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Exact Models for Open Field Layout Problem with l_2 and l_1 Distances

Gergely Kovács Béla Vizvári

Abstract. In layout problem of manufacturing cells, cells have rectangular shapes and must be positioned without overlapping. The objective is to minimize the total transportation cost.

1 Introduction

A large class of layout problem is when the objects to be placed are represented by rectangles. The rectangles must be placed without overlap. The objective is to minimize the total transportation among the objects. The transportation quantity depends on the flow among the objects and their distances. The rectangles may represent very different kinds of things. A typical example is that the position of machines and/or cells of a factory are modeled in this way. The objective is to minimize the total transportation of the semi-finished products.

A general restriction is that the rectangles must have vertical and horizontal edges and can be rotated by 90, 180, and 270 degrees. Each cell has a pick-up point which is the middle point of a fixed edge. Transportation is carried out between the pick-up points. There are four main types of layout in this case. *Open field* layout does not have any further restriction. It is the most general type. Vertical and horizontal lines divide the layout into rectangular parts in the case of *ladder* layout. Transportation is possible on only these lines and all pick-up points must lie on the lines. If the track of the transportation is also a rectangle and the cells can be inside or outside the track then the layout is called *closed loop* layout. Finally, if the track is on a single line then the name of the layout is *spine*.

In the latter case the distances among the pick-up points can be measured easily and the mathematical models use the exact distances. To the best knowledge of the authors of this paper, no mathematical model is known in the case of open field and ladder layout which uses exact distances. The distance of two pick-up point is measured by their l_1 distance, although the vehicle may need to pass a longer distance. In the case of the closed loop layout there is an exact model [Niroomand-Vizvári 2013].

The main contributions of this paper are models with exact distances. The meaning of the word "exact" is that the distance of two points is exactly the length what the vehicle passes if it goes from one point to other other one. If the distances are measured by Euclidean distance (l_2 distance), then the distance of the points in the layout is their Euclidean distance if and only if there is no obstacle between the two points, i.e. it is possible "to see" the other point from one of the points. If the distances are measured by Manhattan distance (l_1 distance) then the distance of two points in the layout is their Manhattan distance if and only if there is a sequence of adjacent vertical and horizontal intervals such that this sequence goes from one point to the other one and both vertically and horizontally goes always into the same direction, for example it goes always up (vertical motion) and right (horizontal motion). Exact models are provided for both l_2 and l_1 distances. Both of them is based on the classical model of [Das 1993]. The model with l_1 distance can be considered as an approximation of the ladder layout.

The latter model is discussed in Section 2, which does not contain new results, however some necessary notations are introduced even this section. Section 3 gives the model for l_2 distances and l_1 distance is discussed in section 4.

2 General Constraints of Layout Problems

The constraints which must be satisfied in all layout problems are discussed based on [Das 1993]. A very similar model can be found in [Tsai *et al.* 2011].

It is assumed that the rectangles of the cells have only horizontal and vertical edges and that the cells are not rotated in any other way. In what follows the words "rectangle" ans "cell" will be used as synonims.

An exact model of the layout of the rectangular cells must satisfy the following constraints:

- the cells must not overlap,
- the cells can be rotated by 90, 180 or 270 degrees.

The model uses the coordinates of the central points of the rectangles as variables. However, the corner points are which must not be covered by other rectangles. The coordinates of the corner points depend on the rotation of the rectangle. The coordinates of the pick-up points are also important as the transportation is carried out among them. These coordinates also depend on the central points and the rotations of the cells. The four corner points and the pick-up point of a cell are called the *critical points* of the cell.

2.1 Contstans and variables

The summary of the notations can be found in the next three tables. Each notation is introduced in the text when it is used first time.

Notations

Table 1. Indices

i, j, p	indices of cells
k, l	positional index (between 1 and 5)
i_k	index of a vertex or pick-up point of cell i
j_l	index of a vertex or pick-up point of cell j
q, r	rotational index (between 1 and 4)

Table 2. Constants and Parameters

n	the number of cells
s_i	the length of the vertical edge of cell i in the basic position
t_i	the length of the horizontal edge of cell i in the basic position
M	a large positive number
$\mathbf{e}_{i_5}$	unit vector of cell i
f_{ij}	the flow value between cells i and j
$\mathbf{A}$	the node-arc adjacency matrix of complete graph of the critical points of the cells

Table 3. Variables

x_i (y_i)	horizontal (vertical) coordinate of the center of cell i
z_i	binary variable; it is 1 (0) if cell i is in a vertical (horizontal) position
x_{i_k} (y_{i_k})	horizontal (vertical) coordinate of a vertex or pick-up point of cell i
λ_{i_q}	binary variables describing the position of the pick-up point of cell i according to its rotation
e_{ij}	$\max\{0, x_i - x_j\}$
f_{ij}	$\max\{0, x_j - x_i\}$
g_{ij}	$\max\{0, y_i - y_j\}$
h_{ij}	$\max\{0, y_j - y_i\}$
α_{ij}	a binary variable; it is 1 if $x_i \geq x_j$
β_{ij}	a binary variable; it is 1 if $y_i \geq y_j$
μ_{ij}	a binary variable; if it is 1, then cells i and j are not overlapping vertically, and if it is 0, then cells i and j are not overlapping horizontally
$d_{i_k j_l}$	distance of points i_k and j_l
$M_{i_k j_l pr}$	penalty variable; it is M if the line $i_k j_l$ crosses the r-th edge of cell p, and it is 0 otherwise
$\gamma_{i_k j_l pr}$, $\delta_{i_k j_l pr}$	auxiliary variables for calculating the value of the penalty variable
$u_{i_k j_l pr}$, $v_{i_k j_l pr}$, $w_{i_k j_l pr}$, $z_{i_k j_l pr}$, $m_{i_k j_l pr}$	binary variables for calculating the value of the penalty variable
$\mathbf{e}_{i_5}$	unit vector of cell i
$\mathbf{g}_{ij}$	path vector between pick up points i_5 and j_5

2.2 Vertices of cells

Cell i is a rectangle of size $s_i \times t_i$ such that in the basic position s_i (t_i) is the length of the vertical (horizontal) edge. Basic rotation means 0 degree of rotation. The cells are represented by their center. The position of a cell is vertical (horizontal) if the position of its edge of length t_i is vertical (horizontal). The position is described by the binary variable z_i.

All vertices of cell i have an index i_k, where $k = 1, 2, 3, 4$. The index of the upper left vertex is i_1, and so on, clockwise. The index of the lower left vertex is i_4.

The horizontal coordinates of the vertices of cell i are described by the following con-

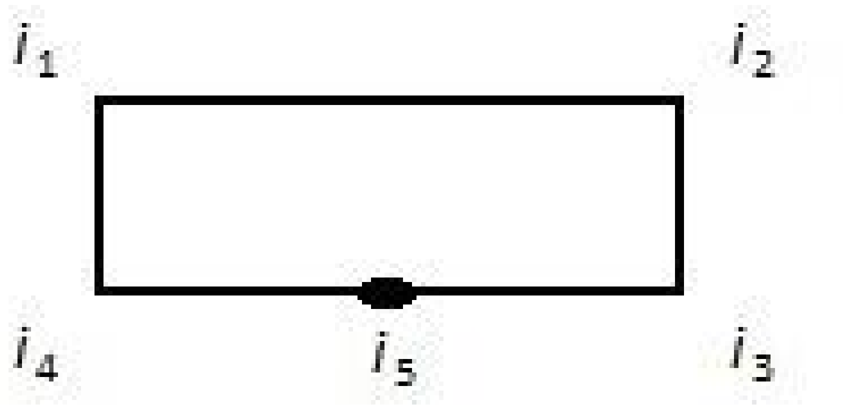

Figure 1: The indices of the vertices and the pick-up point of cell i.

straints:

$$x_{i_1} = x_{i_4} = x_i - \frac{1-z_i}{2}t_i - \frac{z_i}{2}s_i \quad (1)$$

$$x_{i_2} = x_{i_3} = x_i + \frac{1-z_i}{2}t_i + \frac{z_i}{2}s_i \quad (2)$$

The vertical coordinates of the vertices of cell i are described similarly:

$$y_{i_1} = y_{i_2} = y_i + \frac{1-z_i}{2}s_i + \frac{z_i}{2}t_i \quad (3)$$

$$y_{i_3} = y_{i_4} = y_i - \frac{1-z_i}{2}s_i - \frac{z_i}{2}t_i \quad (4)$$

2.3 Pick-up points of cells

This case the coordinates of the pick-up point of cell i depends only on the rotation of the cell in the layout. The rotation of cell i is described by four binary variables, λ_{i_1}, λ_{i_2}, λ_{i_3}, and λ_{i_4} defined as follows:

$$\lambda_{i_1} = \begin{cases} 1 & \text{if the pick-up point is on the right side of the center} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{i_2} = \begin{cases} 1 & \text{if the pick-up point is below the center} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{i_3} = \begin{cases} 1 & \text{if the pick-up point is on the left side of the center} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{i_4} = \begin{cases} 1 & \text{if the pick-up point is above the center} \\ 0 & \text{otherwise} \end{cases}$$

The λ and z variables are not independent. Two equations must hold between them. The cell is in a vertical position if the pick-up point is below or above the center. Thus,

$$z_i = \lambda_{i_2} + \lambda_{i_4} \tag{5}$$

implying that

$$1 - z_i = \lambda_{i_1} + \lambda_{i_3}. \tag{6}$$

The index of the pick-up point of the cell is i_5; i.e. $k = 5$. The two coordinates of the pick-up point of cell i are

$$x_{i_5} = x_i + (\lambda_{i_1} - \lambda_{i_3})\frac{s_i}{2} \tag{7}$$

and

$$y_{i_5} = y_i + (\lambda_{i_4} - \lambda_{i_2})\frac{s_i}{2}. \tag{8}$$

2.4 Non-overlapping constraints

Two cells are overlapping if and only if their centers are too close to each other. The minimal required horizontal (vertical) distance such that two cells are not overlapping is half the sum of the length of their edges in the horizontal (vertical) position. The sum depends on the rotation of the cells. Notice that $e_{ij} + f_{ij}$ ($g_{ij} + h_{ij}$) is the horizontal(vertical) distance of the centers of the cells i and j. If there is no horizontal (vertical) overlap, then the distance must be at least as long as the sum of the two horizontal (vertical) half edges. This requirement is described by the following inequalities (for all i, j, $i \neq j$):

$$e_{ij} + f_{ij} - \frac{1 - z_i}{2}t_i - \frac{z_i}{2}s_i - \frac{1 - z_j}{2}t_j - \frac{z_j}{2}s_j \geq -M\mu_{ij} \tag{9}$$

and

$$g_{ij} + h_{ij} - \frac{1 - z_i}{2}s_i - \frac{z_i}{2}t_i - \frac{1 - z_j}{2}s_j - \frac{z_j}{2}t_j \geq -M(1 - \mu_{ij}). \tag{10}$$

It is difficult to use the formulae of e_{ij}, f_{ij}, g_{ij}, and h_{ij} explicitly in an optimization problem; therefore, they are described implicitly by the following constraints:

$$x_i - x_j = e_{ij} - f_{ij} \tag{11}$$
$$y_i - y_j = g_{ij} - h_{ij} \tag{12}$$
$$e_{ij} \leq M\alpha_{ij} \tag{13}$$
$$f_{ij} \leq M(1 - \alpha_{ij}) \tag{14}$$
$$g_{ij} \leq M\beta_{ij} \tag{15}$$
$$h_{ij} \leq M(1 - \beta_{ij}). \tag{16}$$

To complete the model, the technical constraints defining the type of the variables must be mentioned. Without loss of generality, we may assume that the cells are in the nonnegative quarter of the plane:

$$x_i, y_i, x_{i_k}, y_{i_k} \geq 0. \tag{17}$$

The distance variables are also nonnegative:

$$e_{ij}, f_{ij}, g_{ij}, h_{ij} \geq 0. \tag{18}$$

All other variables are binary:

$$z_i, \lambda_{i_q}, \mu_{ij}, \alpha_{ij}, \beta_{ij} = 0 \text{ or } 1. \tag{19}$$

Notice that the inequalities (9) and (10) handle both overlapping and rotation. Constraints (13) and (14) with nonnegativity ensure that at least one of e_{ij} and f_{ij} is equal to zero.

3 Exact Mathematical Model for l_2 Distances

In our main model we use the Euclidean distance for the i_k and j_l points if there are no cells between the points. Otherwise the distance will be M.

3.1 Distance of two vertices of the same cell

The distance between two adjacent vertices of cell i is the length of the edge:

$$d_{i_1 i_2} = d_{i_3 i_4} = z_i s_i + (1 - z_i)t_i, \tag{20}$$
$$d_{i_2 i_3} = d_{i_1 i_4} = z_i t_i + (1 - z_i)s_i. \tag{21}$$

Moving directly between opposite vertices is not feasible, thus

$$d_{i_1 i_3} = d_{i_2 i_4} = M. \tag{22}$$

3.2 Distance of a vertex and the pick-up point of the same cell

The distance of the pick-up point of cell i and its next vertices is the half of the length of the edge, i.e. $\frac{t_i}{2}$. Moving directly between the pick-up point and the opposite vertices of the same cell is not feasible, thus in this case the distance is M:

$$d_{i_5 i_1} = (\lambda_{i_1} + \lambda_{i_2})M + (\lambda_{i_3} + \lambda_{i_4})\frac{t_i}{2}, \tag{23}$$

$$d_{i_5 i_2} = (\lambda_{i_2} + \lambda_{i_3})M + (\lambda_{i_1} + \lambda_{i_4})\frac{t_i}{2}, \tag{24}$$

$$d_{i_5 i_3} = (\lambda_{i_3} + \lambda_{i_4})M + (\lambda_{i_1} + \lambda_{i_2})\frac{t_i}{2}, \tag{25}$$

$$d_{i_5 i_4} = (\lambda_{i_1} + \lambda_{i_4})M + (\lambda_{i_2} + \lambda_{i_3})\frac{t_i}{2}. \tag{26}$$

3.3 Distance of two points of two different cells

The distance of points i_k and j_l is the Euclidean distance if and only if there is a feasible segment between the two points, i.e. there are no cells between the points. Otherwise the distance will be increased by M, thus

$$d_{i_k j_l} = \sqrt{(x_{i_k} - x_{j_l})^2 + (y_{i_k} - y_{j_l})^2} + \sum_{p=1}^{n} \sum_{r=1}^{4} M_{i_k j_l p r}, \tag{27}$$

where $M_{i_k j_l pr}$ is a penalty variable; it is M if the line $i_k j_l$ crosses the r-th edge of cell p and it is 0 otherwise.

If $r = 1$, then the r-th edge of cell p is the vertical edge between points p_2 and p_3. Then there exists the unique value $\gamma_{i_k j_l p1}$ for which:

$$x_{p_2} = x_{p_3} = \gamma_{i_k j_l p1} x_{i_k} + (1 - \gamma_{i_k j_l p1}) x_{j_l}. \tag{28}$$

Moreover there exists the value $\delta_{i_k j_l p1}$ which satisfies the following equation:

$$\gamma_{i_k j_l p1} y_{i_k} + (1 - \gamma_{i_k j_l p1}) y_{j_l} = \delta_{i_k j_l p1} y_{p_2} + (1 - \delta_{i_k j_l p1}) y_{p_3}. \tag{29}$$

If the segment between i_k and j_l crosses the 1st edge of cell p, then both of the values $\gamma_{i_k j_l p1}$ and $\delta_{i_k j_l p1}$ are between 0 and 1, and (28) determines the horizontal, and (29) determines the vertical coordinate of the cross-point.

Similarly, if $r = 3$, then there exist the same $\gamma_{i_k j_l p3}$ and $\delta_{i_k j_l p3}$, thus:

$$x_{p_1} = x_{p_4} = \gamma_{i_k j_l p3} x_{i_k} + (1 - \gamma_{i_k j_l p3}) x_{j_l} \tag{30}$$

$$\gamma_{i_k j_l p3} y_{i_k} + (1 - \gamma_{i_k j_l p3}) y_{j_l} = \delta_{i_k j_l p3} y_{p_1} + (1 - \delta_{i_k j_l p3}) y_{p_4}. \tag{31}$$

If $r = 2$ or $r = 4$, then the edge of cell p is horizontal, thus the vertical equation is described first. If $r = 2$, then there exist the only $\delta_{i_k j_l p2}$ and $\gamma_{i_k j_l p2}$ which satisfy the

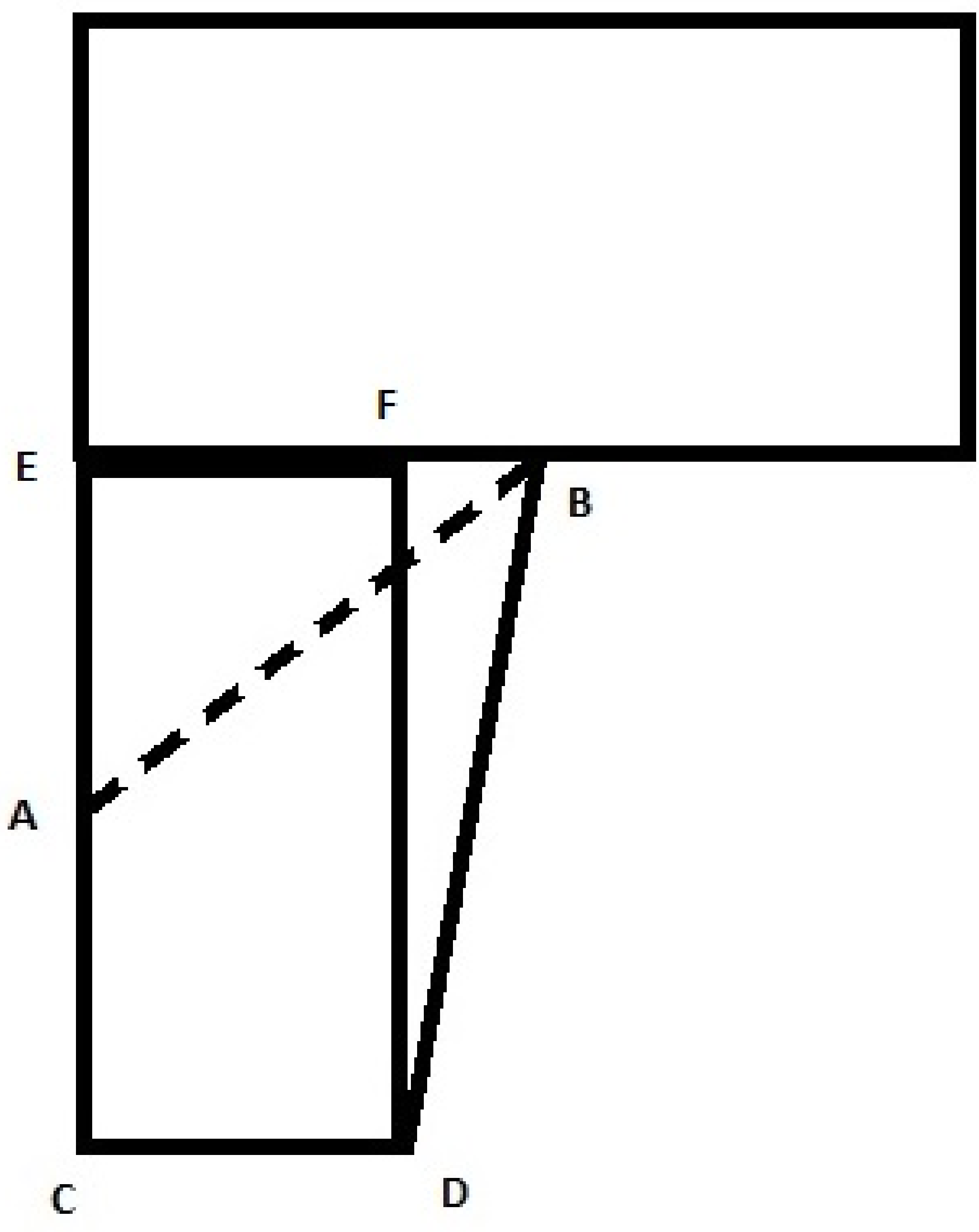

Figure 2: The vehicle cannot use a track like the dashed line as it intersects a cell. There are two ways to go from A to B: (i) AEFB, and (ii) ACDB.

following:

$$y_{p_4} = y_{p_3} = \delta_{i_k j_l p2} y_{i_k} + (1 - \delta_{i_k j_l p2}) y_{j_l} \tag{32}$$

$$\delta_{i_k j_l p2} x_{i_k} + (1 - \delta_{i_k j_l p2}) x_{j_l} = \gamma_{i_k j_l p2} x_{p_4} + (1 - \gamma_{i_k j_l p2}) x_{p_3}. \tag{33}$$

Similarly, if $r = 4$, then there exist the same $\delta_{i_k j_l p4}$ and $\gamma_{i_k j_l p4}$, thus:

$$y_{p_1} = y_{p_2} = \delta_{i_k j_l p4} y_{i_k} + (1 - \delta_{i_k j_l p4}) y_{j_l} \tag{34}$$

$$\delta_{i_k j_l p4} x_{i_k} + (1 - \delta_{i_k j_l p4}) x_{j_l} = \gamma_{i_k j_l p4} x_{p_1} + (1 - \gamma_{i_k j_l p4}) x_{p_2}. \tag{35}$$

For example if $r = 4$, then the segment between i_k and j_l crosses the 4th edge of cell p if and only if, then both of the values $\delta_{i_k j_l p4}$ and $\gamma_{i_k j_l p4}$ are between 0 and 1.

If $0 < \gamma_{i_k j_l pr} < 1$, then for feasibility of the following two inequalities the values of the two binary variables need to be equal with 1. Otherwise need not.

$$1 - \gamma_{i_k j_l pr} - u_{i_k j_l pr} M \leq 0 \tag{36}$$

$$\gamma_{i_k j_l pr} - v_{i_k j_l pr} M \leq 0 \tag{37}$$

Similarly, in the following inequalities w and z are both equal with 1 if $0 < \delta_{i_k j_l pr} < 1$.

$$1 - \delta_{i_k j_l pr} - w_{i_k j_l pr} M \leq 0 \tag{38}$$

$$\delta_{i_k j_l pr} - z_{i_k j_l pr} M \leq 0 \tag{39}$$

If the segment between i_k and j_l crosses the r-th edge of cell p, then all four binary variables are 1, otherwise they need not. Therefore in the following inequality the value of m is 1 if the segment crosses the edge:

$$u_{i_k j_l pr} + v_{i_k j_l pr} + w_{i_k j_l pr} + z_{i_k j_l pr} \leq 3 + m_{i_k j_l pr}. \tag{40}$$

Thus the penalty is determined by the following:

$$M_{i_k j_l pr} = m_{i_k j_l pr} M. \tag{41}$$

3.4 Minimal cost flow

The final main step is the formulation of the objective function. It is the minimization of the sum of the flow between cells weighted by the distance of the pick-up points of the cells, where the edges of the graph are the edge points and the pick-up points.

The following constraint determines a path from pick-up point i_5 to pick-up point j_5, for all non-negative flow value f_{ij}:

$$\mathbf{A}\mathbf{g}_{ij} = \mathbf{e}_{i_5} - \mathbf{e}_{j_5}, \tag{42}$$

where $\mathbf{A}$ is the node-arc adjacency matrix of the complete graph consisting of the critical points of all cells.

Finally the distance weighted with the flow among the cells is minimized, i.e., the objective function is

$$\min \sum_{i,j} f_{ij} \mathbf{d}^T \mathbf{g}_{ij}, \tag{43}$$

where $\mathbf{d}$ is the distance vector from values $d_{i_k j_l}$.

To complete the model, the technical constraints defining the type of the variables must be mentioned. The distance variables are also nonnegative:

$$d_{i_k j_l} \geq 0. \tag{44}$$

The path variables are also nonnegative:

$$g_{i_k j_l} \geq 0. \tag{45}$$

All other variables are binary:

$$\mu_{ij}, u_{i_j k_l pr}, v_{i_j k_l pr}, w_{i_j k_l pr}, z_{i_j k_l pr}, m_{i_j k_l pr} = 0 \text{ or } 1. \tag{46}$$

4 Model for l_1 Distances

In the second model the positions of the cells are similar, but distances are measured by vertical and horizontal moves. Thus the non-overlapping constraints which ensure the feasibility of the solutions are not discussed again. Similarly, if the distances of the points of the model are determined then the minimal paths can be obtained in the same way as in the previous model. The model is based on a new notion of *turning point*. First, this notion is discussed. After that the new notations are introduced. The exact l_1 distances are discussed at the end of this section.

4.1 Turning points

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ two points on the plane which are not on the same vertical or horizontal lines, i.e. $a_1 \neq b_1$ and $a_2 \neq b_2$. Assume that there is no any obstacle on the plane, thus the l_1-walker can go without any restriction. There are two paths of minimal l_1 length from A to B such that the direction of the move changes only once. Either the l_1-walker goes first horizontally to the (b_1, a_2) point and then vertically from (b_1, a_2) to $(b_1, b_2) = B$, or he goes first vertically to the point (a_1, b_2) and then from here to B. The points (b_1, a_2) and (a_1, b_2) are the *turning points* determined by A and B.

The meaning of the turning point from the point of view of engineering is that it is a point where the vehicle may turn +90 or -90 degree. There are infinite many turning points, however there is an optimal solution in which the turning points are determined by the critical points of the cells.

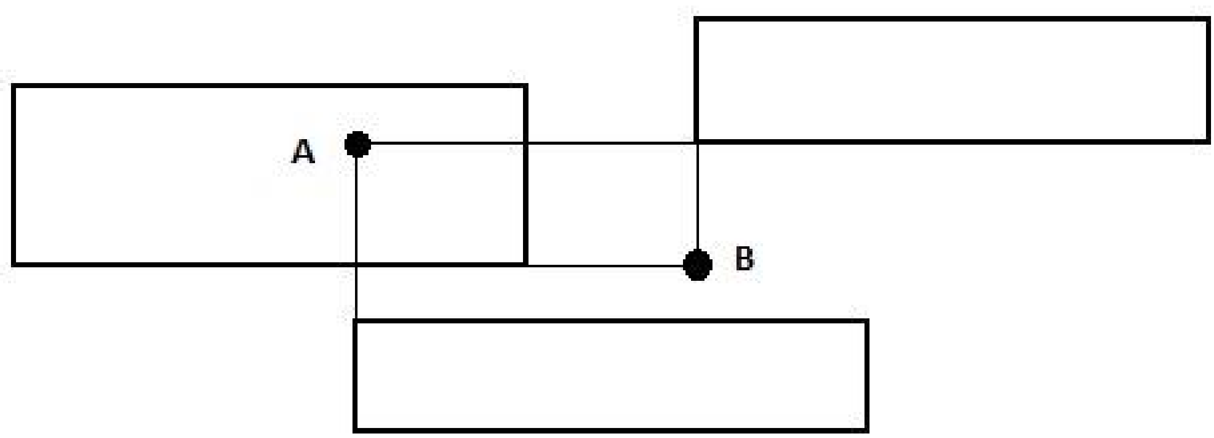

Figure 3: The route of the l_1 walker via point A is infeasible, however the use of point B is possible.

The coordinates of the vertices and pick-up points are the same as at the main model. These critical points are also turning points by definition. The coordinates of all other turning points are chosen from the coordinates of vertices and pick-up points. Cell i gives at most three different horizontal and at most three different vertical value: x_{i_1}, x_{i_3}, x_{i_5}, and y_{i_1}, y_{i_3}, y_{i_5}. (And of course one of x_{i_5} and y_{i_5} is not different from the value of vertex i_1 or i_3.)

All possible turning points $(x_{i_k}; y_{j_l})$ are constructed, where $k, l \in \{1, 3, 5\}$. It means that there are $3n \cdot 3n$ turning points with multiplicity (from the equation of one coordinate of the pick-up points, or from the equation of coordinates of separate vertices or pick-up points). Of course the vertices and the pick-up points all are elements of turning points, for example the pick-up points are the turning points, where $i = j$ and $k = l = 5$.

Some of the turning points are inside cells. These points cannot be used, i.e. any distance between such a point and any other turning point must be at least M. Similarly, if there is an obstacle on both l_1 track of the turning point then their distance must be at least M.

4.2 New or modified constants and variables

Notations

Table 1. New or modified constants and variables used in the model

$\varphi, \psi, \vartheta, \varrho, i$	indices of cells	index
η, ω, ξ, ζ	positional indices (between 1 and 5)	index
$d_{\varphi_\eta \psi_\omega \varrho_\zeta \vartheta_\xi}$	distance of turning points	variable
$e_{\varphi_\eta \psi_\omega}$	$\max\{0, x_{\varphi_\eta} - x_{\psi_\omega}\}$	variable
$f_{\varphi_\eta \psi_\omega}$	$\max\{0, x_{\psi_\omega} - x_{\varphi_\eta}\}$	variable
$g_{\varphi_\eta \psi_\omega}$	$\max\{0, y_{\varphi_\eta} - y_{\psi_\omega}\}$	variable
$h_{\varphi_\eta \psi_\omega}$	$\max\{0, y_{\psi_\omega} - y_{\varphi_\eta}\}$	variable
$V_{\psi_\omega \vartheta_\xi \varrho_\zeta}$ $H_{\varphi_\eta \vartheta_\xi \varrho_\zeta}$	penalty variables	variable
$\gamma_{\psi_\omega \vartheta_\xi \varrho_\zeta}$, $\delta_{\varphi_\eta \vartheta_\xi \varrho_\zeta}$	auxiliary variables for calculating the value of the penalty variable	variable
$u_{\psi_\omega \vartheta_\xi \varrho_\zeta}$, $v_{\psi_\omega \vartheta_\xi \varrho_\zeta}$, $w_{\varphi_\eta \vartheta_\xi \varrho_\zeta}$, $z_{\varphi_\eta \vartheta_\xi \varrho_\zeta}$, $m_{\psi_\omega \vartheta_\xi \varrho_\zeta}$, $o_{\varphi_\eta \vartheta_\xi \varrho_\zeta}$	binary variables for calculating the value of the penalty variables	variable
$e_{\varphi_\eta \psi_\omega i}$	$\max\{0, x_{\varphi_\eta} - x_i\}$	variable
$f_{\varphi_\eta \psi_\omega i}$	$\max\{0, x_i - x_{\varphi_\eta}\}$	variable
$g_{\varphi_\eta \psi_\omega i}$	$\max\{0, y_i - y_{\psi_\omega}\}$	variable
$h_{\varphi_\eta \psi_\omega i}$	$\max\{0, y_{\psi_\omega} - y_i\}$	variable
$M_{\varphi_\eta \psi_\omega i}$	penalty variable	variable
$\mu_{\varphi_\eta \psi_\omega i}$, $\nu_{\varphi_\eta \psi_\omega i}$ $p_{\varphi_\eta \psi_\omega i}$	binary variables for calculating the value of the penalty variables	variable

4.3 Distances of turning points

Two turning points are neighbours, if their horizontal or vertical coordinates are the same and there is no any turning point between these points. In the modified model, the horizontal or vertical distances of neighbours are used only. Otherwise the distances will be at least M.

4.3.1 The distance of two turning points with different indices

If the two turning points $(x_{\varphi_\eta}, y_{\psi_\omega})$ and $(x_{\varrho_\zeta}, y_{\vartheta_\xi})$ have not the same horizontal or vertical coordinates than the distance is equal with M, i.e. for all $\varphi_\eta \neq \varrho_\zeta$ and $\psi_\omega \neq \vartheta_\xi$:

$$d_{\varphi_\eta \psi_\omega \varrho_\zeta \vartheta_\xi} = M. \tag{47}$$

4.3.2 Vertical distance

The distance of the two turning points $(x_{\varphi_\eta}, y_{\psi_\omega})$ and $(x_{\varphi_\eta}, y_{\vartheta_\xi})$ is the following. For all φ_η and $\psi_\omega \neq \vartheta_\xi$:

$$d_{\varphi_\eta\psi_\omega\varphi_\eta\vartheta_\xi} = g_{\psi_\omega\vartheta_\xi} + h_{\psi_\omega\vartheta_\xi} + \sum_{\varrho_\zeta \neq \psi_\omega, \vartheta_\xi} V_{\psi_\omega\vartheta_\xi\varrho_\zeta} + \sum_i M_{\varphi_\eta\psi_\omega i} + \sum_i M_{\varphi_\eta\vartheta_\xi i}, \tag{48}$$

where $g+h$ is the vertical distance of the points. Notice, that describing of g and h is similar to $(11-16)$. $V_{\psi_\omega\vartheta_\xi\varrho_\zeta}$ is a penalty variable which value is equal with M, if y_{ϱ_ζ} is between y_{ψ_ω} and y_{ϑ_ξ}, i.e. the turning points are not neighbours; and it is equal with 0 otherwise. $M_{\varphi_\eta\psi_\omega i}$ $(M_{\varphi_\eta\vartheta_\xi i})$ is a penalty variable which value is equal with M, if point $(x_{\varphi_\eta}, y_{\psi_\omega})$ $((x_{\varphi_\eta}, y_{\vartheta_\xi}))$ is inner point of cell φ.

There exists the unique value $\gamma_{\psi_\omega\vartheta_\xi\varrho_\zeta}$ for which:

$$y_{\varrho_\zeta} = \gamma_{\psi_\omega\vartheta_\xi\varrho_\zeta} y_{\psi_\omega} + (1 - \gamma_{\psi_\omega\vartheta_\xi\varrho_\zeta}) y_{\vartheta_\xi}. \tag{49}$$

If y_{ϱ_ζ} is strictly between y_{ψ_ω} and y_{ϑ_ξ}, then the value $\gamma_{\psi_\omega\vartheta_\xi\varrho_\zeta}$ is strictly between 0 and 1. At this case for feasibility of the following two inequalities the values of the two binary variables need to be equal with 1 and otherwise need not.

$$1 - \gamma_{\psi_\omega\vartheta_\xi\varrho_\zeta} - u_{\psi_\omega\vartheta_\xi\varrho_\zeta} M \leq 0 \tag{50}$$

$$\gamma_{\psi_\omega\vartheta_\xi\varrho_\zeta} - v_{\psi_\omega\vartheta_\xi\varrho_\zeta} M \leq 0 \tag{51}$$

Hence, the value of $m_{\psi_\omega\vartheta_\xi\varrho_\zeta}$ is 1 if y_{ϱ_ζ} is strictly between y_{ψ_ω} and y_{ϑ_ξ}:

$$u_{\psi_\omega\vartheta_\xi\varrho_\zeta} + v_{\psi_\omega\vartheta_\xi\varrho_\zeta} \leq 1 + m_{\psi_\omega\vartheta_\xi\varrho_\zeta}. \tag{52}$$

Thus the penalty is determined by the following:

$$V_{\psi_\omega\vartheta_\xi\varrho_\zeta} = m_{\psi_\omega\vartheta_\xi\varrho_\zeta} M. \tag{53}$$

If y_{ϱ_ζ} is not strictly between y_{ψ_ω} and y_{ϑ_ξ}, then the value $\gamma_{\psi_\omega\vartheta_\xi\varrho_\zeta}$ is either at least 1 or at most 0. Hence, at least one of $u_{\psi_\omega\vartheta_\xi\varrho_\zeta}$ and $v_{\psi_\omega\vartheta_\xi\varrho_\zeta}$ can be 0. The objective function decides on the value of the variable in an optimal way.

The calculation of the inner point penalty for turning point $(x_{\varphi_\eta}, y_{\psi_\omega})$ and cell i is the following:

$$e_{\varphi_\eta\psi_\omega i} + f_{\varphi_\eta\psi_\omega i} - \frac{1 - z_i}{2} t_i - \frac{z_i}{2} s_i \geq -M \mu_{\varphi_\eta\psi_\omega i} \tag{54}$$

and

$$g_{\varphi_\eta\psi_\omega i} + h_{\varphi_\eta\psi_\omega i} - \frac{z_i}{2} t_i - \frac{1 - z_i}{2} s_i - \frac{z_i}{2} t_i \geq -M \nu_{\varphi_\eta\psi_\omega i}. \tag{55}$$

The definitions of e, f, g, and h is similar to $(11-16)$, again. From this

$$\mu_{\varphi_\eta\psi_\omega i} + \nu_{\varphi_\eta\psi_\omega i} \leq 1 + p_{\varphi_\eta\psi_\omega i}. \tag{56}$$

Thus the penalty is determined by the equation:

$$M_{\varphi_\eta\psi_\omega i} = p_{\varphi_\eta\psi_\omega i} M. \tag{57}$$

The calculation of inner point penalty for turning point $(x_{\varphi_\eta}, y_{\vartheta_\xi})$ and cell i is the same.

4.3.3 Horizontal distance

The distance of the two turning points $(x_{\varphi_\eta}, y_{\psi_\omega})$ and $(x_{\vartheta_\xi}, y_{\psi_\omega})$ is the following. For all $\varphi_\eta \neq \vartheta_\xi$ and ψ_ω:

$$d_{\varphi_\eta\psi_\omega\vartheta_\xi\psi_\omega} = e_{\varphi_\eta\vartheta_\xi} + f_{\varphi_\eta\vartheta_\xi} + \sum_{\varrho_\varsigma \neq \varphi_\eta, \vartheta_\xi} H_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} + \sum_i M_{\varphi_\eta\psi_\omega i} + \sum_i M_{\vartheta_\xi\psi_\omega i}, \tag{58}$$

where $e+f$ is the vertical distance of the points, describing of e and f is similar to $(11-16)$.

$H_{\varphi_\eta\vartheta_\xi\varrho_\varsigma}$ is a penalty variable which value is equal to M, if x_{ϱ_ς} is between x_{φ_η} and x_{ϑ_ξ}, i.e. the turning points are not neighbours; and it is equal with 0 otherwise. $M_{\varphi_\eta\psi_\omega i}$ (or $M_{\vartheta_\xi\psi_\omega i}$) is a penalty variable which value is equal with M, if point $(x_{\varphi_\eta}, y_{\psi_\omega})$ (or $(x_{\vartheta_\xi}, y_{\psi_\omega})$) is inner point of cell i.

There exists the unique value $\delta_{\varphi_\eta\vartheta_\xi\varrho_\varsigma}$ for which:

$$x_{\varrho_\varsigma} = \delta_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} x_{\varphi_\eta} + (1 - \delta_{\varphi_\eta\vartheta_\xi\varrho_\varsigma}) x_{\vartheta_\xi}. \tag{59}$$

If x_{ϱ_ς} is between x_{φ_η} and x_{ϑ_ξ}, then the value $\delta_{\varphi_\eta\vartheta_\xi\varrho_\varsigma}$ is between 0 and 1. At this case for feasibility of the following two inequalities the values of the two binary variables need to be equal to 1 and otherwise need not.

$$1 - \delta_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} - w_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} M \leq 0 \tag{60}$$

$$\delta_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} - z_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} M \leq 0 \tag{61}$$

Therefore in the following inequality the value of $o_{\varphi_\eta\vartheta_\xi\varrho_\varsigma}$ is 1 if x_{ϱ_ς} is between x_{φ_η} and x_{ϑ_ξ}:

$$w_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} + z_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} \leq 1 + o_{\varphi_\eta\vartheta_\xi\varrho_\varsigma}. \tag{62}$$

Thus the penalty is determined by the following:

$$H_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} = o_{\varphi_\eta\vartheta_\xi\varrho_\varsigma} M. \tag{63}$$

If x_{ϱ_ς} is not strictly between y_{φ_η} and y_{ϑ_ξ}, then the value $\gamma_{\psi_\omega\vartheta_\xi\varrho_\varsigma}$ is either at least 1 or at most 0. At least one of $w_{\varphi_\eta\vartheta_\xi\varrho_\varsigma}$ and $z_{\varphi_\eta\vartheta_\xi\varrho_\varsigma}$ can be 0. The objective function decides on the value of the variable in an optimal way.

Calculating of inner point penalty for turning point $(x_{\varphi_\eta}, y_{\psi_\omega})$ and cell i is similar to vertical case.

4.4 Minimal cost flow

The structure of the objective function is identical to the structure of the first model.

5 Conclusions

This paper contains two exact models for open field version of the rectangular layout problem. The open field version is the most general version. The expression "exact model" means that not only the contraints are correct but even the objective function is. The objective function was approximated in the earlier models by an l_1 type distance measuring the lengths of infeasible routes. The non-exactness of the previous models is also reflected by the fact that some known solutions obtained by meta-heuristics can be improved by obvious ways. In the new models the objective function contains the lengths of feasible routes of the vehicle which can move in the models in l_2 or l_1 way, repectively. To the best knowledge of the authors, these are the first exact model for the open field case. The only exact model so far is [Niroomand-Vizvári 2013] concerning to the closed loop case when the track of the vehicle is also a rectangle. The new models can be the starting points of the development of new meta-heuristic methods.

References

[Chae-Peters 2006] Chae, J. and Peters, B.A., A simulated annealing algorithm based on a closed loop layout for facility layout design in flexible manufacturing systems. Int. J. Prod. Res., 44, 2006, 2561-2572.

[Das 1993] Das, S.K., A facility layout method for feaible manufacturing systems, International Journal of Production Research, 31(1993), No. 2, 279-297.

[Kim and Kim 2000] Kim, J.-G. and Kim, Y.-D., Layout planning for facilities with fixed shapes and input and output points. Int. J. Prod. Res., 38, 2000, 46354653.

[Lasardo and Nazzal 2011] Lasardo, V., Nazzal, D., Design of a manufacturing facility layout with a closed loop conveyor with shortcuts using queuing theory and genetic algorithm. Proceedings of the 2011 Winter Simulation Conference, 2011, 1964-1975.

[Luggen 1991] Luggen, W.W., Flexible Manufacturing Cells and Systems, Prentice-Hall, Englewood Cliffs, 1991.

[Niroomand-Vizvári 2013] Mixed Integer Linear Programming Formulation of Closed Loop Layout with Exact Distances, Journal of the Chinese Institute of Industrial Engineers (to appear)

[Quantum computer] http://en.wikipedia.org/wiki/Quantum_computer

[Rajasekharan *et al.* 1998] Rajasekharan, M., Peters, B.A. and Yang, T., A genetic algorithm for facility layout design in flexible manufacturing systems. Int. J. Prod. Res., 36, 1998, 95-110.

[Samarghandi *et al.* 2010] Samarghandi, H., Taabayan, P., Firouzi Jahantigh, F., A practical swarm optimization for the single row facility problem. Computer & Industrial Engineering, 58, 2010, 529-534.

[Sherali *et al.* 2003] Sherali, H.D., Fraticelli, B.M.P., Meller, R.D., Enhanced Model Formulation for Optimal Layout, Operations Research, 51(2003), 629-644.

[Tavakkoli-Moghaddam and Panahi 2007] Tavakkoli-Moghaddam, R., Panahi, H., Solving a new mathematical model of a closed-loop layout problem with unequal-sized facilities by a genetic algorithm. Proceedings of the 2007 IEEE IEEM, 2007, 327-331.

[Ting, J.-H., Tanchoco 2001] Ting, J.-H., Tanchoco, J. M. A, Optimal bidirectional spine layout for overhead material handling systems. IEEE Transactions on Semiconductor Manufacturing, 14(1), 2001, 57-64.

[Tompkins *et al.* 1996] Tompkins, J.A., White, J.A., Bozer, Y.A., Frazelle, E.H., Tanchoco, J.M.A. and Trevino, J., Facilities Planning, 2nd ed., Wiley, New York, 1996.

[Tsai *et al.* 2011] Jung-Fa Tsai, Pei-Chun Wang, Ming-Hua Lin, An eggicient deterministic approach for rectangular packing problems, Optimzation, 2011, http://www.tandfonline.com/doi/abs/10.1080/02331934.2011.625029

[White 2011] White, Ch.,Breakthrough: The Secret to Making Processors 1,000 Times Faster, http://mashable.com/2011/09/09/breakthrough-the-secret-to-making-processors-1000-times-faster-video/