \mathbb{Q}^m$. Let B be a basis of A that correspond to a basic feasible solution of the linear programming relaxation. Then we can rewrite P_I as follows

$$P_I \;=\; \left\{ \begin{pmatrix} x_B \\ x_N \end{pmatrix} \in \mathbb{Z}^n \;\middle|\; \begin{array}{c} x_B = B^{-1}b - B^{-1}Nx_N \\ x_B \geq 0, x_N \geq 0 \end{array} \right\}. \tag{2}$$

Next, we relax the nonnegativity of basic variables and the integrality of nonbasic variables, and obtain the so called *corner polyhedron*, introduced by Gomory [7]:

$$P_C \;=\; \operatorname{conv}\left\{ \begin{pmatrix} x_B \\ x_N \end{pmatrix} \in \mathbb{Z}^m \times \mathbb{R}^{n-m} \;\middle|\; \begin{array}{c} x_B = B^{-1}b - B^{-1}Nx_N, \\ x_N \geq 0 \end{array} \right\}$$

To simplify notation, let us reserve $x = x_B$ for the basic variables, denote by $s = x_N$ the nonbasic ones, and introduce $f = B^{-1}b$ and $(-B^{-1}N) = [r_1, r_2, ..., r_{n-m}]$ to get

$$P_C = \operatorname{conv}\left\{ \begin{pmatrix} x \\ s \end{pmatrix} \in \mathbb{Z}^m \times \mathbb{R}^{n-m} \;\middle|\; x = f + \sum_{i=1}^{n-m} r_i s_i, s \geq 0 \right\} \tag{3}$$

Assuming that f is not (yet) integral, we are interested in valid inequalities that separate the fractional corner f from P_C. Since variables x depend in a unique way on the nonbasic variables, according to the system of equations in (3), it is enough to consider inequalities involving only s. An inequality of the form

$$\sum_{j=1}^{n-m} \alpha_j s_j \geq \beta$$

is called a *valid inequality* for the corner polyhedron if it is satisfied by all solutions to (3). We will be only interested in valid inequalities, which are violated by f. The following useful lemma, shown in [2], claims that we do not loose generality by assuming $\beta > 0$ and $\alpha_j \geq 0$ for all $j = 1, ..., n-m$.

Lemma 1.1. *Every non-trivial valid inequality for* (3)*, that is tight at some* $(\bar{x}, \bar{s}) \in P_I$ *can be written in the form*

$$\sum_{j=1}^{n-m} \alpha_j s_j \geq 1 \tag{4}$$

for some $\alpha_j \geq 0$, $j = 1, \ldots, n-m$. □

A non-trivial valid inequality of the form (4) is called *dominated* if there exists another valid inequality $\sum_{j=1}^{n-m} \alpha'_j s_j \geq 1$ for P_C such that $\alpha'_j \leq \alpha_j$ for all $j = 1, \ldots, n-m$, with strict inequality for some indices.

2 Intersection Cuts

Balas [3] introduced the intersection cut for the corner polyhedron (3). By using Minkowski-Weyl decomposition of a polyhedron, (3) can be decomposed into $f + C$, where C is the polyhedral cone generated by rational rays r_i, $i = 1, \dots, n-m$. Let us now consider a closed convex set $S \subset \mathbb{R}^m$ such that it contains the basic solution $f \in \text{int}(S)$ in its interior, but does not contain any point from $\mathbb{Z}^m$ in its interior. To such a closed convex set we can associate the following function

$$\Phi_S(r) := \inf\{t > 0 \mid f + \frac{r}{t} \in S\}, \tag{5}$$

for $r \in \mathbb{R}^m$. Note that since f is in the interior of S, we have $\Phi_S(r)$ finite for all $r \in \mathbb{R}^m$. Furthermore, $\Phi_S(r) = 0$ only if $f + \lambda r \in S$ for all $\lambda \geq 0$. It was shown in [3] that

$$\sum_{j \in N} \Phi_S(r_j) s_j \geq 1 \tag{6}$$

is a valid inequality for (3) that cuts off the vertex $(f, 0) \in P_C$. Conforti, Cornuejols and Zambelli [6] have shown that the converse holds true, too.

Theorem 2.1 ([6]). *If $P_C \neq \emptyset$, then every minimal valid inequality for it is an intersection cut.* □

The above definition of Φ_S implies also the following claim:

Corollary 2.2. *If S is not a maximal lattice-free convex set (w.r.t. inclusion), then it generates a dominated valid inequality.* □

3 Maximal Lattice-Free Convex Sets

Let us next recall the following characterization of maximal lattice-free convex sets in finite dimensional spaces:

Theorem 3.1 ([9]). *Let V be a rational affine subspace of $\mathbb{R}^n$ containing an integral point. A set $S \subset V$ is a maximal lattice-free convex set of V if and only if S is a polyhedron of the form $S = P + L$ where P is a polyhedron, L is a rational linear space, $\dim(S) = \dim(P) + \dim(L) = \dim(V)$, S does not contain any integral point in its interior and there is an integral point in the relative interior of each facet of S.* □

Let us now return to our corner polyhedron P_C and denote by W the linear space spanned by the vectors $\{r_1, \dots, r_{n-m}\}$. As we recalled above, every maximal lattice-free convex set in $f + W$ give rise to a valid linear inequality for P_C. Let S be a maximal lattice-free convex set in $f + W$ containing f in its interior. By Theorem 3.1, S is a polyhedron, and since $f \in \text{int}(S)$, there exists a finite integer ℓ and vectors $a_1, \dots, a_\ell \in \mathbb{R}^m$ such that

$$S = \{x \in \mathbb{R}^m \mid a_i^T (x - f) \leq 1,\ i = 1, ..., \ell\}.$$

The following claim is easy to see (c.f. [5]):

Lemma 3.2. *For all* $r \in \mathbb{R}^m$ *we have*

$$\Phi_S(r) = \max_{i=1,\ldots,\ell} a_i^T r. \tag{7}$$

□

Then (7) readily implies that Φ_S is subadditive and positively homogenous. These properties then provide a short proof for the fact that (6) is a valid inequality for P_C. Namely, we can write for $(x,s) \in P_C$, $x \in \mathbb{Z}^m$ that

$$\sum_{j=1}^{n-m} \Phi_S(r_j)s_j = \sum_{j=1}^{n-m} \Phi_S(r_j s_j) \geq \Phi_S(\sum_{j=1}^{n-m} r_j s_j) = \Phi_S(x-f) \geq 1$$

First equality follows because of positive homogeneity, second inequality follows by subadditivity, the third equality follows by (3), while the final inequality follows from (7) because S is a lattice-free convex set.

4 Cutting off a Fractional Vertex

With the definitions and properties we recalled in the previous sections, we can now prove Theorem 2.1 in an important special case. Let us assume that we consider an integer programming problem of the form

$$\max c^T x$$

$$Ax \leq b$$
$$x \in \mathbb{Z}^n$$

Let $P = \{x \in \mathbb{R}^n | Ax \leq b\}$ be the set of feasible solutions in the continuous relaxation, and let $f \in P$ be a non-integral vertex of P. Let us denote by $Bx \leq d$ the subset of inequalities, which are tight at f. Assuming A is of full column rank, and that there are no redundant inequalities, B is an $n \times n$ invertible matrix. Therefore the particular corner polyhedron we are considering is of the form

$$P_C \;=\; \{(x,s) \in \mathbb{Z}^n \times \mathbb{R}^n \mid x = f + (-B^{-1})s,\ s \geq 0\} \tag{8}$$

where s denotes the slack variables added to the system $Bx \leq d$.

Let us denote by $B^i \in \mathbb{R}^n$ the ith row of B, and by $r_j \in \mathbb{R}^n$ the jth column of $-B^{-1}$. Thus, e.g., we have $B^i r_j = -1$ if $i = j$ and $= 0$ otherwise.

Let us now consider a minimal valid inequality $\alpha^T s \geq 1$ for P_C. We know by Lemma 1.1 that all minimal valid inequalities are of this form, for some $\alpha \geq 0$.

We shall prove that there exists a maximal lattice free set S, such that the corresponding intersection cut (6) is equivalent with $\alpha^T s \geq 1$ (for a more comprehensive result the reader should consult [5]). Note that in this case we have $-B^{-1}s = x - f = x - B^{-1}d$, and hence the above inequality can be written also as $\alpha^T s = \alpha^T(-B)(-B^{-1}s) = (-\alpha^T B)(x-f) \geq 1$.

Theorem 4.1. *Assume that $a_0^T(x - B^{-1}d) \geq 1$ is a facet defining inequality of the corner polyhedron and that $B^{-1}d \notin \mathbb{Z}^n$. Then, there exists a lattice-free convex set S such that (7) is equivalent to $a^T(x - B^{-1}d) \geq 1$.*

Proof. Let $\hat{S} = \{x \in \mathbb{R}^n \mid Bx \leq d,\ a_0^T(x - B^{-1}d) \leq 1\}$. Since $a_0^T(x - B^{-1}d) \geq 1$ is a valid inequality for (8), the set $\hat{S}$ is a lattice-free convex set. We note that the relative interior of any facet of $\hat{S}$ other than $F = \{x \in \hat{S} \mid a_0^T(x - B^{-1}d) = 1\}$ is not containing any integral point. Therefore we can rotate any facet $F' = \{x \in \hat{S} \mid w^T x = u\}$ of $\hat{S}$ along the face $F \cap F'$ such that we keep $B^{-1}d$ inside the resulting new polyhedron, until we hit an integral point $x' \in \mathbb{Z}^n$. Let $F'' \supset F \cap F'$ be the obtained hyperplane. We replace the halfspace corresponding to F' with the one defined by F''. Repeating this with all the facets that do not contain integral points we can arrive to a maximal lattice-free convex set $S \supset \hat{S}$. Note that if $B^{-1}d \notin \mathbb{Z}^n$ we have $B^{-1}d \in \text{int}(S)$. Figure 1 demonstrates this construction in detail.

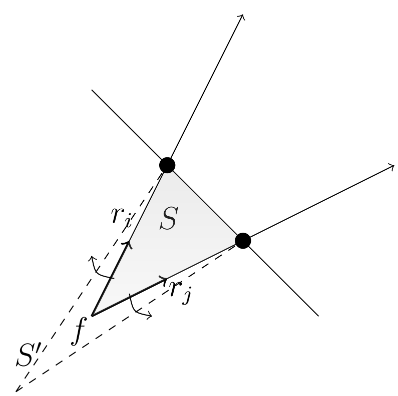

Figure 1: Maximum lattice-free convex set construction

Let us write the obtained maximal lattice free set as $S = \{x \in \mathbb{R}^n \mid a_i^T(x - B^{-1}d) \leq 1,\ i = 0, \ldots, \ell\}$ for some $a_1, \ldots, a_\ell$ obtained in this process. It is then easy to verify that we have

$$\Phi_S(r_j) = \max_{i=0,\ldots,\ell} a_i^T(r_j) = a_0^T r_j.$$

This is because the half line $f + \lambda r_j$, $\lambda \geq 0$ intersects the boundary of S within the facet F, for indices $j = 1, .., n$. Thus, the corresponding intersection cut (3) can be written as

$$\sum_{j=1}^{n} (a_0^T r_j) s_j \geq 1.$$

By using the facts that $s = d - Bx$, and $[r_1, ..., r_n] = -B^{-1}$ we can further rewrite the above as

$$a_0^T(-B^{-1})(d - Bx) \geq 1$$

which is the same as

$$a_0^T(x - B^{-1}d) \geq 1$$

as claimed. □

5 Deepest Cuts

Let us now turn back to a general corner polyhedron (3), and introduce $k = n - m$ to simplify notation. In this section we consider the problem of generating a lattice-free convex set S such that the sum of the coefficient of the corresponding intersection cut (6) is as small as possible. We can state this problem as follows:

$$\begin{array}{lll} \min & \sum_{j=1}^{k} \Phi_S(r_j) & \\ \text{s.t.} & S \subset \mathbb{R}^m & \text{is lattice-free, and} \\ & f \in \text{int}(S), & \end{array} \tag{9}$$

which we can write equivalently, by introducing additional variables λ_j, $j = 1, ..., k$, as

$$\begin{array}{lll} \min & \sum_{j=1}^{k} \lambda_j & \\ \text{s.t.} & \lambda_j = \Phi_S(r_j) & j = 1, ..., k \\ & S \subset \mathbb{R}^m & \text{is lattice-free, and} \\ & f \in \text{int}(S). & \end{array} \tag{10}$$

Using the definition (7) of Φ_S, and representing S as a polyhedral set in terms of some unknown matrix $A \in \mathbb{R}^{\ell \times m}$ we get

$$\begin{array}{lll} \min & \sum_{j=1}^{k} \lambda_j & \\ \text{s.t.} & \lambda_j = \inf\{t > 0 \mid f + \frac{r_j}{t} \in S\} & j = 1, ..., k \\ & S = \{x \in \mathbb{R}^m \mid A(x - f) \leq \mathbf{1}\} & \text{is lattice-free, and} \\ & A \in \mathbb{R}^{\ell \times m}. & \end{array} \tag{11}$$

Finally, we can eliminate inf and represent lattice-freeness with infinitely many constraints, one for each lattice point, we get:

$$\begin{array}{lll} \min & \sum_{j=1}^{k} \lambda_j & \\ \text{s.t.} & Ar_j - \lambda_j \mathbf{1} \leq 0 & j = 1, \ldots, k \\ & \max_{i=1,\ldots,\ell} (A(x - f))_i \geq 1 & \forall\, x \in \mathbb{Z}^m \\ & A \in \mathbb{R}^{\ell \times m} & \\ & \lambda_j \geq 0 & j = 1, \ldots, k \end{array} \tag{12}$$

Note that problem (12) is a semi-infinite mixed-integer disjunctive programming problem, in $\ell m + k + 1$ variables, which are ℓ, the coefficients of matrix A and λ_j, $j = 1, ..., k$.

Let us remark here that to generate useful cuts for an integer programming problem we do not need to use all rows of the simplex tableaux. In fact current practice is to use only one or two rows, corresponding to some fractional components of the basic solution. Hence we can assume that m in the above formulation is a small constant (e.g., $m = 2$.) Furthermore, since m is small, we can fix ℓ at a small value, without restricting much the scope of the model. For instance, for $m = 2$ fixing $\ell = 2$ the above model aims at finding the "deepest" *split cut*. Let us also remark that in the objective function we could introduce coefficients for λ_j, $j = 1, ..., k$, changing the meaning of "deepest."

A common approach to solve semi-infinite optimization problems is to apply cutting-plane or column-generation methods [11]. Initially we use a finite set $X \subseteq \mathbb{Z}^m$ and solve the restricted problem

$$\begin{array}{lll} \min & \sum_{j=1}^{k} \lambda_j & \\ \text{s.t.} & Ar_j - \lambda_j \mathbf{1} \leq 0 & j = 1, \ldots, k \\ & \max_{i=1,\ldots,\ell} (A(x-f))_i \geq 1 & \forall\, x \in X \\ & A \in \mathbb{R}^{\ell \times m} & \\ & \lambda_j \geq 0 & j = 1, \ldots, k \end{array} \tag{13}$$

After we solve (13) and get an optimal matrix A, we try to extend X by solving the following integer programming problem in variables $x \in \mathbb{Z}^m$ and ϵ:

$$\begin{array}{lllll} \max & & \epsilon & & \\ \text{s.t.} & Ax & +\epsilon\mathbf{1} & \leq & \mathbf{1} + Af \\ & & x & \in & \mathbb{Z}^m \\ & & \epsilon & \geq & 0. \end{array} \tag{14}$$

If the optimal value $\bar{\epsilon}$ to (14) is greater than 0, than the optimal $\bar{x}$ in the solution is an integral vector satisfying $A(\bar{x} - f) < 1$. In this case we extend $X \leftarrow X \cup \{\bar{x}\}$ and resolve problem (13). Otherwise we stop.

It is a common practice to formulate disjunctive programs as linear or integer programming problems. To formulate (13) as a linear program we can combine disjunctions into a single disjunction and then using the finiteness of the objective value, we can conclude that disjunctive programming relaxation yields the optimal objective value as discussed by Balas [4]. Otherwise we can simply make copies of the constraints and represent each copy by a single binary decision variable. This approach yields ℓ binary variables for each disjunction, therefore $\ell|X|$ in total.

Let us note that, since integer programming in a fixed number of variables can be solved in polynomial time (see [1, 8]), we can claim

Corollary 5.1. *Problem* (14) *can be solved in polynomial time when* m *is a fixed constant. Furthermore, problem* (13) *can be solved in polynomial time for a fixed number* m *of simplex rows, fixed number* ℓ *of facets and fixed cardinality* $|X|$ *subset of integer lattice around* f*.*

Let us further add that in practice we can start in X with the 2^m integer points we can obtain from f by rounding up and/or down its coordinates. Practical results show that in most cases the above method terminates just with a few calls to problem (14).

6 Numerical Results

We coded all our algorithms in C/C++ environment, using STL library and CPLEX v12.4 C API. We compiled our code using Windows SDK x64 compiler and ran the experiment on a i7 quad-core @3.5GHz computer with 16GB memory installed.

6.1 Example #1

Let us consider the following corner polyhedron ([2]):

$$\begin{aligned} P_I = \{ & (x,s) \in \mathbb{Z}^2 \times \mathbb{R}^5_+ : x = \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} + \\ & \begin{pmatrix} 2 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -3 \\ 2 \end{pmatrix} s_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_4 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} s_5 \} \end{aligned}$$

we solve (12) for a 3-faceted polyhedral region and an initial subset

$$X = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

of $\mathbb{Z}^2$. The following polyhedron (as depicted in Figure 2) is the solution of (12):

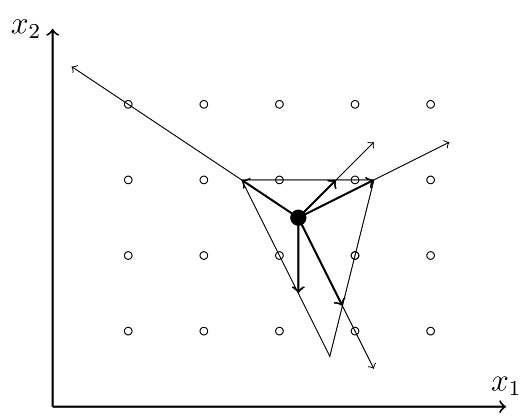

Figure 2: $\bar{A}(x - f) \leq 1$

$$\bar{A} = \begin{bmatrix} 0 & 2 \\ -2 & -1 \\ 1.1429 & -0.2857 \end{bmatrix}$$

where $\bar{A}$ is the optimum solution to (12). This lattice-free convex set generates the inequality:

$$2s_1 + 2s_2 + 4s_3 + s_4 + 1.7143s_5 \geq 1.$$

that is the inequality proposed by the authors drawn on a 4-faceted polyhedron.

6.2 Example #2

Next we consider Nemhauser and Wolsey's example [10] depicted in Figure 3. We carry out the same procedure described in Example 1 and generate a minimal valid inequality using the simplex solution denoted by the black node in Figure 3. Where we note the minimal valid inequality cuts into the CG-closure, and in fact is a facet of the integer hull (then only facet missing from the CG-closure.)

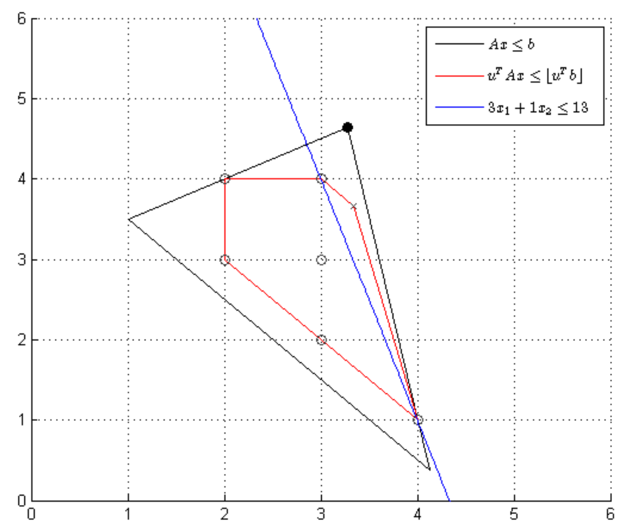

Figure 3: Nemhauser and Wolsey's example, CG closure and MVI cut

6.3 Example #3

We also take a look into a larger data set of partial vertex cover problems. This problem is essentially a weighted vertex cover problem which aims to cover a user-defined cardinality subset of the edges. Given a graph $G = (V, E)$, a vector of vertex weights $w \in \mathbb{R}_+^{|V|}$ and a nonnegative number of edges to be covered $k \leq |E|$, we can formulate partial vertex cover problem as follows:

$$\begin{array}{llcl} \min & \sum_{i\in V} w_i x_i & & \\ \text{s.t.} & x_i + x_j & \geq & y_{ij}, \quad ij \in E \\ & \sum_{ij\in E} y_{ij} & = & k \\ & x_i & \in & \{0,1\}, \quad i \in V \\ & y_{ij} & \geq & 0, \quad ij \in E. \end{array}$$

We uniformly generated partial vertex cover instances over two setups one representing a dense graph the other a sparse graph. Our dense graphs involves of 60 nodes, 12 max vertex degree and 90% cover ratio of the edge set. The sparse instances involve 100 nodes, 5 max vertex degree and similarly 90% cover ratio. We will represent these instances as (60,12,0.9) and (100,5,0.9), respectively. We used branch and cut using various cutting techniques incorporating minimal valid inequalities or mixed integer rounding cuts as follows:

- **MIR**: uses mixed integer rounding inequalities.
- **MVI/MIR**: checks the size of problem (12), if under some threshold, generates a minimal valid inequality, else generates mixed integer rounding inequality.
- **MVI+MIR**: generates mixed integer rounding inequality. It also checks the size of problem (12), if under some threshold, generates a minimal valid inequality.

In this study we are investigating 2-row triangle simplex cuts. We will first take a look into the (60,12,0.9) instances:

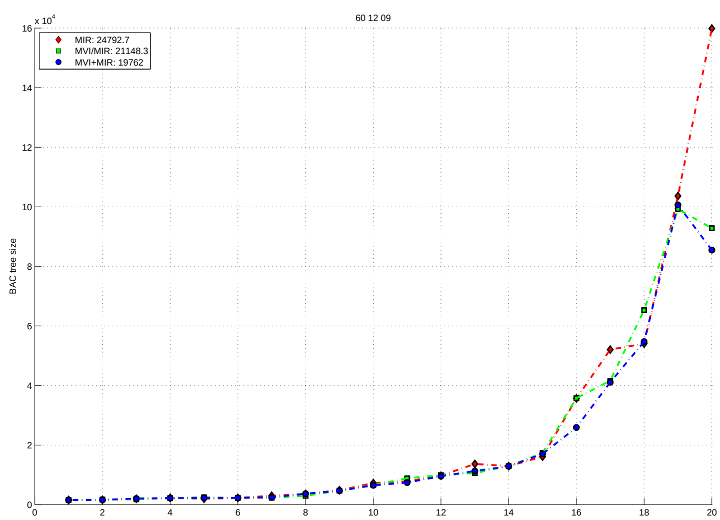

Figure 4: BAC tree size vs. instances

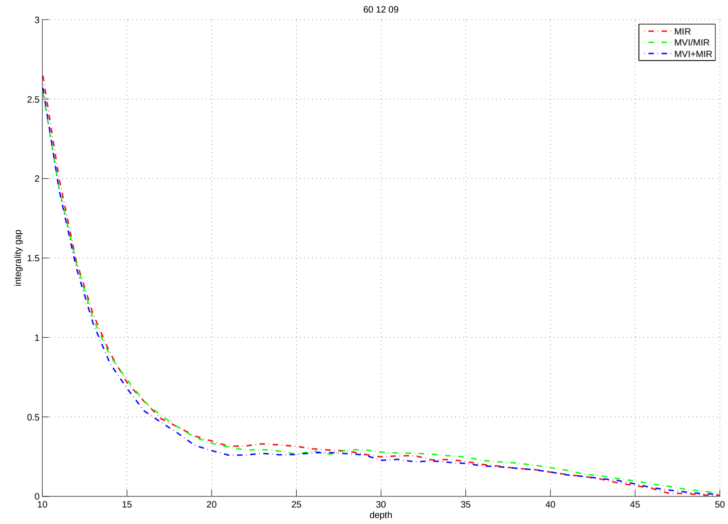

Figure 5: Integrality gap vs. depth

As we can easily observe from the integrality gap data of Figure 5, minimal valid inequalities help with closing the integrality gap over mixed integer rounding cuts. Also we note a sizeable win over the BAC tree size using minimal valid inequalities. Next we take a look into the (100,5,0.9) instances:

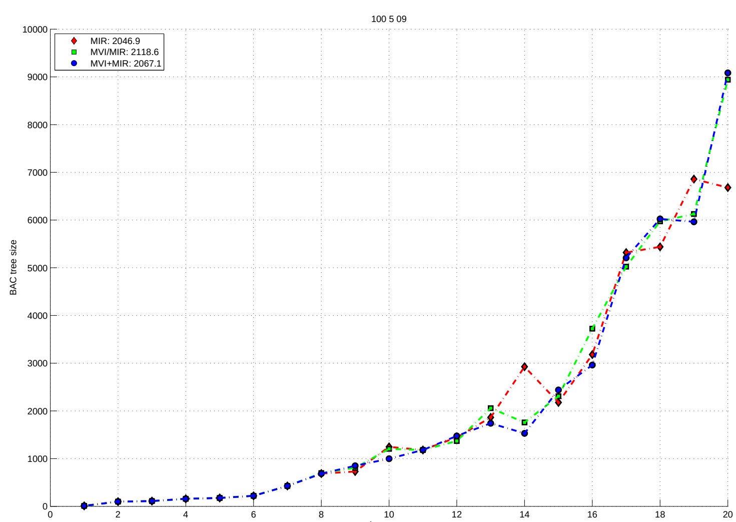

Figure 6: BAC tree size vs. instances

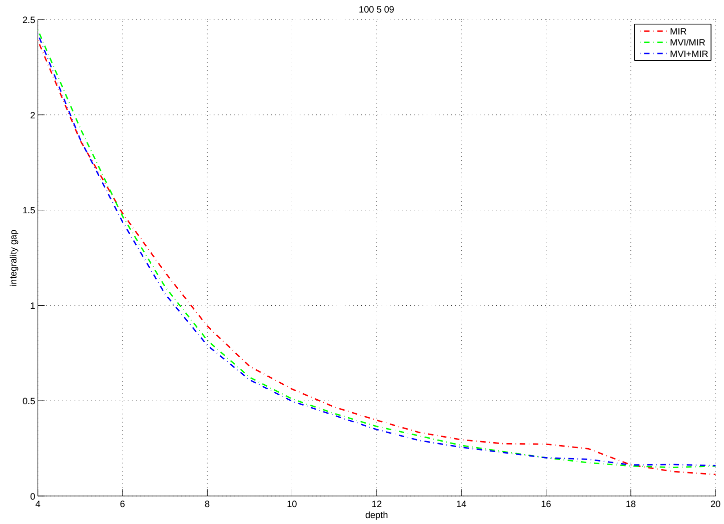

Figure 7: Integrality gap vs. depth

We can still observe the decrease in the integrality gap, however we don't see a win in terms of BAC tree size over these sparse instances. Still we note BAC tree size depends on many other factors in terms of the BAC algorithm. A better indicator of the valid inequality strength, the integrality gap, points to the same conclusion that minimal valid inequalities improves the integrality gap obtained by mixed integer rounding cuts for partial vertex cover problems.

7 Conclusion

We study the on-line generation of maximal lattice-free convex sets in this paper. We develop a fixed size mixed integer programming problem to generate such a set in poly-time and show improved integrality gap over medium sized vertex cover problems employing inequalities that can be generated using the lattice-free convex set generation scheme.

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