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ON NASH-SOLVABILITY OF CHESS-LIKE
GAMES

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RRR 9-2014, DECEMBER 2014

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RUTCOR RESEARCH REPORT
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Abstract. In 2003, the first two authors proved that a chess-like game has a Nash equilibrium (NE) in pure stationary strategies if (A) the number n of players is at most 2, or (B) the number p of terminals is at most 2 and (C) any infinite play is worse than each terminal for every player. In this paper we strengthen the bound in (B) replacing $p \leq 2$ by $p \leq 3$, provided (C) still holds. On the other hand, we construct a NE-free four-person chess-like game with five terminals, which has a unique cycle, but does not satisfy (C). It remains open whether a NE-free example exists for $n = 3$, or for $2 \leq p \leq 4$, or for some $n \geq 3$ and $p \geq 4$ provided (C) holds.

Keywords: stochastic, positional, chess-like, transition-free games; perfect information, Nash equilibrium, directed cycles, terminal position.

Acknowledgements: The first author thanks for partial support the National Science Foundation (Grant and IIS-1161476).

1 Introduction

Zermelo gave his seminal talk on solvability of chess in pure strategies [24] as early as in 1912. Then, König [18] and Kalman [17] strengthen this result showing that in every two-person zero-sum chess-like game, the two players have pure *stationary uniformly optimal* strategies, which define a *subgame-perfect* saddle point. The reader can find the definitions and more details in surveys [23, 9, 22, 2]; see also [5, 10, 11, 15].

Let us note that the same position can appear several times in a chess game, or in other words, the corresponding directed graph has directed cycles.

A *chess-like game* is a finite n -person positional game with perfect information and without moves of chance. The set of its $q = p + 1$ outcomes $A = \{a_1, \dots, a_p, c\}$ consists of p terminal positions, from which there is no move; furthermore all infinite plays are assumed to be equivalent and form a unique special outcome, denoted by c .

The following assumption will play an important role:

- (C) : Outcome c (that is, any infinite play) is worse than each terminal for every player.

In 1951, Nash introduced his fundamental concept of equilibrium for n -person games [20, 21]. After this, it became natural to ask whether one can extend the above solvability results to the n -person chess-like games, replacing the concept of a saddle point by the more general concept of Nash equilibrium (NE), either assuming (C) or not:

- (Q) : Do chess-like games have NE in pure stationary strategies?
- (Q_C) : Do chess-like games that satisfy (C) have NE in pure stationary strategies?

Obviously, the positive answer to (Q) would imply the positive answer to (Q_C) . It was conjecture in [3] that both questions have the same answer and in [16] that it is negative.

In this paper, we answer (Q) in the negative, and answer (Q_C) in the positive for the chess-like games that have at most 3 terminals. Yet, in general, question (Q_C) remains open.

Remark 1 *Both questions (Q) and (Q_C) appear in Table 1 of [3]. That table contains 16 pairs of questions exactly one of which requires (C) in each pair; 30 questions (15 pairs) are answered in [3]. Among the 30 answers, 22 are negative and 8 are positive, but interestingly, no answer ever depends on condition (C) , that is, in each of these 15 pairs either both answers are negative, or both are positive; in other words, condition (C) is always irrelevant.*

Our questions (Q) and (Q_C) form the 16th pair; (Q) will be answered in the negative in this paper, while (Q_C) remains open. Yet, taking the above observations into account, we second [3, 16] and conjecture that the answer to (Q_C) is the same as to (Q) , that is, negative. However, the counterexample for (Q_C) may need to be larger than our example for (Q) .

2 Basic definitions

Chess-like games are finite positional n -person games with perfect information without moves of chance. Such a game can be modeled by a *finite directed graph* (digraph) $G = (V, E)$, whose vertices are partitioned into $n + 1$ subsets, $V = V_1 \cup \dots \cup V_n \cup V_T$. A vertex $v \in V_i$ is interpreted as a *position* controlled by the player $i \in I = \{1, \dots, n\}$, while a vertex $v \in V_T = \{a_1, \dots, a_p\}$ is a *terminal position* (or just a *terminal*, for short), which has no outgoing edges. A directed edge (v, v') is interpreted as a (legal) *move* from position v to v' . We also fix an initial position $v_0 \in V \setminus V_T$.

The digraph G may have *directed cycles* (dicycles). It is assumed that all dicycles of G are equivalent and form a unique outcome c of a chess-like game. Thus, such a game has $q = p + 1$ outcomes that form the set $A = \{a_1, \dots, a_p; c\}$.

Remark 2 In [6], a different approach was considered (for $n = 2$): each dicycle was treated as a separate outcome. Anyway, our main example in Section 4 contains only one dicycle.

To each pair consisting of a player $i \in I$ and outcome $a \in A$ we assign a *payoff* value $u(i, a)$ showing the profit of the player i in case the outcome a is realized. The corresponding mapping $u : I \times A \rightarrow \mathbb{R}$ is called the payoff function. In the literature, the payoff may be called alternatively the reward, utility, or profit.

A *chess-like game in the positional form* is a quadruple (G, D, u, v_0) , where $G = (V, E)$ is a digraph, $D : V = V_1 \cup \dots \cup V_n \cup V_T$ is a partition of the positions, $u : I \times A \rightarrow \mathbb{R}$ is a payoff function, and $v_0 \in V$ is a fixed initial position. The triplet (G, D, v_0) is called a *positional game form*.

To define the *normal form* of a chess-like game we will need the concept of a strategy. A (*pure and stationary*) *strategy* of a player $i \in I$ is a mapping that assigns a move (v, v') to each position $v \in V_i$. (In this paper we restrict ourselves and the players to their pure stationary strategies, so mixed and history dependent strategies will not even be introduced.)

Once each player $i \in I$ chose a (pure and stationary) strategy s^i , we obtain a collection $s = \{s^i \mid i \in I\}$ of strategies, which is called a *strategy profile* or a *situation*. In a chess-like game modeled by a digraph, to define a situation we have to choose, for each vertex $v \in V \setminus V_T$, a single outgoing arc $(v, v') \in E$. Thus, a situation s can be also viewed as a subset of the arcs, $s \subseteq E$. Clearly, in the subgraph $G_s = (V, s)$ corresponding to a situation s there is a unique walk that begins at the initial vertex v_0 and either ends in V_T , or results in a dicycle. In other words, each situation uniquely defines a *play* $P(s)$ that begins in v_0 and either ends in a terminal $a \in V_T$ or consists of an initial path and a dicycle repeated infinitely. In the latter case the play results in the outcome $c \in A$ and is called a *lasso*. Thus, condition (C) can be reformulated as follows:

- (C') The outcome c (that is, any lasso play) is worse than each of the terminal plays for every player.

Let d_v^+ denote the out-degree of the vertices $v \in V_i$ controlled by the player $i \in I$. Then, player i has $\prod_{v \in V_i} d_v^+$ many different strategies. We denote this set of strategies by $S_i = \{s_j^i \mid j = 1, \dots, \prod_{v \in V_i} d_v^+\}$.

Thus, we obtain a *game form*, that is, a mapping $g : S \rightarrow A$, where $S = S_1 \times \dots \times S_n$ is the direct product of the sets of strategies of all players $i \in I$. By this definition, $g(s) = c$ whenever $P(s)$ is a lasso and $g(s) = a$ whenever $P(s)$ terminates in $a \in V_T$. The *normal form* of a chess-like game (G, D, u, v_0) is defined as the pair (g, u) , where $u : I \times A \rightarrow \mathbb{R}$ is a payoff function.

A situation $s \in S$ is called a *Nash equilibrium* (NE) if for each player $i \in I$ and for each situation s' that may differ from s only in the coordinate i , we have $u(i, g(s)) \geq u(i, g(s'))$. In other words, s is a NE if no player $i \in I$ can profit by replacing his/her strategy s^i in s by a new strategy s'^i , assuming that the other $n - 1$ players keep their strategies unchanged.

In our constructions and proofs it will be more convenient to use instead of a payoff function $u : I \times A \rightarrow \mathbb{R}$ a *preference profile* $o = (\succ_1, \dots, \succ_n)$, where \succ_i is a complete linear order over A that defines the preference of player i , for all $i \in I$. Accordingly, for $a, a' \in A$ and $i \in I$ we write that $a \succ_i a'$ if either $a \succ_i a'$ or $a = a'$. With these notations, the positional and normal form of a game are represented by the quadruple (G, D, o, v_0) and pair (g, o) , respectively.

3 Main results

We study the existence of Nash-equilibria in the chess-like games that are realized by pure stationary strategies. Our first result is the negative answer to question (Q).

Theorem 1 *A NE (in pure stationary strategies) may fail to exist in a Chess-like game.*

In Section 4 we give a NE-free example with $n = 4$ players, $p = 5$ terminals, a single directed cycle, and with $|V_i| \leq 3$ for all players $i \in I$; yet, condition (C) does not hold. Let us recall that acyclic chess-like games always have a (subgame-perfect) NE [19]. Thus, our example shows that even a single dicycle can spoil Nash-solvability.

Our second result strengthens Theorem 6 of [4] weakening the bound $p \leq 2$ to $p \leq 3$.

Theorem 2 *A chess-like games satisfying condition (C) and with at most three terminals has NE in pure and stationary strategies.*

We prove this theorem in Section 6. To prepare for the proof, we recall that the chess-like games with at most two players, $n \leq 2$, are Nash-solvable, as well as the games that satisfy condition (C) and have at most two terminals, $p \leq 2$; see Section 5, the beginning of Section 6, also Section 3 of [4] and Section 12 of [6].

Remark 3 *It was shown in [11] that for each $\epsilon > 0$, a subgame-perfect ϵ -NE in pure but history dependent strategies exists, even for n -person backgammon-like games, in which positions of chance are allowed. Moreover, it was shown in [10, 11] that for chess-like games, the above result holds even for $\epsilon = 0$, that is, a standard NE exists too.*

Yet, our example in Section 4 shows that pure stationary strategies may be insufficient to ensure the existence of a NE in any n -person chess-like game, when $n \geq 4$; not to mention the existence of a subgame-perfect NE, for which counterexamples, satisfying (C) with $n = 2$ and $n = 3$, were obtained earlier [4, 1, 3].

4 An example of a NE-free chess-like game

In this section we prove Theorem 1 by presenting an example that answers question (Q) in the negative. The corresponding game in the positional form was found with the aid of a computer. Yet, due to its relatively small size we can describe here the normal form of the game explicitly and verify that it has no NE, indeed.

In this game there are four players $I = \{1, 2, 3, 4\}$ and $p = 5$ terminals, that is, the set of outcomes is $A = \{a_1, a_2, a_3, a_4, a_5, c\}$, where, as before, c denotes the single outcome assigned to all infinite plays. Let us underline that the digraph of this particular example has only one dicycle and recall that any dicycle-free (acyclic) chess-like game always has a NE [19]. The game is presented in its positional form in Figure 4 and in its normal form in Table 4.

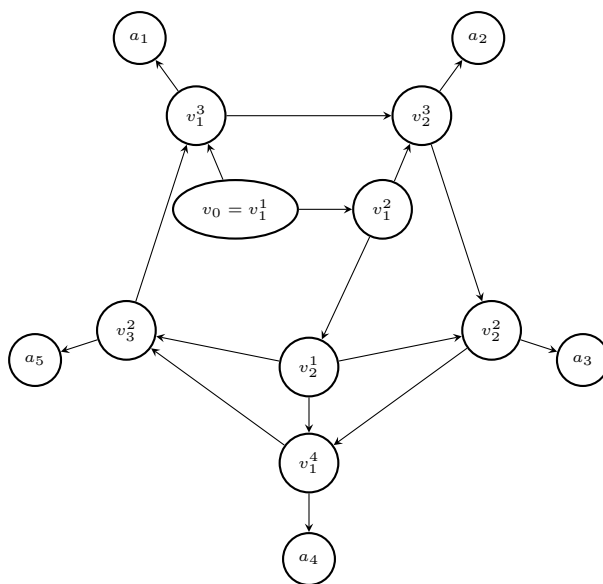


Figure 1: A NE-free chess-like game. It has $p = 5$ terminals, respectively, $V_T = \{a_1, a_2, a_3, a_4, a_5\}$; four players and the vertices controlled by player $i \in I = \{1, 2, 3, 4\}$ are labeled as v_j^i ; the initial vertex is $v_0 = v_1^1$ in the center. This chess-like game does not have a NE (in pure stationary strategies) if the preferences of the four players satisfy conditions (1).

This example, depicted in Figure 4, has no NE if the players preferences satisfy the following relations:

$$a_2 \succ_1 a_4 \succ_1 a_3 \succ_1 a_1 \succ_1 a_5 \quad (1a)$$

$$\{a_1, c\} \succ_2 a_3 \succ_2 \{a_4, a_5\} \succ_2 a_2 \quad (1b)$$

$$\{a_5, c\} \succ_3 a_1 \succ_3 a_2 \succ_3 \{a_3, a_4\} \quad (1c)$$

$$\{a_1, a_2, a_3, a_5\} \succ_4 a_4 \succ_4 c, \quad (1d)$$

where for any $a, a', a'' \in A$ the notation $\{a', a''\} \succ a$ means that both $a' \succ a$ and $a'' \succ a$ hold. Each of the players have multiple linear orders that agree with (1). For instance, player 1 has six such orders that satisfy (1a), etc.

To see that each game satisfying conditions (1) is NE-free, let us consider its normal form and demonstrate that in each possible situation at least one of the players can change his/her strategy and improve the resulting outcome from his/her point of view. To do so, let us list the strategies of each players, first.

Player 1 controls two nodes $V_1 = \{v_1^1, v_2^1\}$ of out-degrees 2 and 3, respectively, and, hence, has the following 6 strategies $S_1 = \{s_j^1 \mid j = 1, \dots, 6\}$:

$$s_1^1 = \{(v_1^1, v_1^2), (v_2^1, v_2^2)\},$$

$$s_2^1 = \{(v_1^1, v_1^3), (v_2^1, v_2^2)\},$$

$$s_3^1 = \{(v_1^1, v_1^2), (v_2^1, v_1^4)\},$$

$$s_4^1 = \{(v_1^1, v_1^3), (v_2^1, v_1^4)\},$$

$$s_5^1 = \{(v_1^1, v_1^2), (v_2^1, v_3^2)\},$$

$$s_6^1 = \{(v_1^1, v_1^3), (v_2^1, v_3^2)\}.$$

Player 2 controls three nodes $V_2 = \{v_1^2, v_2^2, v_3^2\}$ of out-degrees 2 each and, hence, has 8 strategies $S_2 = \{s_j^2 \mid j = 1, \dots, 8\}$:

$$s_1^2 = \{(v_1^2, v_2^1), (v_2^2, v_1^4), (v_3^2, v_1^3)\},$$

$$s_2^2 = \{(v_1^2, v_2^3), (v_2^2, v_1^4), (v_3^2, v_1^3)\},$$

$$s_3^2 = \{(v_1^2, v_2^1), (v_2^2, a_3), (v_3^2, v_1^3)\},$$

$$s_4^2 = \{(v_1^2, v_2^3), (v_2^2, a_3), (v_3^2, v_1^3)\},$$

$$s_5^2 = \{(v_1^2, v_2^1), (v_2^2, v_1^4), (v_3^2, a_5)\},$$

$$s_6^2 = \{(v_1^2, v_2^3), (v_2^2, v_1^4), (v_3^2, a_5)\},$$

$$s_7^2 = \{(v_1^2, v_2^1), (v_2^2, a_3), (v_3^2, a_5)\}$$

$$s_8^2 = \{(v_1^2, v_2^3), (v_2^2, a_3), (v_3^2, a_5)\}.$$

Player 3 controls two nodes $V_3 = \{v_1^3, v_2^3\}$ of out-degree 2 each and, thus, has 4 strategies $S_3 = \{s_j^3 \mid j = 1, \dots, 4\}$:

$$\begin{aligned}
s_1^3 &= \{(v_1^3, v_2^3), (v_2^3, v_2^2)\}, \\
s_2^3 &= \{(v_1^3, a_1), (v_2^3, v_2^2)\}, \\
s_3^3 &= \{(v_1^3, v_2^3), (v_2^3, a_2)\}, \\
s_4^3 &= \{(v_1^3, a_1), (v_2^3, a_2)\}.
\end{aligned}$$

Finally, player 4 controls one node $V_4 = \{v_1^4\}$ of out-degree 2 and, hence, has two strategies $S_4 = \{s_1^4, s_2^4\}$:

$$\begin{aligned}
s_1^4 &= \{(v_1^4, v_3^4)\}, \\
s_2^4 &= \{(v_1^4, a_4)\}.
\end{aligned}$$

The normal form of the game $g : S_1 \times S_2 \times S_3 \times S_4 \rightarrow A$ is described in Table 4. For every situation $s = (s_{\ell_1}^1, s_{\ell_2}^2, s_{\ell_3}^3, s_{\ell_4}^4) \in S_1 \times S_2 \times S_3 \times S_4 = S$ the outcome $g(s)$, which is either a terminal a_j or the dicycle c , is shown in the entry $(\ell_1, \ell_2, \ell_3, \ell_4)$ of the table. Upper indices of these outcomes indicate the players who can improve the situation s . It is not difficult, although time consuming, to verify that each of those upper indices indeed correspond to a player who can improve on situation s , assuming that players' preferences satisfy (1).

Thus, a situation s is a NE if and only if the corresponding entry has no upper indices. As Table 4 shows, there is no such situation and, hence, the game does not have a NE.

5 Two-person chess-like games are Nash-solvable

In this section we recall for completeness the following known result:

Theorem 3 *Every 2-person chess-like game has a NE in pure stationary strategies.*

The proof of this result can be found in Section 3 of [4]; see also in Section 12 of [6]. Since the proof is short, we will repeat it here for the convenience of the reader. It is based on an important property of two-person game forms that seems to be not generalizable for $n > 2$.

A two-person game form g is *Nash-solvable* if for every payoff $u = (u_1, u_2)$ the game (g, u) has a NE; furthermore, g is called *zero-sum-solvable* if for every payoff $u = (u_1, u_2)$ that satisfies $u_1(a) + u_2(a) = 0$ for all $a \in A$ the game (g, u) has a NE; finally, g is called *± 1 -solvable* if the corresponding game (g, u) has a NE for every payoff $u = (u_1, u_2)$ such that $u_1(a) + u_2(a) = 0$ for each outcome $a \in A$ and both u_1 and u_2 take only values $+1$ or -1 .

Theorem 4 *A two-person chess-like game form g is Nash-solvable if and only if it is zero-sum-solvable if and only if it is ± 1 -solvable.* \square

		s_1^4								s_2^4							
		s_1^2	s_2^2	s_3^2	s_4^2	s_5^2	s_6^2	s_7^2	s_8^2	s_1^2	s_2^2	s_3^2	s_4^2	s_5^2	s_6^2	s_7^2	s_8^2
s_{31}	s_1^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_3^2	a_3^{23}	a_4^2	a_4^{23}	a_3^1	a_3^3	a_4^{24}	a_4^{234}	a_3^1	a_3^3
	s_2^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_3^{23}	a_3^{23}	a_4^{23}	a_4^{23}	a_3^{13}	a_3^3	a_4^{234}	a_4^{234}	a_3^{13}	a_3^3
	s_3^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_5^{12}	a_3^{23}	a_4^2	a_4^{23}	a_4^{24}	a_3^3	a_4^{24}	a_4^{234}	a_4^{24}	a_4^3
	s_4^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_3^{23}	a_3^{23}	a_4^{23}	a_4^{23}	a_3^{13}	a_3^3	a_4^{234}	a_4^{234}	a_4^{13}	a_4^3
	s_5^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_5^{12}	a_3^{23}	a_4^{23}	a_4^{23}	a_3^{13}	a_3^3	a_5^{12}	a_4^{234}	a_5^{12}	a_5^3
	s_6^1	c^4	c^4	a_3^{23}	a_3^{23}	a_5^2	a_5^2	a_3^{23}	a_3^{23}	a_4^{23}	a_4^{23}	a_3^{13}	a_3^3	a_4^{234}	a_4^{234}	a_4^{13}	a_4^3
s_{32}	s_1^1	a_1^3	a_1^3	a_3^{23}	a_3^{23}	a_5^{12}	a_5^{12}	a_3^2	a_3^{23}	a_4^{24}	a_4^{234}	a_3^1	a_3^3	a_4^{24}	a_4^{234}	a_3^1	a_3^3
	s_2^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
	s_3^1	a_1^3	a_1^3	a_1^1	a_3^{23}	a_5^{12}	a_5^{12}	a_5^{12}	a_3^{23}	a_4^{24}	a_4^{234}	a_4^{24}	a_3^3	a_4^{24}	a_4^{234}	a_4^{24}	a_4^3
	s_4^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
	s_5^1	a_1^3	a_1^3	a_1^1	a_3^{32}	a_5^{12}	a_5^{12}	a_5^{12}	a_3^{23}	a_1^1	a_4^{234}	a_1^1	a_3^{23}	a_5^{12}	a_4^{234}	a_5^{12}	a_5^{23}
	s_6^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^3	a_1^3	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
s_{33}	s_1^1	a_2^{23}	a_2^{23}	a_3^1	a_2^2	a_5^{12}	a_2^{23}	a_3^1	a_2^2	a_4^{124}	a_2^2	a_3^1	a_2^2	a_4^{124}	a_2^2	a_3^1	a_2^2
	s_2^1	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3
	s_3^1	a_2^{23}	a_2^{23}	a_2^{23}	a_2^2	a_1^5	a_2^{23}	a_1^5	a_2^2	a_4^{14}	a_2^2	a_4^{14}	a_2^2	a_4^{14}	a_2^2	a_4^{14}	a_2^2
	s_4^1	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3
	s_5^1	a_2^{23}	a_2^{23}	a_2^{23}	a_2^2	a_1^5	a_2^{23}	a_1^5	a_2^2	a_2^{23}	a_2^2	a_2^{23}	a_2^2	a_1^5	a_2^2	a_1^5	a_2^2
	s_6^1	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3
s_{34}	s_1^1	a_1^3	a_2^{23}	a_2^3	a_2^2	a_5^{12}	a_2^{23}	a_3^2	a_2^2	a_4^{24}	a_2^2	a_3^1	a_2^2	a_4^{24}	a_2^2	a_3^1	a_2^2
	s_2^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
	s_3^1	a_1^3	a_2^{23}	a_1^1	a_2^2	a_5^{12}	a_2^{23}	a_5^{12}	a_2^2	a_4^4	a_2^2	a_4^4	a_2^2	a_4^4	a_2^2	a_4^4	a_2^2
	s_4^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1
	s_5^1	a_1^3	a_2^{23}	a_1^1	a_2^2	a_5^{12}	a_2^{23}	a_5^{12}	a_2^2	a_1^1	a_2^2	a_1^1	a_2^2	a_5^{12}	a_2^2	a_5^{12}	a_2^2
	s_6^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^3	a_1^{13}	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1	a_1^1

Table 1: This table represents the example of Figure 4 in normal form. The strategies of player $i \in I$ are labeled as s_i^j . The strategy combinations of players 1 and 3 are represented as rows, while the strategy combinations of players 2 and 4 are the columns of this table. The intersection of a row and a column represents a situation s and the corresponding element in the table is the outcome $g(s) \in A = \{a_1, a_2, a_3, a_4, a_5, c\}$. The upper indices over these outcomes denote the players who could improve on this situation. Since every entry of this table has at least one upper index, none of them is a NE.

The equivalence of zero-sum solvability and ± 1 -solvability was shown in 1970 by Edmonds and Fulkerson [8], and independently in [12]. The further equivalence with Nash-solvability was shown in [13]; see also [14]. In the latter paper it was also shown that a similar statement fails for three-person game forms.

To describe the proof of Theorem 3 we need to define one more commonly used notion of *forcing*. Given a subset $Y \subseteq A$ of outcomes, we say that $X \subseteq V$ is a subset of positions from which player $i \in I$ can force Y , if player i has a strategy s^i such that for $v_0 \in X$ and for all strategies s^{-i} of all the other players the resulting outcome $g(s^i, s^{-i})$ belongs to Y . It is well known that given a set $Y \subseteq A$ of terminal positions, the unique maximal subset $X \subseteq V$ from which player i can force Y can be constructed by the following recursive procedure starting with $X = Y$. If $v \in V_i \setminus X$ and there exists an outgoing arc (v, v') with $v' \in X$, then add arc (v, v') to s^i , and add v to X . If $v \in V_j \setminus X$ for some $j \neq i$ and all outgoing arcs (v, v')

are such that $v' \in X$ then add v to X .

Proof of Theorem 3. By Lemma 4 it is sufficient to prove ± 1 -solvability.

Hence, we can assume that each outcome $a \in A = V_T \cup \{c\}$ is either winning for player 1 and losing for player 2, or vice versa. Without any loss of generality, we can also assume that c is a winning outcome for player 1.

Then, let $V_T = V_T^1 \cup V_T^2$ be the partition of all terminals into outcomes winning for players 1 and 2, respectively. Furthermore, let $X^2 \subseteq V$ denote the set of all positions from which player 2 can enforce $Y = V_T^2$; in particular, $V_T^2 \subseteq X^2$. Finally, let us set $X^1 = V \setminus X^2$; in particular, $V_T^1 \subseteq X^1$. By the above definitions, in every position $v \in V_1 \cap X^1$ player 1 can stay out of X^2 , that is there exists a move (v, v') such that $v' \in X^1$. Let us fix a strategy $s^1 \in S_1$ for player 1 that chooses such a move in each position $v \in V_1 \cap X^1$ and an arbitrary move in positions $v \in V_1 \cap X^2$.

Then, for any strategy $s^2 \in S_2$ of player 2, the situation $s = (s^1, s^2)$ is a NE.

If we have $v_0 \in X^1$ then the play stays within X^1 (by the definition of X^2) and, thus, the corresponding outcome $g(s^1, s^2)$ is winning for player 1, since the play ends in either V_T^1 or in c . Player 2 cannot improve on this, since by the definition of X^2 , there is no move (v, v') with $v \in V_2 \cap X^1$ and $v' \in X^2 \supseteq V_T^2$.

If we have $v_0 \in X^2$, then the corresponding play must stay within X_2 and terminate in V_T^2 , since from positions in X^2 player 2 can force $Y = V_T^2$ and, hence, $g(s^1, s^2)$ is winning for player 2. Player 1 cannot improve in this situation, since by the definition of X^2 there is no move (v, v') such that $v \in X_1 \cap X^2$ and $v' \notin X^2$, and player 2 can force $Y = V_T^2$ from every position in X^2 . In particular, player 1 cannot create a dicycle or reach V_T^1 . \square

6 A chess-like game with at most three terminals and satisfying (C) is Nash-solvable

A NE is called *proper* if the corresponding outcome is a terminal, not a cycle. A game will be called *connected* if there is a directed path from v_0 to a terminal position $a \in V_T$.

Obviously, in a non-connected game c is a unique possible outcome and, hence, any situation is a NE, but there exists no proper NE. Let us remark, however, that even a connected chess-like game may have a cyclic NE, as the example in figure 2 shows.

We strengthen slightly Theorem 2 and prove here the following statement.

Theorem 5 *A connected chess-like games satisfying condition (C) and with at most three terminals has a proper NE in pure stationary strategies.*

It was first proven in the preprint [7]. For the convenience of the reader we reproduce this proof here, in a slightly simplified form.

Let us note that if the number of terminals is only one, then due to condition (C) any situation leading to the terminal is a NE and, hence, any connected chess-like game with a single terminal has a NE.

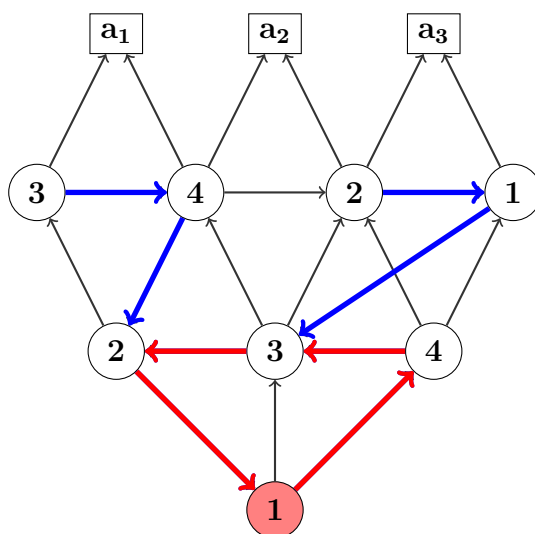


Figure 2: A connected chess-like game with $V_T = \{a_1, a_2, a_3\}$ and $I = \{1, 2, 3, 4\}$; the players' IDs are shown inside the nodes they control, and with the shaded node at bottom as the initial vertex v_0 . Here, thick lines form a situation s resulting in a cycle. Yet, it is easy to verify that s is a NE.

For the case of two terminals we present two simple proofs of Nash-solvability.

Proof 1 for the case of $p = 2$: Let us partition the set of players into two coalitions $I = K_1 \cup K_2$ such that all players of K_1 prefer a_1 to a_2 and all players of K_2 prefer a_2 to a_1 . Without loss of generality, we may assume that all preferences are strict. Furthermore, by (C), all players in each of the two coalitions have the same preference over the set of outcomes $A = \{a_1, a_2, c\}$. Let us consider the corresponding two-person game. According to the previous section, it has a NE and it is easily seen that the same situation is a NE in the original game too. \square

Proof 2 for the case of $p = 2$ ([4] Section 4): Let us recall another proof that derives a contradiction from the negation of the claim.

Let us consider a minimal NE-free game satisfying (C) and having two terminals. Let us introduce the same partition $I = K_1 \cup K_2$ as in the previous proof. Without loss of generality, we can assume that the considered game is connected and that for $i = 1, 2$ any player of K_i has no move to a_i and is not forced to move to a_{3-i} . Indeed, otherwise the game could be reduced, while all its properties listed above are kept, which is a contradiction with the minimality of the chosen game. Hence, a non-terminating move exists in every non-terminal position.

Let us consider now an arbitrary play (path) P from v_0 to one of the terminals. We can define a situation s by using the arcs in P from positions along P , while choosing an arbitrary non-terminating move in all other positions. Then, obviously, any deviation by a

single player from s will result either in returning to P (and thus, to the same terminal) or in cycling (that is, in c). Thus, by condition (C) we conclude that s is a NE. \square

Proof of Theorem 5. We prove the claim by induction on the number n of players.

We can use $n = 1$ as a base case, since then the claim is trivial. We could start with $n = 2$, since for this case the claim is proven in [4]; see also Section 5.

In what follows, first we show that it is enough to prove the theorem for a special case, and then that we can fix a strategy of a player, say, $n \in I$ such that the resulting $(n - 1)$ -person chess-like game has a proper NE that, together with the fixed strategy of player n , constitutes a proper NE in the original game.

Recall that $N^+(v)$ denotes the set of out-neighbors of vertex $v \in V$ and $d^+(v) = |N^+(v)|$. We call a position *dummy* (regardless of the controlling player) if $d^+(v) = 1$ and a *terminal* if $d^+(v) = 0$. Recall that $V_T \subseteq V$ denotes the set of terminals.

Let us also denote by $a^*(i) \in V_T$ the best terminal of player $i \in I$, that for all $a \in A$ such that $a \neq a^*(i)$ we have $a^*(i) \succ_i a$.

Let us first note that it is enough to prove Theorem 5 for the special case of the chess-like games in which no non-dummy position can reach directly the best terminal of the controlling player, and in which no non-dummy position is forced to move to V_T :

$$a^*(i) \succ_i V_T \cap N^+(v) \quad \text{for all players } i \in I \text{ and vertices } v \in V_i \text{ with } d^+(v) > 1 \quad (2a)$$

$$N^+(v) \not\subseteq V_T \quad \text{for all vertices } v \in V \text{ with } d^+(v) > 1. \quad (2b)$$

Let us remark that however natural the above conditions look, it needs a careful proof that they do not restrict generality. For instance, in the case of condition (2a) we may have an edge (v, a_t) in the graph, where $v \in V_i$ and where a_t is the best terminal for player i . Still, it is possible to have a NE in which player i has to choose a different outgoing edge at vertex v in order to prevent another player on the play from improving. See Figure 3 for such an example.

Still, it can be shown quite simply that conditions (2) can be assumed without any restriction, due to the following lemma:

Lemma 1 *Assume that all connected chess-like games $\mathcal{G} = (G, D, u, v_0)$ with at most p terminals and satisfying conditions (2) have a proper NE. Then, all connected chess-like games with at most p terminals have a proper NE, even those that do not satisfy conditions (2).*

Proof Assume indirectly that there are counterexamples, and consider a minimal counterexample $\mathcal{G} = (G, D, u, v_0)$, where minimality is with respect to the number of non-dummy nodes, that is nodes $v \in V$ that have $d^+(v) \geq 2$. Since \mathcal{G} is a connected chess-like game which does not have a proper NE, by our assumption it must violate one of the conditions (2). Thus, we must have a player $i \in I$ and a non-dummy node $v \in V_i$ such that either $(v, a^*(i)) \in E$, or $N^+(v) \subseteq V_T$. In both cases let us denote by a_t the terminal which is the best reachable one for player i from position v .

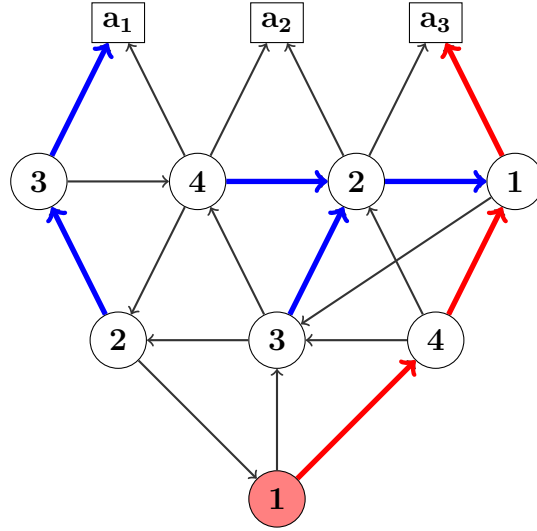


Figure 3: An example of a connected chess-like game with $V_T = \{a_1, a_2, a_3\}$, $I = \{1, 2, 3, 4\}$, players ID-s shown inside nodes they control, and with the shaded node at bottom as the initial vertex v_0 . Here, thick lines form a strategy S ending at terminal a_3 . It is easy to verify that if $u_1(a_2) < u_1(a_3) < u_1(a_1)$ then this is a NE, regardless of the preferences of the other players. Let us note that even if a_2 is player 2's best outcome, he cannot switch to directly reaching a_2 without destroying the current equilibrium (player 1 could then improve).

Let us now delete all outgoing arcs from vertex v , except (v, a_t) . This way v becomes a dummy node and, since it was not before, the resulting game \mathcal{G}' is smaller than \mathcal{G} by our inductive measure.

Since \mathcal{G}' is still a connected chess-like game with one less non-dummy vertex than \mathcal{G} and, since \mathcal{G} was a minimal counterexample, we can conclude that \mathcal{G}' must have a situation s that is a NE in \mathcal{G}' . Since $V(\mathcal{G}') = V(\mathcal{G})$ and $E(\mathcal{G}') \subset E(\mathcal{G})$, situation s is also a situation in game \mathcal{G} . Since the only difference between \mathcal{G} and \mathcal{G}' are the arcs leaving vertex v , and they all go to a terminal not preferred by player i to a_t , we conclude that s is also a NE in \mathcal{G} , completing the proof of the claim. \square

In the rest of this section we will assume conditions (2) and (C).

The following claim, interesting on its own, will be instrumental in proving Theorem 5.

Lemma 2 *Assume that $\mathcal{G} = (G, D, u, v_0)$ is a connected chess-like game with $p \leq 3$ satisfying conditions (2) and (C), and that it has a proper NE. Then, it has a proper NE, s , which does not enter a terminal from non-dummy nodes except the outcome $g(s) \in V_T$. In other words, it satisfies*

$$v' \notin V_T \setminus \{g(s)\} \text{ for all } (v, v') \in s, \quad d^+(v) > 1. \quad (3)$$

Proof If $p < 3$, this claim follows from the cited results of [4], thus, we can assume that $p = 3$. Let us denote by s^* a proper NE of \mathcal{G} , which exists by our assumptions. Let us

define $F = \{(v, v') \in s^* \mid v \text{ is non-dummy and } v' \in V_T, v' \neq g(s^*)\}$. By (2a) we have $N^+(v) \setminus V_T \neq \emptyset$ for all non-dummy nodes $v \in V$. Let us fix one of these, denote it by $w(v)$ for all non-dummy nodes v , and define

$$s = (s^* \setminus F) \cup \{(v, w(v)) \mid (v, v') \in F\}$$

According to property (2a), the above definition is feasible and yields a situation s for \mathcal{G} . Let us also note that $g(s) = g(s^*)$, and they both have the same play $P = P^* \subseteq s \cap s^*$.

We claim that s is a proper NE of \mathcal{G} satisfying property (3). Since the latter is obvious by our construction, we only need to argue that s is a NE.

To simplify our notations in proving this claim, let us call edges of $s \setminus s^*$ *blue*, and assume indirectly that there is a player $i \in I$ such that $P \cap V_i \neq \emptyset$ and that player i can change the strategy at nodes of V_i so that to achieve a better outcome $a' \succ_i g(s) = g(s^*)$. Since s^* is a NE, the new play P' must contain some blue edges. Let (v, v') be the first blue edge along the play P' in situation s' when traversing from the initial point v_0 . Since (v, v') is a blue edge, we must have $(v, a'') \in s^*$ for some $a'' \neq g(s^*)$. Let us also note that player i could divert play $P = P^*$ from strategy s^* to reach a'' by changing this strategy along the vertices in P up to vertex v , since this initial segment of P does not involve any blue edges. Since s^* is a NE, this implies that we have $g(s^*) \succ_i a''$. Consequently, we have $a' \succ_i g(s) = g(s^*) \succ_i a''$, implying that a' is player i 's best outcome, since we have $p = 3$. Due to the construction of s , the last move along play P' must be controlled by player i , meaning that (s)he can directly reach the best outcome a' , contradicting condition (2a).

This contradiction proves that no player can improve on situation s and thus completes the proofs of our claim. \square

For a subset $X \subseteq V_T$ of the terminal set and a player $i \in I$ let us denote by $W(i, X) \subseteq V$ the set of vertices from which player i can *force* an outcome in X .

Now, we are ready to prove Theorem 5. Let us assume, without any loss of generality, that $a_1 \succ_n a_2 \succ_n a_3$, and consider the set $W = W(n, \{a_1, a_2\})$. Let us now fix a strategy for player n . Let us recall that W is obtained from $\{a_1, a_2\}$ by a closure operation. For vertices $v \in V_n \cap W$ choose $(v, v') \in E$ that we used in that closure operation to join v to W . For $v \in V_n \setminus W$ let us choose an arc $(v, v') \in E$ such that the initial vertex v_0 is not cut from the terminal set V_T . Such a choice is possible since \mathcal{G} is connected. Let us now denote by F the set of arcs chosen in this way from vertices of V_n .

Denoting by $E_n = \{(v, v') \in E \mid v \in V_n\}$ the set of edges controlled by player n , let us delete next all edges in $E_n \setminus F$. In the obtained game \mathcal{G}' all nodes in V_n are dummy, so de facto \mathcal{G}' is a game involving $n - 1$ players. Due to our careful selection of edges in F , \mathcal{G}' it is again a connected $(n - 1)$ -person chess-like game. Due to the inductive hypothesis, \mathcal{G}' has a proper NE and, thus, by Lemma 2, it has one, say s' , satisfying conditions (3).

Note that by our construction $F \subseteq s'$ and, hence, s' is a situation in \mathcal{G} as well. We shall show either s' itself is a NE of \mathcal{G} , or we can obtain another situation s'' from s' that is a NE of \mathcal{G} , depending on $g(s')$.

Case 1: $g(s') = a_1$. In this case s' is a NE of \mathcal{G} , since only player n could improve in s , but a_1 is already his best outcome.

Case 2: $g(s') = a_2$. In this case we claim that s' is again a NE of \mathcal{G} . Clearly, only player n could improve on situation s' , and only if it can reach terminal a_1 . For this he would need a vertex $v \in V_n$ such that $(v, a_1) \in E$. However, according to condition (2a), such an edge does not exist.

Case 3: $g(s') = a_3$. Let us note that by the definition of subset W , no edge in s' can leave W and, since $a_3 \notin W$, the entire play $P' \subseteq s'$ must be disjoint from W . Let us also note that the set $F \subseteq s'$ may include some edges (v, a_2) for some $v \in W \cap V_n$. Let us remark that in this case s' may not be a NE of \mathcal{G} , since there may be edges in s' entering W , and player n may be able to change this strategy to reaching such an edge, from which the play will reach W , and by extension terminal a_2 .

Let us now change situation s' as follows. Let us replace every edge $(v, a_2) \in F$ (where $v \in W \cap V_n$) by (v, v') for some $v' \in N^+(v) \setminus V_T$. By property (2b), this is always possible. Furthermore, let us replace every edge $(v, v') \in s$, where $v \notin W$ and $v' \in W$, by an edge (v, v'') , where $v'' \in N^+(v) \setminus W$. According to our definition of W , this is also possible. Let us call the new edges (v, v'') introduced in this way *blue edges* and denote by s'' the resulting situation of \mathcal{G} .

We claim that s'' is a proper NE of \mathcal{G} completing our proof of the theorem.

To see our claim, note that our changes did not change the play P and, hence, we have $g(s'') = g(s) = a_3$. Let us also note that since $N^+(v) \subseteq V \setminus W$ for all $v \in V_n \setminus W$ by the definition of W , and since no edge of s'' enters the set W , player n cannot improve on s'' . Thus, our claim will follow if we can show that no other player can improve on s'' , either. To this end, let us assume indirectly that player $i \in \{1, \dots, n-1\}$ can change his strategy and change the play P'' to P''' ending in a terminal $a' \in \{a_1, a_2\}$ such that

$$a' \succ_i a_3. \quad (4)$$

Since player i could not improve on situation s'' , the new play P''' must involve some blue edges. Let $(v, v'') \in s''$ be the first blue edge when we traverse P''' from the initial vertex v_0 . By the definition of blue edges, we must have $(v, v') \in s'$ for some $v' \in W$. Since the induced subgraph (W, s') is acyclic by our construction, there is a unique path Q in s' connecting v' to a terminal node $a'' \in \{a_1, a_2\}$. Concatenating the path segment of P''' from v_0 to v , then the edge (v, v') , and then the path Q from v' to a'' , we get a path R from v_0 to a'' such that all edges of $P''' \setminus s'$ are controlled by player i . Since player i could not improve on situation s' , we must have

$$a_3 \succ_i a''. \quad (5)$$

Therefore inequalities (4) and (5) imply that a' is player i 'th best outcome (since $p = 3$). Since s'' has no edges entering terminals a_1 or a_2 , the last edge of P''' must be controlled by player i , contradicting property (2a). This contradiction proves our claim and thus concludes the proof of our main theorem. \square

7 Summary and open ends

We say that a family of the chess-like games is Nash-solvable if each of these games has at list one NE in pure stationary strategies. Nash-solvability (NS) obviously holds when $n = 1$ or $p = 1$. The following results concerning NS are known, including the results of this paper:

- (i) $n = 2 \implies NS$ (see [4, 6] or Section 5 above);
- (ii) $p \leq 3$ and $(C) \implies NS$ (see [4] and Section 6 above);
- (iii) $|V_i| = 1$ for all $i \in I \implies NS$ (see [4]);
- (iv) $n = 4, p = 5, |V_i| \leq 3$ for all $i \in I \not\Rightarrow NS$ (see Section 4).

These leave the following questions open:

- (v) $n = 3 \implies NS?$
- (vi) $p = 2 \implies NS?$ $p = 3 \implies NS?$ $p = 4 \implies NS?$
- (vii) $|V_i| \leq 2$ for all $i \in I \implies NS?$
- (viii) $(C) \implies NS?$

It was conjectured in [16] that the answer to (viii) is negative. The corresponding example, if exists, would strengthen simultaneously the above example of Section 4 and the main example of [16]; see Figure 2 and Table 2 there.

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