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Sufficient conditions for the existence of Nash equilibria in bimatrix games in terms of forbidden 2×2 subgames

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Abstract. In 1964 Shapley observed that a matrix has a saddle point in pure strategies whenever every its 2×2 submatrix has one. In contrast, a bimatrix game may have no pure strategy Nash equilibrium (NE) even when every 2×2 subgame has one. Nevertheless, Shapley's claim can be extended to bimatrix games as follows. We partition all 2×2 bimatrix games into fifteen classes $C = \{c_1, \dots, c_{15}\}$ depending on the preferences of two players. A subset $t \subseteq C$ is called a NE-theorem if a bimatrix game has a NE whenever it contains no subgame from t. We suggest a general method for getting all minimal (that is, strongest) NE-theorems based on the procedure of joint generation of transversal hypergraphs given by a special oracle. By this method we obtain all (six) strongest NE-theorems.

Key words: matrix and bimatrix games, saddle point, Nash equilibrium, Nash-solvability, dual hypergraphs, transversal hypergraphs, dualization
JEL Classification : C02, C62, C65, C72

1 Introduction

1.1 Basic concepts

1.1.1 Bimatrix games and Nash equilibria

Let I and J be finite sets of strategies of the row and column players. The pairs of strategies $(i,j) \in I \times J$ are called the *situations*. A *bimatrix game* $U = (a,b)$ is a pair of real-valued matrices $a : I \times J \to \mathbb{R}$, $b : I \times J \to \mathbb{R}$ with the common set of entries $I \times J$. Values $a(i,j)$ and $b(i,j)$ are interpreted as the utility functions (also called payoffs) of the row and column players in the situation (i,j). A situation $(i,j) \in I \times J$ is called a Nash equilibrium (NE) if

$$a(i',j) \leq a(i,j) \; \forall i' \in I \;\text{ and }\; b(i,j') \leq b(i,j) \; \forall j' \in J;$$

in other words, if no player can make a profit choosing a new strategy if the opponent keeps the old one.

In this paper we restrict the players by their pure strategies; the mixed ones are not mentioned, so we skip the adjective "pure", since referring to a NE we always mean one in pure strategies; in particular, it may not exist.

1.1.2 Games and configurations

Note that in the above definition of a NE the real values $a(i,j)$ and $b(i,j)$ are irrelevant; only the preferences of the players matter. Moreover, the preferences of a player between outcomes that differ in the choice the opponent are irrelevant too. To decide whether a situation $(i,j) \in I \times J$ is a NE in a game $U = (a,b)$, it is sufficient to know only two orderings: in the row i with respect to b and in the column j with respect to a.

Let us briefly recall the concept of ordering. Given a set Y and a mapping $P : Y^2 \to \{<, >, \sim\}$ that assigns one of these three symbols to each ordered pair $y, y' \in Y$, we say that y is *less or worse* than y' if $y < y'$, that y is *more or better* than y' if $y > y'$, and finally, that y and y' are *equivalent or they make a tie* if $y \sim y'$. Standardly, P is called an *ordering* if it is transitive and symmetric.

The set of orderings P_i for all rows $i \in I$ and P_j for all columns $j \in J$ is called a *configuration*. Each game $U = (a,b)$ naturally defines a unique configuration $P(U)$ as follows: the orderings P_i and P_j are introduced by the real values $b(i,*)$ and $a(*,j))$ for each row $i \in I$ and and each column $j \in J$, respectively.

Note that each configuration is realized by multiple games. Nevertheless, the set of all NE of a game U is uniquely defined by its configuration $P(U)$.

1.1.3 Subgames and subconfigurations, minimality and local minimality

Standardly, we define a subgame $U' = (a', b')$ of a game $U = (a,b)$ as a restriction a' and b' of the mappings a and b from $I \times J$ to $I' \times J'$, where $I' \subseteq I$ and $J' \subseteq J$ are arbitrary fixed subsets of strategies.

A property (family) p of games is called *monotone* if it satisfies the following implication: if p holds for a game U then it holds for any its subgame too.

Furthermore, we say that U is a *minimal* (respectively, *locally minimal*) game satisfying a monotone property p if p holds for U, but fails for any proper subgame of U (respectively, for any subgame obtained from U by deleting exactly one strategy, either from I or from J). Obviously, the minimality implies the local minimality.

Furthermore, we define a *subconfiguration* as the configuration of a subgame. Then, the above definitions of monotonicity and (local) minimality are naturally applicable to configurations, as well.

1.2 Generalizing Shapley's "2×2 Theorem" for saddle points to Nash equlibria; main goals of the paper

A bimatrix game U is called a *zero sum or matrix* game if $a(i,j) + b(i,j) = 0$ for every situation (i,j). In this case the game is well-defined by one of two matrices (say, by a) and a NE is called a *saddle point* (SP).

Shapley (1964) noticed that a matrix game has a SP whenever every its 2×2 submatrix has one. Then, obviously, any submatrix has a SP too. In other words, all minimal SP-free matrices are of size 2×2. Boros et al. (2009) strengthened this result showing that any locally minimal SP-free matrix is of size 2×2, as well; in other words, any SP-free matrix of a larger size has a row or column whose elimination results in a SP-free submatrix.

Other generalizations of Shapley's theorem were obtained by Gurvich and Libkin (1990) and by Kukushkin (2007).

In contrast, criteria of the absence (or existence) of a NE are not that simple. The "naive generalization" of Shapley's claim to bimatrix games fails: a 3×3 game might have no NE even if every its 2×2 subgame has one; see Example 1 in Gurvich and Libkin (1990) or in Boros et al. (2009).

Moreover, for each $n \geq 3$ a $n \times n$ bimatrix game might have no NE even if every its subgame has one; Boros et al. (2009), see Lemma 1 and Theorem 2 below. In other words, a minimal NE-free game may be arbitrarily large.

The goal of this paper is to obtain the weakest conditions sufficient for the existence of a NE in a bimatrix game, in terms of forbidden 2×2 subgames. In a sense, our analysis will be complete, that is, we obtain all such conditions. The completeness is of independent interest; it is based on the theory of transversal hypergraphs, which is discussed below in the introduction. Technically, our approach is based on a characterization of the locally minimal NE-free games recently obtained by Boros at al. (2009); see Section 3.

Note that many other types of conditions sufficient for the existence of NE in bimatrix games are known in the literature; see Section 2 for more details.

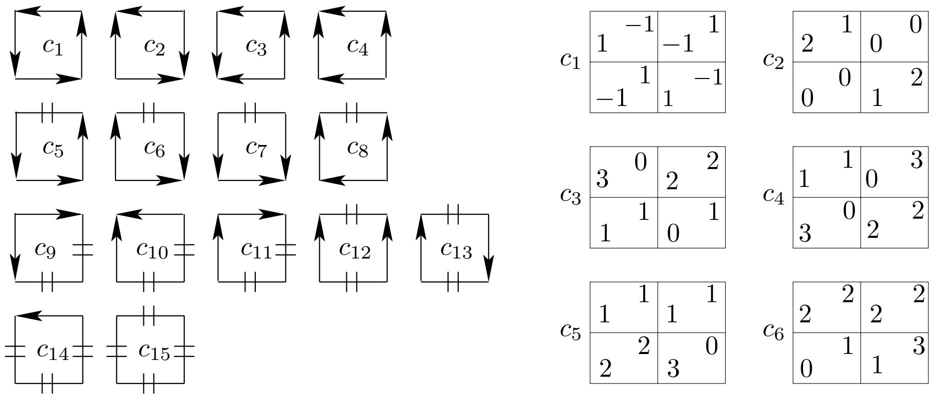


Figure 1: Fifteen 2-squares. In Figures 1-7 we use the following notation:
Configurations are represented by planar grids whose nodes correspond to the situations of $I \times J$. A line with two dashes between two nodes means that the corresponding two situations make a tie, while an arrow from one node to another means that the second situation is better than the first one for the player choosing among them. Games are represented by tables whose rows and columns are the strategies I and J, respectively. Thus, situations $(i, j) \in I \times J$ are represented by cells of these tables, where $a(i, j)$ and $b(i, j)$ are located respectively in the bottom-left and top-right corners of the cell (i, j).

1.3 An alphabet formed by 2-squares

For brevity, we will refer to a 2×2 configuration as a 2-*square*. Up to the transposition and permutations of the rows and columns, there exist only fifteen different 2-squares. They are listed in Figure 1; for the first six of them the examples of corresponding bimatrix games are given too.

Remark 1 *If a configuration is obtained from another by permutations of the rows and columns, then they are naturally identified, since a priori the strategies of a game are not enumerated. In this paper, we identify also the transposed* 2*-squares. In contrast, this assumption is essential; distinguishing them would change our main results. Note that seven* 2*-squares* $c_1, c_2, c_4, c_9, c_{10}, c_{11}$, *and* c_{15} *are symmetric, but the remaining eight are not and would be duplicated.*

Four 2-squares c_1, c_2, c_3, c_4 have no ties; another four, c_5, c_6, c_7, c_8, and the next five, c_9, c_{10}, c_{11}, c_{12}, c_{13}, have, respectively, one and two ties; finally, c_{14} and c_{15} have 3 and 4 ties, respectively.

The considered fifteen 2-squares have $0, 2, 1, 1, 1, 2, 1, 2, 3, 2, 2, 2, 2, 3$, and 4 NE, respectively. Thus, only c_1 has none. Shapley's theorem asserts that every c_1-free *zero-sum* game has a SP.

Note also that the first six configurations c_1 - c_6 are frequent in the literature. For example, the non-zero-sum bimatrix games realizing c_2 and c_4 may represent classical "battle of the sexes" and "prisoner's dilemma"; c_5 and c_6 illustrate concepts of the "promise" and "threat".

1.4 Complete analysis based on duality

1.4.1 Transversal or dual hypergraphs

Let C be a finite set whose elements we denote by $c \in C$. A *hypergraph* H (*on the ground set* C) is a family of subsets $h \subseteq C$ called the *hyperedges* (or just *edges*, in short) of H. A hypergraph H is called a Sperner hypergraphs if the containment $h \subseteq h'$ holds for no two distinct edges of H. Given two hypergraphs T and E on the common ground set C, they are called *transversal* or *dual* if the following properties hold:

- (i) $t \cap e \neq \emptyset$ for every $t \in T$ and $e \in E$;
- (ii) for every subset $t' \subseteq C$ such that $t' \cap e \neq \emptyset$ for each $e \in E$ there exists an edge $t \in T$ such that $t \subseteq t'$;
- (iii) for every subset $e' \subseteq C$ such that $e' \cap t \neq \emptyset$ for all $t \in T$ there exists an edge $e \in E$ such that $e \subseteq e'$.

Here (i) means that the edges of E and T are transversal, while (ii) and (iii) mean that T contains all minimal transversals to E and E contains all minimal transversals to T, respectively. It is well-known, and easily seen, that (ii) and (iii) become equivalent when (i) holds. Although for a given hypergraph H there exist multiple dual hypergraphs, yet, only one of them, which we will denote by H^d, is a Sperner hypergraph. Thus, within the family of Sperner hypergraphs duality is well-defined; moreover, it is an involution, that is, equations $T = E^d$ and $E = T^d$ are equivalent, implying $T^{dd} = T$ and $E^{dd} = E$. It is also easy to verify that dual Sperner hypergraphs have the same ground sets. For example, the following two hypergraphs are dual:

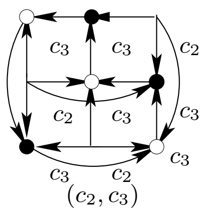
$$E' = \{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6)\}, \tag{1}$$

$$T' = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_6, c_9), (c_1, c_3, c_9)\}; \tag{2}$$

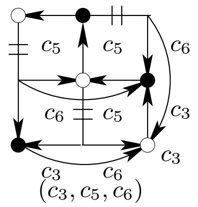
as well as the following two:

$$E = \{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6), (c_2, c_4, c_5, c_6)\}, \tag{3}$$

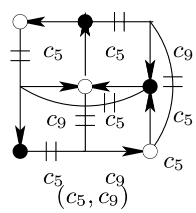
$$T = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_3, c_9), (c_1, c_2, c_6, c_9), (c_1, c_3, c_4, c_9), (c_1, c_3, c_6, c_9)\}. \tag{4}$$



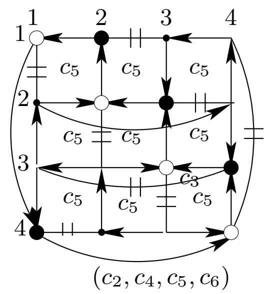
1, 2	2, 1	0, 0
0, 0	1, 2	2, 1
2, 1	0, 0	1, 2



0, 1	1, 0	0, 0
0, 0	0, 2	2, 1
1, 1	0, 0	1, 2



0, 1	1, 0	0, 0
0, 0	0, 1	1, 0
1, 0	0, 0	0, 1



	12	13	14	23	24	34
12	c_5	c_6	c_5	c_5	c_4	c_5
13	c_4	c_2	c_6	c_6	c_2	c_4
14	c_5	c_4	c_5	c_5	c_6	c_5
23	c_5	c_4	c_5	c_5	c_6	c_5
24	c_6	c_2	c_4	c_4	c_2	c_6
34	c_5	c_6	c_5	c_5	c_4	c_5

1, 2	2, 1	0, 1	1, 0
1, 0	1, 2	2, 1	0, 1
0, 1	1, 0	1, 2	2, 1
2, 1	0, 1	1, 0	1, 2

Figure 2: NE-examples: four NE-free games and the corresponding configurations with the families of 2-cycles $(c_2, c_3), (c_3, c_5, c_6), (c_5, c_9)$ and $(c_2, c_4, c_5, c_6)\}$. The family of 2-cycles of any NE-free game contains one of these four sets or c_1.

In Figures 2-7 we keep the same legend as for Figure 1. Note, however, that two nodes may be not connected when the relation between the corresponding situations follows from transitivity.

Note also that each configuration and game in Figures 2-7 is a (locally) minimal NE-free one. Hence, by Theorem 2 (see Section 3), it is of size $n \times n$ for some $n \geq 2$ and contains a strong improvement cycle C_n. The corresponding $2n$ situations are denoted by n white and n black small discs.

1.4.2 Hypergraphs of examples and theorems

Given the alphaber $C = \{c_1, \dots, c_{15}\}$, we call a subset $e \subseteq C$ a *NE-example* if there is a NE-free configuration P that has all 2-squares from P and no others; respectively, a subset $t \subseteq C$ is called a *NE-theorem* if a configuration has a NE whenever it has no 2-square from t. Obviously, $e \cap t \neq \emptyset$ for any NE-example e and NE-theorem t, since otherwise e is a counter-example to t. Moreover, it is both well-known and easily seen that the hypergraphs of all inclusion-minimal (that is, strongest) NE-examples E_{NE} and NE-theorems T_{NE} are transversal.

For example, consider c_1 and four configurations of Figure 2. It is easy to verify that all five are minimal NE-free configurations, chosen because they contain few types of 2-squares; the corresponding sets are given in Figure 2; they form the hypergraph E defined by (3). Figure 2 shows that each edge of E is a NE-example.

Consider the dual hypergraph T given by (4). We will prove that every edge $t \in T$ is a NE-theorem, thus, showing that the "research is complete", that is, E_{NE} and T_{NE} are the hypergraphs of all strongest NE-examples and theorems.

Given a family of NE-examples E', the dual hypergraph T' should be viewed as a hypergraph of "conjectures" rather than "theorems". Indeed, some inclusion-minimal examples might be missing in E' and some examples of E' might be reducible. In this case some conjectures from the dual hypergraph $T' = E'^d$ will fail, being too strong. For instance, let us consider E' given by (1) in which the NE-example (c_2, c_4, c_5, c_6) is missing. (It is not that easy to obtain a 4×4 example without computer.) Respectively, the conjecture (c_1, c_3, c_9) appears in $T' = E'^d$. This conjecture fails, being too strong. So, in $T = T_{NE}$ it is replaced by three weaker (but correct) NE-theorems (c_1, c_3, c_9, c_2), (c_1, c_3, c_9, c_4), and (c_1, c_3, c_9, c_6).

In general, to guarantee that the analysis of a given property in terms of a given set of attributes is complete, we have to

- (i) construct some hypergraphs E' of examples and T' of conjectures,
- (ii) verify their duality, $T' = E'^d$, and
- (iii) to prove all conjectures.

Part (ii) is purely technical, unlike (i) and (iii). If it seems too difficult to prove a conjecture, one should look for new examples, etc. Several such iterations may be required for a difficult problem.

1.4.3 Joint generation of examples and theorems

This approach can be applied not only to NE-free bimatrix games.

In general, given a set of objects Q (in our case, configurations), list C of subsets (properties) $Q_c \subseteq Q$, $c \in C$ (in our case, c-free configurations), the target subset $Q_0 \subseteq Q$ (configurations that have a NE), we introduce a pair of hypergraphs $E = E(Q, Q_0, C)$ and $T = T(Q, Q_0, C)$ (examples and theorems) defined on the ground set C as follows:

(i) every set of properties assigned to an edge $t \in T$ (a theorem) implies Q_0, that is, $q \in Q_0$ whenever q satisfies all properties of t, or in other words, $\cap_{c \in t} Q_c \subseteq Q_0$; in contrast,

(ii) each set of properties corresponding to the complement $C \setminus e$ of an edge $e \in E$ (an example) does not imply Q_0, i.e., there is an object $q \in Q \setminus Q_0$ satisfying all properties of $C \setminus e$, or in other words, $\cap_{c \notin e} Q_c \not\subseteq Q_0$.

If hypergraphs E and T are dual then we can say that "our understanding of Q_0 in terms of C is perfect", that is, every new example $e' \subset C$ (theorem $t' \subseteq C$) is a superset of some "known" example $e \in E$ (theorem $t \in T$).

Without loss of generality, we can assume that examples of $e \in E$ and theorems $t \in T$ are inclusion-wise minimal in C; or in other words, both E and T are Sperner hypergraphs.

Given Q, Q_0 and C, we try to generate hypergraphs E and T jointly [13]. Of course, the oracle may be a problem: Given a subset $C' \subseteq C$, it may be difficult to decide whether C'

is a theorem (i.e., if $q \in Q_0$ whenever q satisfies all properties of C') or an example (i.e., if there is a $q \in Q \setminus Q_0$ satisfying all properties of $C \setminus C'$). However, the stopping criterion, $E^d = T$, is well-defined and, moreover, it can be verified in quasi-polynomial time [8].

1.4.4 Several remarks

Remark 2 *Containment $\cap_{c \in t} Q_c \subseteq Q_0$ may be strict. In other words, theorem t gives sufficient but not always necessary conditions for $q \in Q_0$. We can also say that theorems $t \in T$ give all optimal "inscribed approximations" of $Q_0 \subseteq Q$ in terms of C. One can easily replace the above containments by equalities, just replacing Q_c by $Q_c \cap Q_0$ for all $c \in C$.*

Remark 3 *Each NE-theorem actually implies the existence of a NE in any subgame, which is in general a stronger property than the existence of a NE in the original game.*

The anonymous reviewer noticed that this is a way to understand why the containment $\cap_{c \in t} Q_c \subseteq Q_0$ of the previous remark may be strict, and also pointed out that such total Nash-solvability in turn implies the weak acyclicity in the two-player games (Takahashi and Yamamori (2002)) and the existence of piecewise linear fictitious play processes (Hofbauer (1995a, 1995b) and Berger (2007)).

Remark 4 *Gvishiani and Gurvich (1983) illustrated this approach by a simple model problem in which Q is the set of 4-gons, Q_0 is the set of squares, and C is a set of six properties of a 4-gon. Two dual hypergraphs of all strongest theorems T and examples E were constructed. Chvatal et al. (1990) applied the same approach to a more serious problem related to families of Berge's graphs.*

1.5 Main results

We will prove the following criteria of Nash-solvability.

Theorem 1 *A bimatrix game has a NE whenever the set of its 2-squares contains none of the NE-examples:*

$\{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6), (c_2, c_4, c_5, c_6)\}$.

Conversely, a game can be NE-free if the set of its 2-squares contains a NE-example.

This statement can be reformulated in other words as follows. A bimatrix game has a NE whenever the set of its 2-squares is disjoint from at least one of the NE-theorems:

$\{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_3, c_9), (c_1, c_2, c_6, c_9), (c_1, c_3, c_4, c_9), (c_1, c_3, c_6, c_9)\}$.

Conversely, a game can be NE-free if it meets all these sets.

Two statements of Theorem 1 are equivalent, because the hypergraphs E_{NE} of the NE-examples and T_{NE} of the NE-theorems are transversal, $E^d_{NE} = T_{NE}$. In fact, we will prove slightly stronger results; see Section 4.

2 Related results on Nash-solvability in terms of forbidden 2×2 subgames

The next observation is due to the anonymous reviewer. Let us forbid any 2×2 configuration in which two "vertical sides" are oppositely directed. Then, the rows can be naturally ordered such that the strategy $i \in I$ is dominated by $i' \in I$ whenever $i < i'$. Obviously, a NE exists in this case. In the alphabet C the corresponding forbidden configurations are $(c_1, c_2, c_3, c_5, c_6, c_{13})$. They do form a theorem, but not a strongest one, since (c_1, c_2, c_5) and (c_1, c_3, c_5) are already theorems.

Any two transposed 2×2 configurations are identified in the considered alphabet C. As an alternative, we could distinguish them; see Remark 1. Then, the alphabet becomes larger: the 2-squares $c_1, c_2, c_4, c_9, c_{10}, c_{11}$, and c_{15} remains unchanged, due to their symmetry, but the remaining 8 would be duplicated. The families of theorems and examples would change too. In particular, the above theorem, as well as "the transposed one", might become strongest.

Monderer and Shapley (1996, Theorem 2.8) showed that whether a bimatrix game admits an exact potential (which also ensures the existence of a NE) is determined by its 2×2 subgames too. Yet, this theorem takes into account the cardinalities, while we restrict ourselves by the ordinal preferences.

A fruitful approach to Nash-solvability is based on separation of the structure of the game and payoffs. A bimatrix game (g, u) is defined as a *game form* $g : I \times J \to C$ and a pair of *utility functions* $u = (a, b)$, where $a : C \to \mathbb{R}$, $b : C \to \mathbb{R}$, and $C = \{c_1, \ldots, c_p\}$ is a set of outcomes. The rows and columns of a game form g naturally define a pair of hypergraphs $H_r(g)$ and $H_c(g)$ on the same ground set C. A game form g is called *tight* if these two hypergraphs are transversal, $H_r(g) = H_c^d(g)$. Notice that property (i) of Section 1.4.1 holds for $H_r(g)$ and $H_c(g)$ automatically and, hence, (ii) and (iii) are equivalent.

The following three properties of a game form g are equivalent with tightness: the game (g, u) has a NE for

- (j) any $u = (a, b)$;
- (jj) any zero-sum u, i.e., such that $a(c) + b(c) = 0$ for each $c \in C$;
- (jjj) any zero-sum u, in which a and b take only values ± 1.

The equivalence of (jj), (jjj), and tightness was proven by Edmonds and Fulkerson (1970) and independently by Gurvich (1973); then, (j) was added by Gurvich (1975, 1988). These results demonstrate one more application of the transversal hypergraphs in game theory.

The property of tightness was strengthened in terms of forbidden 2-squares as follows: g is called *totally tight* if every its 2×2 subform has a constant line, row or column. Kukushkin (2007) and Boros et al. (2010) proved that a game (g, u) has no improvement cycle (which also ensures the existence of a NE) for any u if and only if g is totally tight.

Among our fifteen 2-squares all but the first four are totally tight. Nevertheless (c_1, c_2, c_3, c_4) is not a theorem, since (c_5, c_9) is an example. We leave the explanation to the careful reader.

Finally, let us remark that Nash-solvability of bimatrix games in pure strategies is an important topic of the non-cooperative game theory and many types of sufficient conditions, not in terms of forbidden 2×2 subgames, are known. For example, a NE always exists in the compact convex games, Debreu (1952), Glicksberg (1952), and Fan (1952); in potential games, Monderer and Shapley (1996); in submodular games, Topkis (1979); etc.

3 Locally minimal NE-free bimatrix games

3.1 On the (locally) minimal saddle point free matrix games

According to Shapley's theorem, all minimal SP-free matrix games are of size 2×2. Boros et al. (2009) strengthened this result showing that the same holds not only for all minimal, but also for all locally minimal SP-free matrix games.

It is well-known and obvious that a 2×2 matrix has no saddle point if and only if both numbers of one of its diagonals are strictly greater than both numbers of the other. This observation and Shapley's theorem result in a very simple criterion for the existence of saddle points in matrix games.

In contrast, locally minimal NE-free bimatrix game may be arbitrary large. Yet, they also admit the following explicit and simple characterization.

3.2 Strong improvement cycles

Given an integer $n \geq 2$, we will say that an $n \times n$ bimatrix game $U = (a, b)$, has a *strong improvement n-cycle C_n* if the strategies of each player can be numbered by $\{1, \ldots, n\}$ such that

- (a) for each situation $(1, 2), (2, 3), \ldots, (n-1, n), (n, 1))$ the value of a is the unique largest in the corresponding column, while the value of b is the second largest (not necessarily unique) in the corresponding row;

- (b) for every situation $(1, 1), (2, 2), \ldots, (n-1, n-1), (n, n)$ the value of b is the unique largest in the corresponding row, while the value of a is the second largest (not necessarily unique) in the corresponding column.

Formally,

$$\begin{aligned} a(i-1, i) &> a(i, i) \geq a(j, i) \quad \text{for all } j \neq i-1; & (5) \\ b(i, i) &> b(i, i+1) \geq b(i, j) \quad \text{for all } j \neq i. & (6) \end{aligned}$$

Standardly, we set $i + 1 = 1$ for $i = n$ and $i - 1 = n$ for $i = 1$.

Obviously, the $2n$ situations of (a) and (b) define n best responses of the column player and n best responses of the row player, respectively. In particular, there is a unique best response to each strategy. Hence, a game may have at most one strong improvement cycle.

Many examples can be found in Figures 2-7, where the above $2n$ situations are denoted by n black and n white disks for the cases (a) and (b), respectively.

3.3 Criteria of the local minimality for NE-free games

Lemma 1 *(Boros et al. (2009)) Every $n \times n$ game U with a strong improvement cycle C_n is a locally minimal NE-free one, i.e., U has no NE, while each subgame obtained from U by eliminating either one row or one column has a NE.*

For completeness, we repeat here the proof, seeing it is immediate.

Game U is NE-free, since the n best responses of the column player and n best responses of the row player (given by (a) and (b), respectively) are disjoint.

Yet, if we eliminate the first column, $(1, 2)$ becomes a NE in the reduced game. Indeed, it becomes a best response in its row, after $(1, 1)$ is eliminated, and it remains the (unique) best response in its column. Due to symmetry, the same arguments work if we replace the first column by any column or row. □

Interestingly, the inverse, in a sense, statement holds too.

Theorem 2 *(Boros et al. (2009)) Each locally minimal NE-free game contains a locally minimal NE-free square subgame with a strong improvement cycle.* □

3.4 Comments to Theorem 2

This statement will be sufficient for our purposes but, in fact, the following stronger claims were proven in Boros et al. (2009).

Let in Theorem 2 the game U be of size $m \times \ell$ and the subgame U' be of size $n \times n$ such that $2 \leq n \leq \min(m, \ell)$. First, we may claim that for the strong improvement cycle C_n in U', the inequalities of (a) and (b) of Section 3.2 hold not only for the situations of U' but for all situations of U as well.

Secondly, if we eliminate all rows and columns of U' from U, the remaining subgame U'' will be a locally minimal NE-free one or empty.

Applied successively, the last claim results in the following decomposition. Each locally minimal NE-free bimatrix game U is square (of size $n \times n$) and for some integer $k \geq 1$ its rows and columns can be partitioned into k subsets $I = \cup_{\alpha=1}^{k} I_\alpha$ and $J = \cup_{\alpha=1}^{k} J_\alpha$ (whose sizes satisfy the equalities $|I_\alpha| = |J_\alpha| = n_\alpha \geq 2$ and $\sum_{\alpha=1}^{k} n_\alpha = n$) such that for each $\alpha \in [k] = \{1, \ldots, k\}$ the "diagonal" subgame defined by $I_\alpha \times J_\alpha$ has a strong improvement cycle C_{n_α}. Again, we may claim that for this cycle, the inequalities of (a) and (b) of Section 3.2 hold not only for the situations of $I_\alpha \times J_\alpha$ but for all situations of $I \times J$, as well.

Obviously, the above decomposition is unique. It is also clear that the inverse statement holds too: any bimatrix game that has the above structure is a locally minimal NE-free one. The proof is similar to the proof of Lemma 1.

In a natural way, Theorem 2 can be reformulated for the configurations. In particular, it implies that (locally) minimal NE-free games and configurations can be arbitrary large.

3.5 On the minimal NE-free games and configurations

Clearly, a minimal NE-free bimatrix game or configuration is locally minimal, but not vice versa. It is also clear that for a minimal one, the parameter k of the previous subsection must be equal to 1. However, this condition is only necessary, but not sufficient for the minimality, because a NE-free game U (configuration P) may have a NE-free subgame U' (subconfiguration P') disjoint from the (unique) strong improvement cycle of U (of P).

For example, every game or configurations in Figures 2-7 is a minimal NE-free one. Although, according to Boros et al. (2009), it seems difficult to characterize or recognize the minimal NE-free games and configurations, yet, the characterization of the *locally* minimal ones given by Theorem 2 will be sufficient for our purposes.

4 Strengthening Theorem 1

We have to prove all six NE-theorems $t \in T_{NE}$. In fact, we will get stronger results. Although, in a sense, a NE-theorem t cannot be strengthened (indeed, t' is not a NE-theorem whenever $t' \subset t \in T_{NE}$ and the containment $t' \subset t$ is strict), but still, we can obtain stronger statements in slightly different terms.

For any subset $e \subseteq C$ the class of e-free configurations (games) is hereditary: if a configuration (game) is e-free then, obviously, every its subconfiguration (subgame) is e-free too. Hence, we can restrict ourselves by the locally minimal NE-free examples. by Theorem 2, each such configuration is of size $n \times n$ for some $n \geq 2$, and it contains a strong improvement cycle C_n.

Let us consider three NE-theorems (c_1, c_2, c_5), (c_1, c_3, c_5), and (c_1, c_2, c_6, c_9). Formally, since the 2-square c_1 itself has no NE, it must be eliminated. Yet, in a sense, it is the only exception. More precisely, we can strengthen the above three NE-theorems as follows.

Theorem 3 *The set of types of 2-squares of any locally minimal NE-free configuration, except c_1, meets (c_2, c_5), (c_3, c_5), and (c_2, c_6, c_9).*

Furthermore, the remaining three theorems (c_1, c_3, c_9, c_2), (c_1, c_3, c_9, c_4), and (c_1, c_3, c_9, c_6) can be strengthened too. Namely, we will characterize explicitly all locally minimal NE-free configurations that are also c_3- and c_9-free. This family of configurations is sparse but still infinite. We will obtain the following result. Let $C(P) \subseteq C$ denote the set of all types of 2-squares of a configuration P; furthermore, let us set $C' = \{c_1, c_2, c_4, c_5, c_6, c_7, c_8, c_{13}\}$ and $C'' = C' \cup \{c_{12}\}$.

Theorem 4 *Let P be a locally minimal NE-free $n \times n$ configuration that is also c_3- and c_9-free. Then*

- *(i) n is even;*
- *(ii) if $n = 2$ then P is c_1;*
- *(iii) if $n = 4$ then P is a unique (c_2, c_4, c_5, c_6)-configuration in Figure 2;*
- *(iv) if $n = 6$ then $C(P) = C'$;*
- *(v) if $n = 8$ then $C' \subseteq C(P) \subseteq C''$ and there is a P with $C(P) = C'$;*
- *(vi) finally, if $n \geq 10$ then $C(P) = C''$.*

Obviously, this statement implies the remaining three NE-theorems: (c_1, c_3, c_9, c_2), (c_1, c_3, c_9, c_4), and (c_1, c_3, c_9, c_6). Thus, Theorems 3 and 4 imply Theorem 1.

5 Proofs of Theorems 3 and 4

5.1 Preliminaries

By Theorem 2, we can restrict ourselves to a locally minimal NE-free configuration P of size $n \times n$ (for some $n \geq 2$) with a strong improvement cycle C_n. Without loss of generality, we assume that n strategies of I and J are enumerated by $\{1, \ldots, n\}$ and (5) and (6) hold for any game $U = (a, b)$ corresponding to P.

Obviously, for $n = 2$, the 2-square c_1 is a unique NE-free configuration, which is obviously a strong 2-cycle. Hence, we can assume that $n \geq 3$. Additionally, we assume that P is t-free and consider successively the following subsets t : (c_2, c_5), (c_3, c_5), (c_2, c_6, c_9), and (c_3, c_9). Theorem 3 will follow, since in the first three cases we get a contradiction. For $t = (c_3, c_9)$ we will characterize the corresponding configurations explicitly, thus, proving Theorem 4.

5.2 Locally minimal NE-free configurations that are also c_2- and c_5-free

Consider C_n in Figure 3 (where $n = 7$). By (6), $b(i,i) > b(i,j)$ whenever $j \neq i$; in particular, $b(i,i) > b(i,i-1)$ for $i \in [n] = \{1, \ldots, n\}$; standardly, we set $i - 1 = n$ for $i = 1$. Similarly, $a(i,i) \geq a(j,i)$ whenever $j \neq i - 1$; in particular, $a(i,i) \geq a(i+1,i)$ for $i \in [n] = \{1, \ldots, n\}$, where standardly, we set $i + 1 = 1$ for $i = n$. Moreover, the latter n inequalities are also strict, since otherwise c_5 would appear.

By similar arguments we show that $b(i,i+1) > b(i,i+2)$ and $a(i,i+1) > a(i-1,i+1)$ for $i = 1, \ldots, n-1$; see Figure 3.

Next, let us notice that $a(i,i) = a(i-2,i)$ for $i = 2, \ldots n$. Indeed, $a(i,i) \geq a(i-2,i)$, since C_n is a strong cycle, and c_2 would appear in case $a(i,i) > a(i-2,i)$.

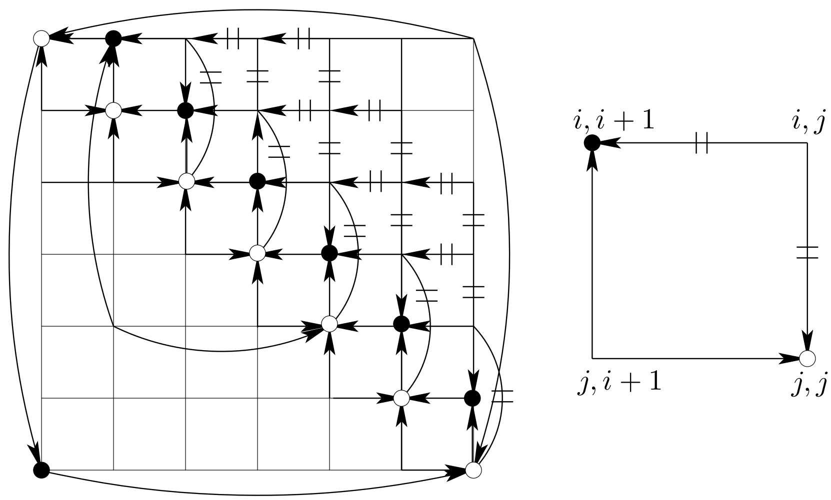
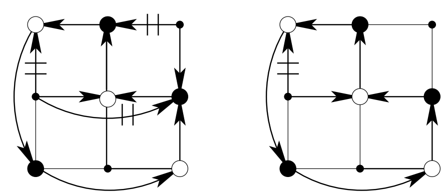


Figure 3: Except c_1, there is no locally minimal NE-free and also c_2- and c_5-free configuration. In Figures 3-7 we keep the legend of Figures 1 and 2. In addition, an arrow with two dashes from one situation to another means that either these situations make a tie, or one is better than the other for the player choosing among them.



(a) (b)

Figure 4: Except c_1, there is no locally minimal NE-free configuration that is also (a) c_2-, c_6-, and c_9-free or (b) c_3- and c_5-free.

Furthermore, $b(i, i+2) \geq b(i, i+3)$ for $i = 1, \ldots, n-3$, since otherwise $(i, i+2)$, $(i, i+3)$, $(i+2, i+2)$, $(i+2, i+3)$ would form a c_5.

Next, let us notice that $a(i, i+3) = a(i+1, i+3)$ for $i = 1, \ldots, n-3$. Indeed, $a(i, i+3) \leq a(i+3, i+3) = a(i+1, i+3)$, and if $a(i, i+3) < a(i+1, i+3)$ then $(i, i+1)$, $(i, i+3)$, $(i+3, i+1)$, $(i+3, i+3)$ would form a c_2, by (5) and (6).

Similarly, by induction on j, we show that $b(i, i+j) \geq b(i, i+j+1)$ and $a(i, i+j) = a(i+1, i+j)$ for $1 \leq i \leq n-3$ and $2 \leq i+j \leq n-1$.

In particular, $a(n, n) = a(n-2, n) = a(n-3, n) = \ldots = a(2, n) = a(1, n)$ in contradiction with the strict inequality $a(n, n) > a(1, n)$ obtained before. □

5.3 Locally minimal NE-free configurations that are also c_2-, c_6-, and c_9-free or c_3- and c_5-free

These two cases are easy. Consider C_n in Figures 4 (a) and (b), with $n = 3$, corresponding respectively to the two cases in question: (c_2, c_6, c_9) and (c_3, c_5). By definition, in both cases $b(2,2) > b(2,1)$ and $a(1,1) \geq a(2,1)$. In case (b) we already have a contradiction, since the obtained four situations form c_3 or c_5.

In case (a) we have to proceed a little further. Clearly, $b(2,3) \geq b(2,1)$, $b(1,2) \geq b(1,3)$, $a(2,3) > a(1,3)$, and again we get a contradiction, since four situations $(1,1)$, $(1,3)$, $(2,1)$, $(2,3)$ form c_9 if two equalities hold, c_6 if exactly one, and c_2 if none. □

5.4 Locally minimal NE-free configurations that are also c_3- and c_9-free

Consider C_n in Figure 5 (with $n = 8$). By (5) and (6), for all i we have:

$$\begin{aligned} b(i,i) > b(i,i+1),\ a(i,i+1) &> a(i+1,i+1), \\ b(i+1,i+1) \geq b(i+1,i),\ a(i+1,i+2) &\geq a(i,i+2). \end{aligned}$$

Then, by induction we prove that

$$a(i,i) = a(i+1,i) \text{ and } b(i,i+1) = b(i,i+2), \tag{7}$$

since otherwise c_3 appears, while

$$a(i,i) > a(i-2,i) \text{ and } b(i,i+1) > b(i,i-1), \tag{8}$$

since otherwise c_9 appears; see Figure 5.

We prove all four claims in (7) and (8) by induction introducing situations in the following (alternating diagonal) order:

$(2,1)$, $(1,3)$, $\ldots$, $(i,i-1)$, $(i-1,i+1)$, $\ldots$, $(n,n-1)$, $(n-1,1)$, $(1,n)$, $(n,2)$.

Furthermore, $b(1,1) = b(2,1) \geq b(4,1)$ unless $n < 5$. Also, $b(2,1) = b(4,1)$, since otherwise the four situations $(2,1)$, $(4,1)$, $(2,4)$, and $(4,4)$ form c_3.

Similarly, we prove that $b(1,3) = b(1,5)$ unless $n < 5$.

Then, let us recall that $b(4,5) \geq b(4,1)$ and conclude that $b(4,5) > b(4,1)$, since otherwise situations $(1,1)$, $(4,1)$, $(1,5)$, and $(4,5)$ form c_9.

Similarly, from (5), (6), (7), and (8) we derive that $a(4,1) = a(6,1)$. Indeed, we have $a(4,1) = a(2,1) = a(1,1) \geq a(6,1)$, $b(4,6) = b(4,5) > b(4,1)$, $a(6,6) > a(4,6)$, and $b(6,6) > b(6,1)$. Hence c_3 appears whenever $a(4,1) > a(6,1)$, implying that $a(4,1) = a(6,1)$.

From this we derive further that $b(6,7) > b(6,1)$. Indeed, $a(1,1) = a(1,2) = a(1,4) = a(1,6)$, $b(1,1) > b(1,2) = b(1,3) = b(1,5) = b(1,7)$, $a(6,7) > a(1,7)$, and $b(6,7) \geq b(6,1)$. Hence c_9 appears whenever $b(6,7) = b(6,1)$, implying that $b(6,7) > b(6,1)$.

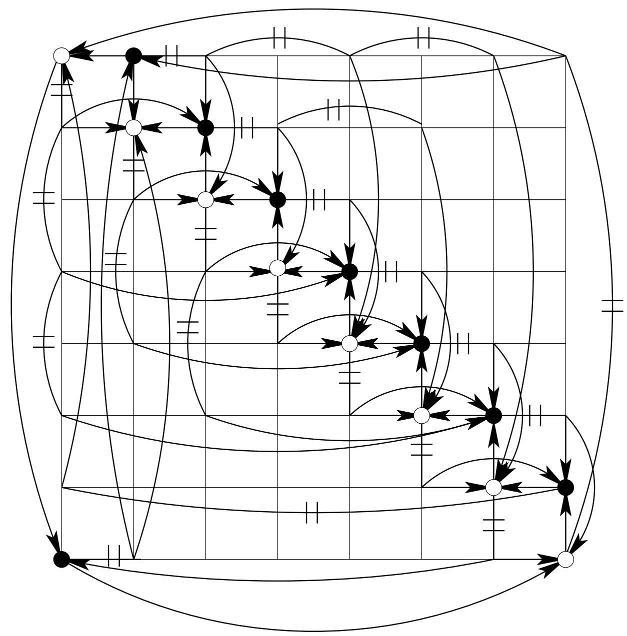


Figure 5: Locally minimal NE-free and (c_3, c_9)-free configurations.

In general, we prove similarly by induction that

$$a(i,i) = a(i+1,i) = a(i+3,i) = \ldots = a(i+2j-1,i), \text{ while } a(i-1,i) > a(i,i) > a(i-2j,i); \tag{9}$$

$$b(i,i+1) = b(i,i+2) = b(i,i+4) = \ldots = b(i,i+2j), \text{ while } b(i,i) > b(i,i+1) > b(i,i-2j+1). \tag{10}$$

In both cases each sum is taken $\mod (n)$ (in particular, $n \equiv 0$) and $1 \leq j < n/2$ (in particular, j takes values $1, 2$, and 3 for $n = 7$ and $n = 8$).

If $n = 1$, the configuration cannot be NE-free. If $n > 1$ is odd we immediately obtain a contradiction, since in this case, by (9), $a(1,1) = a(n-1,1)$, while, by (8), $a(1,1) > a(n-1,1)$ for all $n > 1$. This implies part (i) of Theorem 4.

In contrast, for each even n, the family F_n of all locally minimal NE-free and (c_3, c_9)-free configurations is not empty.

Up to an isomorphism, F_2 and F_4 each consists of a unique configuration: c_1 in Figure 1 and (c_2, c_4, c_5, c_6) in Figure 2, respectively; the claim is obvious for F_2 and follows from (5), (6), (7) and (8) for F_4. This implies part (ii) and (iii) of Theorem 4.

Two configurations from F_6 and F_8, are given in Figures 6 and 7, respectively. We already know that each configuration $P \in F_{2k}$ must satisfy (5) - (10). In particular, it should look like the configuration in Figure 6; see the caption for the explanation. Yet, P has one more important property:

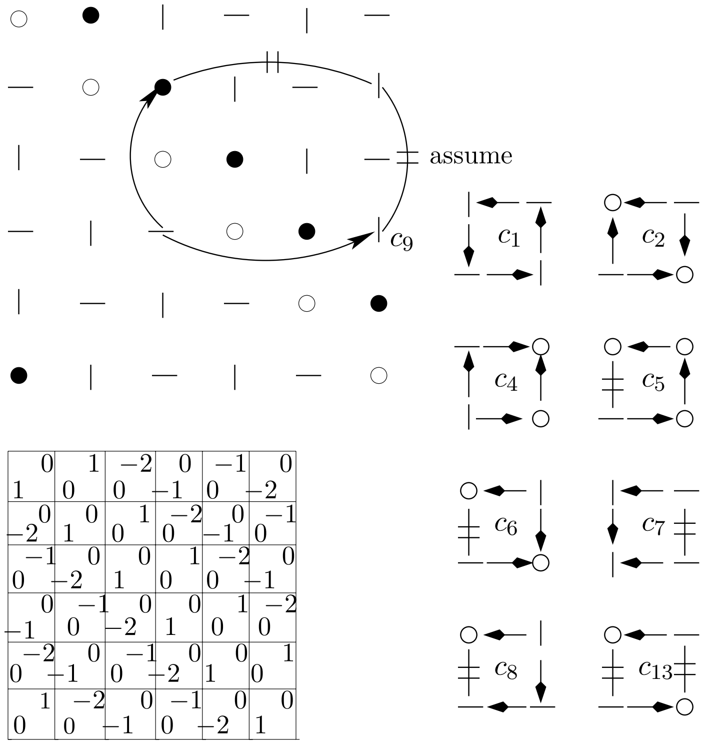


Figure 6: An example from F_6. The horizontal and vertical bars indicate the second largest elements with respect to a and b, respectively. It is not difficult to verify that the eight 2-squares $c_1, c_2, c_4, c_5, c_6, c_7, c_8$, and c_{13} appear in this configuration (they are realized by the horizontal and vertical bars and by the black and white disks, as shown in the right-hand side), while none of the remaining seven 2-squares appears.

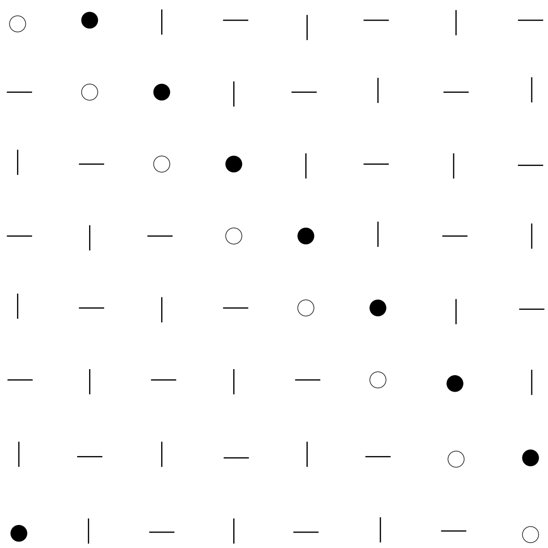
$$b(i, i+2j+1) \neq b(i, i+2j'+1), \; a(i+2j, i) \neq a(i+2j', i) \tag{11}$$

In other words, in every row, the b-values of the entries marked with a black disk and vertical bars are equal, while all marked by horizontal bars are pairwise distinct; respectively, in every column, the a-values of the entries marked with a white disk and horizontal bars are equal, while all marked by vertical bars are pairwise distinct. for all $i \in [n]$ and for all positive distinct $j, j' < n/2$, since otherwise c_9 appears; see Figure 6.

Denote by G_n the family of all configurations satisfying (5) - (11). We have already know that $F_n \subseteq G_n$ and also that $F_n = G_n = \emptyset$ if n is odd, and $F_n = G_n = \{c_1\}$ for $n = 2$.

Now, let us demonstrate that $F_n = G_n$ for every even n. For $n = 2$ and $n = 4$ this was already shown.

For $n = 6$ and $n = 8$ the configurations satisfying all conditions (5) - (11) are described by the Figures 6 and 7, respectively. It is not difficult to verify that each configuration of G_n contains eight 2-squares $C' = \{c_1, c_2, c_4, c_5, c_6, c_7, c_8, c_{13}\}$ whenever $n \geq 6$ and to find these 2-squares in Figure 6. The 2-square c_{12} may also appear for $n = 8$, but Figure 7 gives an



0 / 1	1 / 0	−1 / 0	0 / −1	−2 / 0	0 / −2	−3 / 0	0 / −3
0 / −1	0 / 1	1 / 0	−1 / 0	0 / −2	−2 / 0	0 / −3	−3 / 0
−1 / 0	0 / −1	0 / 1	1 / 0	−3 / 0	0 / −3	−2 / 0	0 / −2
0 / −2	−1 / 0	0 / −3	0 / 1	1 / 0	−3 / 0	0 / −1	−2 / 0
−2 / 0	0 / −2	−3 / 0	0 / −3	0 / 1	1 / 0	−1 / 0	0 / −1
0 / −3	−2 / 0	0 / −2	−3 / 0	0 / −1	0 / 1	1 / 0	−1 / 0
−3 / 0	0 / −3	−2 / 0	0 / −2	−1 / 0	0 / −1	0 / 1	1 / 0
1 / 0	−3 / 0	0 / −1	−2 / 0	0 / −3	−1 / 0	0 / −2	0 / 1

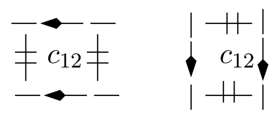


Figure 7: An example from F_8. The legend is the same as in the previous figure and the same eight 2-squares always appear, while c_{12} may appear or not. For the game in the lower left corner it does not. The remaining six 2-squares cannot appear.

example when it does not. It is easily seen that c_{12} always appears when $n \geq 10$. Indeed, in this case there exist two rows and three columns such all six intersections are vertical bars.

On the other hand, no configuration $P \in G_n$ contains c_9, c_{10}, c_{11}, c_{14}, or c_{15}, since no 2-square in P can have two adjacent equalities. Thus, P can contain only nine 2-squares of $C'' = C' \cup \{c_{12}\}$. In particular, each $P \in G_n$ is (c_3, c_9)-free; in other words, $G_n \subseteq F_n$ and, hence, $G_n = F_n$ for all even n. This implies Theorem 4, parts (v) and (vi) and provides an explicit characterization for family F_n of locally minimal NE- and (c_3, c_9)-free configurations.

Let us summarize: for even n each configuration $P \in F_n = G_n$ contains the same set C'' of 2-squares if $n \geq 10$; for $P \in G_8$ there are two options C'' or C' (see example in Figure 6, where c_{12} does not appear); for $P \in G_6$ the only option is C'; furthermore, G_4 consists of a unique configuration (c_2, c_4, c_5, c_6) in Figure 2 and G_2 only of c_1; finally, $F_n = G_n$ is empty if n is odd. □

6 Plans for future research and partial results

One can replace Nash-solvability by dominance-solvability, as an alternative legitimate target property. In this case we obtained the following list of examples:

$E_{DE} = \{(c_1),\ (c_2),\ (c_3,\ c_5,\ c_6),\ (c_4,\ c_5,\ c_6),\ (c_5,\ c_9),\ (c_5,\ c_{10})\ (c_6,\ c_{11})\}$.

The transversal hypergraph, which represents the list of conjectures, is

$E^d_{DE} = \{(c_1, c_2, c_3, c_4, c_9, c_{10}, c_{11}), (c_1, c_2, c_5, c_6), (c_1, c_2, c_5, c_{11}), (c_1, c_2, c_6, c_9, c_{10})\}$.

We managed to prove only the first and the last conjectures, while (c_1, c_2, c_5, c_6) and (c_1, c_2, c_5, c_{11}) remain open. It is also possible that our list of examples is incomplete, but only examples of size 4×5 or larger might be missing, since all examples of size at most 4×4 were found out by a computer code.

Improvement cycles in a bimatrix game $U = (a, b)$ with $a : I \times J \to \mathbb{R}$ and $b : I \times J \to \mathbb{R}$ are standardly defined as follows. Given an integer $n \geq 2$, we say that U has a *weak improvement n-cycle* if there are n strategies in I and n strategies in J such that enumerating them by $[n] = \{1, \ldots, n\}$ and setting $i + 1 = 1$ for $i = n$, we obtain

$$b(i, i) \geq b(i, i + 1) \quad \text{and} \quad a(i, i) \geq a(i + 1, i) \;\; \text{for all} \;\; i \in [n]. \tag{12}$$

Furthermore, a weak improvement n-cycle is called *strict* (respectively, *semi-weak*) if every (respectively, at least one) inequality in (12) is strict. Among the 2-squares in Figure 1, c_1, c_5, c_{11}, c_{13}, c_{14}, and c_{15} are weak improvement cycles; among them c_1 is strict, and all, except c_{15}, are semi-weak.

It is both obvious and well-known that a bimatrix game has a NE if it has no strict improvement cycles. (Note that the concepts of the strict and strong (Section 1.2) improvement n-cycles differ.)

We tried to study the *improvement acyclicity* as a target property in the alphabet C. For the strict, semi-weak, and weak cycles we got, respectively, the following three families of examples.

$E_{St} = \{(c_1), (c_2, c_3), (c_2, c_4, c_5, c_6), (c_3, c_5, c_6), (c_5, c_9), (c_5, c_{10}),$
$(c_3, c_4, c_6, c_{13}, c_7, c_8), (c_3, c_4, c_6, c_{13}, c_7, c_{11}), (c_3, c_4, c_6, c_{13}, c_8, c_{11}),$
$(c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_7, c_8), (c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_7, c_{12}), (c_3, c_4, c_6, c_9, c_{10}, c_{11}, c_8, c_{12})\}$;

$E_{SW} = \{(c_1), (c_5), (c_{11}), (c_{13}), (c_{14}), (c_2, c_3), (c_2, c_4, c_6, c_9), (c_2, c_4, c_6, c_{10}), (c_2, c_4, c_7, c_8, c_9), (c_2, c_4, c_7, c_8, c_{10}), (c_2, c_6, c_7), (c_2, c_6, c_8)\}$;

$E_{We} = \{(c_1), (c_5), (c_{11}), (c_{13}), (c_{14}), (c_{15}), (c_2, c_3), (c_2, c_9), (c_2, c_{10}), (c_2, c_6, c_7), (c_2, c_6, c_8)\}$.

By computing three transversal hypergraphs, we obtain the following families of conjectures.

$E^d_{St} = \{(c_1, c_2, c_3, c_9, c_{10}), (c_1, c_2, c_4, c_5), (c_1, c_2, c_5, c_6), (c_1, c_2, c_5, c_7, c_8), (c_1, c_2, c_5, c_7, c_{11}), (c_1, c_2, c_5, c_8, c_{11}), (c_1, c_2, c_5, c_{13}, c_7, c_{12}), (c_1, c_2, c_5, c_{13}, c_8, c_{12}), (c_1, c_2, c_5, c_{13}, c_9), (c_1, c_2, c_5, c_{13}, c_{10}), (c_1, c_2, c_5, c_{13}, c_{11}), (c_1, c_2, c_6, c_9, c_{10}), (c_1, c_3, c_4, c_9, c_{10}), (c_1, c_3, c_5), (c_1, c_3, c_6, c_9, c_{10})\}$;

$E^d_{SW} = \{(C', c_2), (C', c_3, c_4, c_6), (C', c_3, c_4, c_7, c_8), (C', c_3, c_6, c_7), (C', c_3, c_6, c_8), (C', c_3, c_6, c_9, c_{10}), (C', c_3, c_7, c_8, c_9, c_{10})\}$;

where $C' = \{c_1, c_5, c_{11}, c_{13}, c_{14}\} \subset C$ is the set of all semi-weak 2-squares.

$E^d_{We} = \{(C'', c_2), (C'', c_3, c_6, c_9, c_{10}), (C'', c_3, c_7, c_8, c_9, c_{10})\}$;

where $C'' = C' \cup \{c_{15}\} \subset C$ is the set of all weak 2-squares.

In cases of the weak and semi-weak acyclicity all conjectures were proven. Hence, in these two cases, all obtained theorems and examples are strongest. Somewhat surprisingly, the strict acyclicity appears more difficult. We were able to prove only six conjectures

(c_1, c_2, c_4, c_5), (c_1, c_2, c_5, c_6), $(c_1, c_2, c_6, c_9, c_{10})$, $(c_1, c_3, c_4, c_9, c_{10})$, (c_1, c_3, c_5), $(c_1, c_3, c_6, c_9, c_{10})$,

leaving open the remaining nine. It seems that there may exist more or stronger examples in this case, but such examples (if any) must be of size at least 6×6, since all of size at most 5×5 were found out by a computer code.

As we already mentioned in Section 2, one could expand the alphabet $C = \{c_1, \ldots, c_{15}\}$, distinguishing the transposed 2-squares. Also one can restrict oneself by the tie-free games or, equivalently, by the games (g, u) defined via game forms. In both cases, one can consider all target properties listed above.

The same approach can be applied to n-person normal form games, as well, but then, the alphabet should include the n-dimensional configurations, whose number is typically too large. For example, for $n = 3$ there are already $3 \times 15 = 45$ configurations of sizes $1 \times 2 \times 2$, $2 \times 1 \times 2$, and $2 \times 2 \times 1$, and there are hundreds of different $2 \times 2 \times 2$ configurations. Obviously, there will be many NE-free configurations in any reasonable alphabet and all these configurations must belong to any NE-theorem.

In fact, the proposed method, which could be called *math-pattern recognition*, is very general: any target property can be studied in terms of any set of attribute properties, related to game theory or beyond; see Remark 4.

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