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Extended Complementary Nim

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Extended Complementary Nim

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Abstract. In the standard Nim with n heaps, a player by one move can reduce (by a positive amount) exactly one heap of his choice. In this paper we consider the game of *complementary* Nim (Co-Nim), in which a player by one move can reduce at least one and at most $n-1$ heaps, of his choice. An explicit formula for the Sprague-Grundy (SG) function of Co-Nim was obtained by Jenkyns and Mayberry in 1980. We consider a further generalization, called *extended complementary* Nim (Exco-Nim). In this game there is one extra heap and a player by one move can reduce at least one and at most $n-1$ of the first n heaps, as in Co-Nim, and (s)he can also reduce the extra heap, whenever it is not empty. The $\mathcal{P}$-positions of Exco-Nim are easily characterized for any n. For $n \geq 3$ the SG function of Exco-Nim is a simple generalization of the SG function of Co-Nim. Somewhat surprisingly, for $n = 2$ the SG function of Exco-Nim looks much more complicated and "behaves in a chaotic way". For this case we provide only some partial results and some conjectures. (Note that for $n = 2$, Co-Nim and the standard two-heap Nim coincide.)

Key words: Moore's k-Nim, extended complementary Nim, impartial games, $\mathcal{P}$-positions, Sprague-Grundy function.

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1 The Sprague-Grundy theory for impartial games

In a two-person combinatorial game, the two players take turns moving alternatingly, in accordance with some rules. The game has perfect information (that is, each player knows the state of the game after each move) and there is no element of luck such as dice or card deals. The game terminates whenever one player has no move; then (s)he loses. A game is *impartial* if moves from each position are the same for both players. A game is *short* if every play in it is finite; in particular, each position can be visited at most once. In this paper, we consider only impartial short games, calling them simply games.

The reader can find a comprehensive theory of combinatorial games in [1, 2, 4]. Those who are familiar with it can skip this section.

1.1 Modeling impartial games by directed graphs

An game is modeled by a directed graph (digraph) $\Gamma = (X, E)$, in which a vertex $x \in X$ is a *position* while a directed edge $(x, x') \in E$ is a *legal move* (or simply a *move*, for short) from x to x'. We will also use notation $x \to x'$ and say in this case that x' is *legally and immediately reachable* (or simply *reachable*, for short) from x.

Digraph Γ may be infinite, but we will always assume that any sequence of successive moves (a *play*) $x \to x'$, $x' \to x''$, ... is finite. In particular, this implies that Γ has no directed cycles. Imagine that a token is initially placed in a position and two players alternate turns moving this token from its current vertex x to x' so that $x \to x'$ is a legal move. The game ends when the token reaches a *terminal*, that is, a vertex with no outgoing edges. Then, the player who made the last move wins, respectively, one who is out of legal moves, loses.

1.2 Winning positions and moves

Since any short impartial game terminates, exactly one player wins. It is not difficult to characterize the winning strategies. Customarily, a position $x \in X$ is called a $\mathcal{P}$-position, if the "previous" player can win, that is if the payer moving from position x cannot win if the other player plays correctly. The subset $\mathcal{P} \subseteq X$ of all $\mathcal{P}$-*positions* is uniquely defined by the following two properties:

(1) $\mathcal{P}$ is *independent*, that is, for any $x \in \mathcal{P}$ and move $x \to x'$ we have $x' \notin \mathcal{P}$;

(2) $\mathcal{P}$ is absorbing, that is, for any $x \notin \mathcal{P}$ there is a move $x \to x'$ such that $x' \in P$.

It is easily seen that the set $\mathcal{P}$ can be obtained by the following simple recursive algorithm: (a) include in $\mathcal{P}$ each terminal of Γ; (b) delete from Γ every position $x \in V \setminus \mathcal{P}$ from which there is a move $x \to x'$ to $x' \in \mathcal{P}$; repeat (a) and (b).

It is also clear that any move $x \to x'$ with $x' \in \mathcal{P}$ is a winning move. Indeed, by (1) the opponent must leave $\mathcal{P}$ by the next move, and then, by (2), the player can reenter $\mathcal{P}$.

By definition, all plays of Γ are finite and, by construction, all terminals are in $\mathcal{P}$; sooner or later the opponent will be out of moves. Thus, any move to a $\mathcal{P}$-position is winning while any move from a $\mathcal{P}$-position is losing; in other words, the *previous* player wins.

The complementary set $X \setminus \mathcal{P}$ is called the set of $\mathcal{N}$-positions, since any move to an $\mathcal{N}$-position is losing, or in other words, the *next* player wins.

1.3 Sum of impartial games and the Nim-sum

Given two games Γ_1 and Γ_2, their sum $\Gamma_1 + \Gamma_2$ is played as follows: On each turn, a player chooses either Γ_1 or Γ_2 and plays in it, leaving the other game unchanged. The game ends when no move is possible, neither in Γ_1 nor in Γ_2. Obviously, this operation is commutative and associative. Hence, it allows to define the sum $\Gamma_1 + \cdots + \Gamma_n$ of n summand games for any integer $n \geq 2$.

The simplest example of the sum is the ancient game of Nim played as follows. There are n heaps containing $x_1, \ldots, x_n$ tokens. Two players alternate turns. By one move, a player chooses a heap $i \in N = \{1, \ldots, n\}$ and takes from it δ tokens; $0 < \delta \leq x_i$. The player who takes the last token wins. By these definitions, Nim is the sum of n games, each of which (a one-heap-Nim) is trivial. Yet, Nim itself is not. It was solved by Charles L. Bouton in his seminal paper [3] as follows. The Nim-sum $x_1 \oplus \cdots \oplus x_n$ is defined as the bitwise binary sum. For example,

$$3 \oplus 5 = 011_2 \oplus 101_2 = 110_2 = 6,\ 3 \oplus 6 = 5,\ 5 \oplus 6 = 3, \text{ and } 3 \oplus 5 \oplus 6 = 0.$$

It was shown in [3] that $x = (x_1, \ldots, x_n)$ is a $\mathcal{P}$-position of Nim if and only if $x_1 \oplus \cdots \oplus x_n = 0$.

1.4 The Sprague-Grundy function and its basic properties

To play the sum $\Gamma = \Gamma_1 + \Gamma_2$, it is not sufficient to know the $\mathcal{P}$-positions of Γ_1 and Γ_2, since $x = (x_1, x_2)$ may be a $\mathcal{P}$-position of Γ even when x_1 is not a $\mathcal{P}$-position of Γ_1 and x_2 is not a $\mathcal{P}$-position of Γ_2. For example, $x = (x_1, x_2)$ is a $\mathcal{P}$-position of the two-pile Nim if and only if $x_1 = x_2$ while only $x_1 = 0$ and $x_2 = 0$ are the $\mathcal{P}$-positions of the corresponding one-pile Nim games. To play sums, the concept of a Sprague-Grundy (SG) function is needed, which is a refinement of the concept of $\mathcal{P}$-positions.

We denote by $\mathbb{Z}_{\geq}$ the set of nonnegative integers. Given a finite subset $S \subseteq \mathbb{Z}_{\geq}$, let $\text{mex}(S)$ (the minimum excluded value or the minimum excludant) be the smallest $k \in \mathbb{Z}_{\geq}$ that is not in S. In particular, $\text{mex}(\emptyset) = 0$, by definition.

Given an impartial game $\Gamma = (V, E)$, the SG function of Γ, denoted by $\mathcal{G} = \mathcal{G}_\Gamma$ is a mapping $\mathcal{G} : V \to \mathbb{Z}_{\geq}$ defined recursively by $\mathcal{G}(x) = \text{mex}(\{\mathcal{G}(x') | x \to x'\})$. Since all plays are finite, this recursive definition defines $\mathcal{G} = \mathcal{G}_\Gamma$ uniquely. In particular, we have $\mathcal{G}(x) = 0$ for any terminal x.

This definition easily implies the following important properties of the SG function:

(a) No move keeps the SG value, that is, $\mathcal{G}(x) \neq \mathcal{G}(x')$ for any move $x \to x'$.

(b) The SG value can be arbitrarily (but strictly) reduced by a move, that is, for any integer v such that $0 \leq v < \mathcal{G}(x)$ there is a move $x \to x'$ such that $\mathcal{G}(x') = v$.

In particular, the SG value can be reduced to 0 whenever it is not 0, which implies:

(c) The $\mathcal{P}$-positions are exactly the zeros of the SG function: $\mathcal{G}(x) = 0$ if and only if x is a $\mathcal{P}$-position.

(d) The SG function of NIM is the NIM-sum of the cardinalities of its piles, that is, $\mathcal{G}(x) = x_1 \oplus \cdots \oplus x_n$ for $x = (x_1, \ldots, x_n)$.

(e) In general, the SG function of the sum of n games is the NIM-sum of the n SG functions of the summands. More precisely, let $\Gamma = \Gamma_1 + \cdots + \Gamma_n$ be the sum of n games and $x = (x^1, \ldots, x^n)$ be a position of Γ, where x^i is the corresponding position of Γ_i for $i = 1, 2, \ldots, n$, then $\mathcal{G}_\Gamma(x) = \mathcal{G}_{\Gamma_1}(x^1) \oplus \cdots \oplus \mathcal{G}_{\Gamma_n}(x^n)$.

2 The games CO-NIM and EXCO-NIM

2.1 Moore's NIM and CO-NIM

In 1910, Eliakim Hastings Moore [8] introduced the following generalization of NIM.

Given two integers k and n such that $1 \leq k < n$ and n piles of tokens, two players move alternatingly. By one move, it is allowed to reduce (strictly) at least 1 and at most k piles. The player who makes the last move wins.

We will denote this game by NIM(n, k) while Moore's original notation was NIM$_k$. Let us notice that NIM(n, k) turns into the standard NIM when $k = 1$. We will call NIM(n, k) the *complementary* NIM (or CO-NIM for short) when $n = k + 1$.

Moore characterized the $\mathcal{P}$-positions of NIM(n, k) as follows. Given a position $x = (x_1, \ldots, x_n)$, let us represent its n entries as binary numbers, take their bitwise sum modulo $(k + 1)$, and denote the obtained number by $M(x)$. Then x is a $\mathcal{P}$-position of NIM(n, k) if and only if $M(x) = 0$. We will call $M(x)$ the Moore sum of the position x.

Thus, the positions of SG value 0 are easy to characterize. Yet, no closed formula for the SG function of NIM(n, k) for $1 < k < n$ is known. In this paper, we will obtain such a formula for the case $n = k + 1$, that is, for CO-NIM and its extension, we call EXCO-NIM.

2.2 Game EXCO-NIM and structure of the paper

Given, $n \geq 2$ and $n+1$ piles that contain $x_0, x_1, \ldots, x_n$ tokens, two players move alternatingly. By one move a player can reduce x_0 and at most $n - 1$ of the remaining n piles; also among the $n + 1$ piles at least one must be reduced.

In other words, a legal move $(x_0, x_1, \ldots, x_n) \to (x'_0, x'_1, \ldots, x'_n)$ must satisfy the following:

- $x'_j \leq x_j$ for all indices $j = 0, 1, \ldots, n$;
- $x_0 + x_1 + \cdots + x_n \;>\; x'_0 + x'_1 + \cdots + x'_n$;
- there is an index $1 \leq i \leq n$ such that $x'_i = x_i$.

As usual, the player who makes the last move wins.

Clearly, the game Exco-Nim is the same as Co-Nim if we start from a position with $x_0 = 0$. Though these two games look very similar (and are when $n \geq 3$) but we will see that they are drastically different when $n = 2$.

An explicit formula for the SG function Co-Nim was obtained by Jenkyns and Mayberry [7] as early as 1980. We will extend this result to Exco-Nim. The paper is organized as follows. In the rest of this section we prove some simple properties of Exco-Nim. In the next section we will derive a closed formula for the SG function of Exco-Nim for $n \geq 3$. We also include an alternative proof in Appendix.

Somewhat surprisingly, in case $n = 2$, the game of Exco-Nim becomes much more difficult than for $n \geq 3$. In the last section, we show partial results and suggest some conjectures for this case. Let us remark that Co-Nim with $n = 2$ is trivial; it simply coincide with the standard tw0-pile Nim.

2.3 The $\mathcal{P}$-positions of Exco-Nim

It is very easy to characterize the $\mathcal{P}$-positions of Exco-Nim for any $n \geq 2$.

Proposition 1. *A position $x = (x_0, x_1, \ldots, x_n)$ of* Exco-Nim *is a $\mathcal{P}$-position if and only if $x_0 = 0$ and $x_1 = x_2 = \cdots = x_n$.*

The rules of Exco-Nim together with the equalities $x_0 = 0$ and $x_1 = x_2 \cdots = x_n$ immediately imply properties (1) and (2) of subsection 1.2, characterizing $\mathcal{P}$-positions.

To show (1), let us notice that if the above equalities hold, then no legal move can keep them all. Indeed, $x_0 = 0$ cannot be reduced further and all n numbers $x_1 = \cdots = x_n$ cannot be reduced simultaneously.

To show (2), note that if at least one of the above equalities fails, then all can be re-enforced by a (unique) move. Indeed, defining $x'_0 = 0$ and $x'_i = \min_{1 \leq j \leq n} x_j$ for $i = 1, 2, \ldots, n$, we can see that $x \to x'$ is a legal move, unless we had $x_0 = 0$ and $x_i = \min_{1 \leq j \leq n} x_j$ for all $i = 1, 2, \ldots, n$. □

2.4 Upper and lower bounds for the SG function of Exco-Nim

Proposition 2. *For a position $x = (x_0, x_1, \ldots, x_n)$ of* Exco-Nim*, we have:*

$$\ell(x) = x_0 + \mathcal{G}(0, x_1, \ldots, x_n) \leq \mathcal{G}(x_0, x_1, \ldots, x_n) \leq x_0 + x_1 + \cdots + x_n = u(x).$$

Proof. The upper bound is obvious while the lower bound follows from the following claim.

Lemma 1. *The SG function $\mathcal{G}(x_0, x_1, \ldots, x_n)$ is strictly monotone with respect to x_0.*

Proof. Let $x' = (x'_0, x_1, \ldots, x_n)$ and $x'' = (x''_0, x_1, \ldots, x_n)$. If $x'_0 > x''_0$, then $\mathcal{G}(x') > \mathcal{G}(x'')$, since any position reachable from x'' is reachable from x' as well. □

Thus, $\mathcal{G}(x_0, x_1, \ldots, x_n) \geq x_0 + \mathcal{G}(0, x_1, \ldots, x_n) = \ell(x)$, resulting in Proposition 2.

3 A formula for the SG function of EXCO-NIM for $n \geq 3$

For a position $x = (x_0, x_1, \ldots, x_n)$ of EXCO-NIM let us define

$$m(x) = \min_{1 \leq i \leq n} x_i, \tag{1a}$$

$$u(x) = \sum_{i=0}^{n} x_i, \tag{1b}$$

$$y(x) = u(x) - n\, m(x), \tag{1c}$$

$$z(x) = \binom{y(x)+1}{2} + 1, \tag{1d}$$

$$g(x) \ = \ \begin{cases} u(x), & \text{if } m(x) < z(x); \\ (z(x)-1) + ((m(x) - z(x)) \mod (y(x)+1)), & \text{if } m(x) \geq z(x). \end{cases} \tag{2}$$

Theorem 1. *For $n \geq 3$, the SG function of* EXCO-NIM *is defined by* (2), *i.e.,* $\mathcal{G}(x) = g(x)$.

The proof below is somewhat similar to the proof of Theorem 10 in [7]. However, our claim is slightly more general; also our approach and notation differ a lot from [7]. For this reason, we include the proof for convenience of the reader.

Proof. Note that $g(x)$ is a function on x, but we will derive from its definition that g can be viewed as a function of the arguments m and u, or alternatively, of m and y. Respectively, we will write $g(m, u)$ and $g(m, y)$ along with $g(x)$.

Without loss of generality, we will assume that inequalities $x_1 \leq x_2 \leq \cdots \leq x_n$ hold for the considered positions $x = (x_0, x_1, \ldots, x_n)$, unless it is explicitly said otherwise.

For a given position x, let $\mathcal{L}(x)$ denote the set of pairs (m, u) reachable from x, that is,

$$\mathcal{L}(x) = \{(m(x'), u(x')) \mid x \to x'\} \tag{3}$$

The following lemma shows that $\mathcal{L}(x)$ is uniquely defined by $m(x)$ and $u(x)$.

Lemma 2. *Given a position x, we have $(m, u) \in \mathcal{L}(x)$ if and only if the following two conditions are satisfied:*

(i) *$m \leq m(x)$ and particularly $m < m(x)$ if $y(x) = 0$;*

(ii) $m(x) + (n-1)m \leq u \leq u(x) - \max\{1, m(x) - m\}$.

Proof. For each move $x \to x'$ it is easy to verify that $(m(x'), u(x'))$ satisfy (i) and (ii), and hence, $(m(x'), u(x')) \in \mathcal{L}(x)$. For the inverse direction let us consider three cases.

Case (a): $m = m(x)$. In this case $y(x) > 0$, by (i). Let us consider moves $x \to x'$ such that $0 \leq x'_0 \leq x_0$, $x'_1 = x_1 = m\,(= m(x))$, $m \leq x'_i \leq x_i$ for $i = 2, \ldots, n-1$, and

$m \le x'_n \le x_n - 1$. Clearly, any $m = m(x)$ and u satisfying (i) and (ii) can be realized by such a move $x \to x'$.

Case (b): $m < m(x)$ and $u \le u(x) - x_2 + m$. Consider moves $x \to x'$ such that $0 \le x'_0 \le x_0$, $x'_1 = x_1 \, (= m(x))$, $x'_2 = m \, (< x_2)$, and $m \le x'_i \le x_i$ for $i = 3, \ldots, n$. Then, $m(x) + (n-1)m$ and $u(x) - (x_2 - m)$ are respectively the minimum and maximum u realized by such moves; moreover any integer between them is also realized by such a move. Hence, all m and u that satisfy $m < m(x)$ and $m(x) + (n-1)m \le u \le u(x) - x_2 + m$ are realized by these moves $x \to x'$, too.

Case (c): $m < m(x)$ and $u > u(x) - x_2 + m$. Consider moves $x \to x'$ such that $0 \le x'_0 \le x_0$, $x'_1 = m \, (< x_1 = m(x))$, $x'_2 = x_2$, and $m \le x'_i \le x_i$ for $i = 3, \ldots, n$. Then, $x_2 + (n-1)m$ and $u(x) - (m(x) - m)$ are respectively the minimum and maximum u realized by such a move. Note that inequalities $x_2 + (n-1)m \le u(x) - x_2 + m < u$ hold (and that $n \ge 3$ is required for the first inequality). Hence, all m and u that satisfy $m < m(x)$ and $u(x) - x_2 + m < u \le u(x) - (m(x) - m)$ are realized by these moves $x \to x'$. □

Corollary 1. *For any $m, u \in \mathbb{Z}_{\ge}$, let $V(m, u) = \{x \mid m(x) = m \text{ and } u(x) = u\}$. Then, all $x \in V(m, u)$ define the same set $\mathcal{L}(x)$.*

This corollary enables us to replace $\mathcal{L}(x)$ by $\mathcal{L}(m, u)$ when convenient. Note that (ii) of Lemma 2 can be equivalently reformulated for y as follows.

$$\text{(ii}'\text{)} \quad m(x) - m \le y \le y(x) + n(m(x) - m) - \max\{1, m(x) - m\}. \tag{4}$$

Hence, we can also replace $\mathcal{L}(x)$ by $\mathcal{L}(m, y)$ when convenient.

The following two lemmas are straightforward, but will be important to analyze $g(x)$.

Lemma 3. *For any positive integer v, there exists a unique $y = \gamma(v)$ such that $v = \binom{y+1}{2} + r$ for some $0 \le r \le y$. Furthermore, $\gamma(v)$ is a nondecreasing function of v.* □

Lemma 4. *If a position x satisfies $m(x) \ge z$, then we have $g(x) < m(x) < u(x)$.* □

We present now two main lemmas that will imply Theorem 1. We say that (m, y) *is of type (I)* if $m < z(m, y)$, and *of type (II)* if $m \ge z(m, y)$. By the definition of g in (2), $g(m, y) = u(m, y)$ if (m, y) is of type (I) and $z(m, y) - 1 + ((m - z(m, y)) \mod (y + 1))$ if (m, y) is of type (II).

Similarly to g, we will view u and z as functions of m and y, since $u(x)$ and $z(x)$ in (1) are uniquely defined by $m(x)$ and $y(x)$, and vice versa. Note that for a pair (m, y) of type (II) we have $\gamma(g(m, y)) = y$.

Lemma 5. *For any $m, y \in \mathbb{Z}_{\ge}$, we have $g(m, y) \notin \{g(m', y') \mid (m', y') \in \mathcal{L}(m, y)\}$.*

Proof. If (m, y) is of type (I) then $g(m, y) = u(m, y) \, (= y + nm)$. For any $(m', y') \in \mathcal{L}(m, y)$ we have $u(m', y') < u(m, y)$. Hence, for (m', y') of type (I) we have $g(m', y') < g(m, y)$. On the other hand, if (m', y') is of type (II), then by Lemma 4, we again have $g(m', y') < m' < u(m', y') < u(m, y) = g(m, y)$.

Let (m, y) be of type (II) and $(m', y') \in \mathcal{L}(m, y)$. Assume to the contrary that $g(m', y') = g(m, y) = v$. Then, by Lemmas 2 (ii) and 4, we have $u(m', y') > m > g(m, y)\,(= g(m', y'))$, and hence, (m', y') is of type (II). Also, by Lemma 3 we have $y' = y = \gamma(v)$, which implies that $m' - m = 0 \mod (y + 1)$, by the definition of g. Since $m - m' \leq y$, by (4) we have $m' = m$. Thus, $y' = y$ and $m' = m$, which contradict the upper bound in (4). □

Lemma 6. *For any $m, y \in \mathbb{Z}_{\geq}$ and $v \in \{0, 1, \dots, g(m, y) - 1\}$, there exists a pair $(m', y') \in \mathcal{L}(m, y)$ such that $g(m', y') = v$.*

Proof. Assume first that (m, y) is of type (II), that is, $g(m, y) = z(m, y) - 1 + ((m - z(m, y)) \mod (y + 1))$. Let v be an integer between 0 and $g(m, y) - 1$. Then, by Lemma 3, there exists a unique $\gamma(v)$ such that $\gamma(v) \leq y$ and $v = \binom{\gamma(v)+1}{2} + r$ for $0 \leq r \leq \gamma(v)$. Since $m > g(m, y) > v$, we have $m \geq \binom{\gamma(v)+1}{2} + 1 + r$. Hence, there exists some $\lambda \in \mathbb{Z}_{\geq}$ such that $m' = \binom{\gamma(v)+1}{2} + 1 + r + \lambda(\gamma(v) + 1)$ and $m - \gamma(v) \leq m' \leq m$. By $m' \geq \binom{\gamma(v)+1}{2} + 1$, we conclude that $(m', \gamma(v))$ is of type (II), and hence, $g(m', \gamma(v)) = v$. We claim that $(m', \gamma(v)) \in \mathcal{L}(m, y)$, which completes the proof, for the case when (m, y) is of type (II).

If $m' = m$ then $\gamma(v) < y$ holds, since $v = g(m', \gamma(v)) < g(m, y)$. This implies that $(m', \gamma(v))$ satisfies (i) of Lemma 2 and (ii′) of (4), and hence, $(m', \gamma(v)) \in \mathcal{L}(m, y)$.

On the other hand, if $m' < m$, then (i) of Lemma 2 and (ii′) of (4) are again satisfied by $(m', \gamma(v))$, and hence, $(m', \gamma(v)) \in \mathcal{L}(m, y)$.

Now let us prove the lemma for (m, y) of type (I). Consider the following three cases:

$$\{v \in \mathbb{Z} \mid u(m, y) - m \leq v < u(m, y)\} \subseteq \{g(m', y') \in \mathcal{L}(m, y)\}. \tag{5}$$

$$\{v \in \mathbb{Z} \mid m \leq v < u(m, y) - m\} \subseteq \{g(m', y') \in \mathcal{L}(m, y)\}. \tag{6}$$

$$\{v \in \mathbb{Z} \mid 0 \leq v < m\} \subseteq \{g(m', y') \in \mathcal{L}(m, y)\}. \tag{7}$$

For each value v such that $0 \leq v < u(m, y)$ we will construct a pair $(m', y') \in \mathcal{L}(m, y)$ such that $g(m', y') = v$. For ranges (5)-(6) we obtain pairs of type (I) while for (7) we will need pairs of type (II).

For (5), let us choose $v = u(m, y) - \delta$, where $1 \leq \delta \leq m$, and define $m' = m - \delta$ and $u' = u(m, y) - \delta$. Note that $(m', u') \in \mathcal{L}(m, u)$, since m' and u' satisfy (i) and (ii) of Lemma 2. Furthermore, (m', u') is of type (I), since $m' < m < z(m, u) \leq z(m', u')$. Thus, $g(m', u') = v$.

Now, let us consider (6). Let $m' = 0$ and $m \leq u' < u - m$. Then, $(m', u') \in \mathcal{L}(m, u)$, since m' and u' satisfy (i) and (ii) of Lemma 2. Moreover, (m', u') is of type (I), since $m' = 0$. Thus, $g(m', u') = u'$, implying (6).

Finally, let us consider (7). Let v be an integer between 0 and $m - 1$. By Lemma 3, there exists a unique $\gamma(v)$ such that $v = \binom{\gamma(v)+1}{2} + r$ for $0 \leq r \leq \gamma(v)$. Since $m \geq v + 1 = \binom{\gamma(v)+1}{2} + 1 + r$, there exists $\lambda \in \mathbb{Z}_{\geq}$ such that $m' = \binom{\gamma(v)+1}{2} + 1 + r + \lambda(\gamma(v) + 1)$ and $m - \gamma(v) \leq m' \leq m$. Since $m' \geq \binom{\gamma(v)+1}{2} + 1$, we conclude that $(m', \gamma(v))$ is of type (II) and hence $g(m', \gamma(v)) = v$. Moreover, $(m', \gamma(v)) \in \mathcal{L}(m, y)$, which completes the proof.

Since (m, y) is of type (I), we have $v < m < z(m, y)$. Note that $\gamma(z(m, y)) = y$ and $z(m, y) = \binom{y+1}{2} + 1$. This implies that $v < \binom{y+1}{2}$ and hence $\gamma(v) < y$. It is easily seen that $(m', \gamma(v))$ satisfies (i) of Lemma 2 and (ii′) of (4) and, thus, $(m', \gamma(v)) \in \mathcal{L}(m, y)$. □

To finish the proof of Theorem 1 it remains to note that, by Lemmas 5 and 6, function g defined by (2) satisfies the equation $g(x) = mex\{g(x') \mid x \to x'\}$ defining the SG function and, thus, $\mathcal{G}(x) = g(x)$, as claimed. □

4 The game Exco-Nim in case $n = 2$

In the above proof, we made use of the inequality $n \geq 3$ several times. For $n = 2$ the game Co-Nim is just Nim with only two piles, but in case of Exco-Nim these two piles can "overlap". Somewhat surprisingly, this case is much more difficult; the SG function "behaves unpredictably" and could be hardly expressed by an explicit formula. Below we suggest several partial results and conjectures.

4.1 Lower and upper bounds for the SG function

First, let us note that, by Proposition 1, position $x = (x_0, x_1, x_2)$ of Exco-Nim is a $\mathcal{P}$-position if and only if $x_0 = 0$ and $x_1 = x_2$. Then, since Co-Nim with $n = 2$ is the same as Nim, Proposition 2 turns into inequalities

$$\ell(x) = x_0 + (x_1 \oplus x_2) \leq \mathcal{G}(x_0, x_1, x_2) \leq x_0 + x_1 + x_2 = u(x). \tag{8}$$

If $x_0 = 0$, the lower bound is achieved: $\mathcal{G}(0, x_1, x_2) = \ell(0, x_1, x_2) = x_1 \oplus x_2$. Indeed, the Exco-Nim turns into the standard two-pile Nim when $n = 2$ and $x_0 = 0$.

Obviously, the upper bound is achieved if $x_1 = 0$ or $x_2 = 0$. We shall see that both bounds are achieved in many other cases.

Without loss of generality, from now on we will assume that $x_1 \leq x_2$.

4.2 Shifting x_1 and x_2 by the same power of 2

We suggest simple conditions under which the SG function $\mathcal{G}(x_0, x_1, x_2)$ is invariant with respect to a shift of x_1 and x_2 by the same power of 2.

Lemma 7. *For any $k \in \mathbb{Z}_{\geq}$, let us set $\Delta^k = (0, 2^k, 2^k)$. Then,*

$$\begin{array}{llll} \mathcal{G}(x + \Delta^k) & = \mathcal{G}(x) & \textit{if} & \mathcal{G}(x) < 2^k, \ \textit{and} \\ \mathcal{G}(x + \Delta^k) & \geq 2^k & \textit{if} & \mathcal{G}(x) \geq 2^k. \end{array}$$

Proof. We show this by induction on x. We first note that $\mathcal{G}(0, 0, 0) = \mathcal{G}(0, a, a)$ holds for any positive integer a, which proves the base of the induction.

For a position x, we assume that the statement is true for all x' with $x' \leq x$, $x' \neq x$ and show that it holds for x by separately considering the cases $\mathcal{G}(x) < 2^k$ and $\mathcal{G}(x) \geq 2^k$.

Case 1: $\mathcal{G}(x) < 2^k$. We note that if $x \to x'$ is a legal move then so is $x + \Delta^k \to x' + \Delta^k$. For any v with $0 \le v < \mathcal{G}(x)\,(< 2^k)$, there exists a legal move $x \to x'$ such that $\mathcal{G}(x') = v$. It follows from induction hypothesis that $\mathcal{G}(x' + \Delta^k) = \mathcal{G}(x')\,(= v)$. Since $x + \Delta^k \to x' + \Delta^k$ is a legal move, we have $\mathcal{G}(x + \Delta^k) \ne v$. Since this applies for values $0 \le v < \mathcal{G}(x)$, we can conclude that $\mathcal{G}(x + \Delta^k) \ge \mathcal{G}(x)$.

We will show next that for any position x' obtained from $x + \Delta^k$ by a move $x + \Delta^k \to x'$, the SG functions of x and x' differ, $\mathcal{G}(x') \ne \mathcal{G}(x)$.

Let $x' = (x'_0, x'_1, x_2 + 2^k)$ without loss of generality. If $x'_1 < 2^k$, then we have

$$\mathcal{G}(x') \ge \ell(x') \ge x'_1 \oplus (x_2 + 2^k) \ge 2^k > \mathcal{G}(x).$$

If $x'_1 \ge 2^k$ and $\mathcal{G}(x' - \Delta^k)\,(= \mathcal{G}(x'_0, x'_1 - 2^k, x_2)) \ge 2^k$, then by inductive hypothesis, $\mathcal{G}(x') = \mathcal{G}((x' - \Delta^k) + \Delta^k) \ge 2^k$, which implies that $\mathcal{G}(x') \ne \mathcal{G}(x)$. Finally, if $x'_1 \ge 2^k$ and $\mathcal{G}(x' - \Delta^k) < 2^k$, then $\mathcal{G}(x' - \Delta^k) \ne \mathcal{G}(x)$, since $x \to x' - \Delta^k$ is a legal move. By our inductive hypothesis, we have $\mathcal{G}(x') = \mathcal{G}(x' - \Delta^k) \ne \mathcal{G}(x)$. This completes the proof of the case $\mathcal{G}(x) < 2^k$.

Case 2: $\mathcal{G}(x) \ge 2^k$. By definition, for any integer v with $0 \le v < 2^k\,(\le \mathcal{G}(x))$, there exists a legal move $x \to x'$ such that $\mathcal{G}(x') = v$. By inductive hypothesis, we have $\mathcal{G}(x') = \mathcal{G}(x' + \Delta^k)$. Since $x' + \Delta^k$ is legally reachable from $x + \Delta^k$, we obtain $\mathcal{G}(x + \Delta^k) \ge 2^k$. □

To illustrate the first claim of the lemma, let us note that: $\mathcal{G}(1,0,0) = 1 < 2$, $\mathcal{G}(2,0,0) = 2 < 4$, $\mathcal{G}(3,0,0) = 3 < 4$, $\mathcal{G}(4,0,0) = 4 < 8$, $\mathcal{G}(0,1,2) = 1 \oplus 2 = 3 < 4$, $\mathcal{G}(1,0,1) = 2 < 4$, $\mathcal{G}(1,1,1) = 3 < 4$, $\mathcal{G}(1,1,2) = 4 < 8$, $\mathcal{G}(2,1,3) = 6 < 8$, $\mathcal{G}(3,1,1) = 5 < 8$.

Furthermore, computations show that $\mathcal{G}(1,2,6) = 5 < 8 < 9 = u(1,2,6)$ and $\mathcal{G}(1,5,6) = \mathcal{G}(2,5,5) = 11 < 12 = u(1,5,6) = u(2,5,5) < 16$. Hence, for all $i \in \mathbb{Z}_\ge$ we have

$\mathcal{G}(1,0,0) = \mathcal{G}(1,2,2) = \mathcal{G}(1,4,4) = \cdots = \mathcal{G}(1,2i,2i) = 1;$
$\mathcal{G}(2,0,0) = \mathcal{G}(2,4,4) = \mathcal{G}(2,8,8) = \cdots = \mathcal{G}(2,4i,4i) = 2;$
$\mathcal{G}(3,0,0) = \mathcal{G}(3,4,4) = \mathcal{G}(3,8,8) = \cdots = \mathcal{G}(3,1+4i,1+4i) = 3.$
$\mathcal{G}(4,0,0) = \mathcal{G}(4,8,8) = \mathcal{G}(4,16,16) = \cdots = \mathcal{G}(4,8i,8i) = 4;$
$\mathcal{G}(1,0,1) = \mathcal{G}(1,4,5) = \mathcal{G}(1,8,9) = \cdots = \mathcal{G}(1,4i,1+4i) = 2;$
$\mathcal{G}(0,1,2) = \mathcal{G}(0,5,6) = \mathcal{G}(0,9,10) = \cdots = \mathcal{G}(0,1+4i,2+4i) = (1+4i) \oplus (1+5i) = 3;$
$\mathcal{G}(1,1,1) = \mathcal{G}(1,5,5) = \mathcal{G}(1,9,9) = \cdots = \mathcal{G}(1,1+4i,1+4i) = 3;$
$\mathcal{G}(1,1,2) = \mathcal{G}(1,9,10) = \mathcal{G}(1,17,18) = \cdots = \mathcal{G}(1,1+8i,2+8i) = 4;$
$\mathcal{G}(2,1,3) = \mathcal{G}(2,9,11) = \mathcal{G}(2,17,19) = \cdots = \mathcal{G}(2,1+8i,3+8i) = 6;$
$\mathcal{G}(3,1,1) = \mathcal{G}(3,9,9) = \mathcal{G}(3,17,17) = \cdots = \mathcal{G}(3,1+8i,1+8i) = 5;$
$\mathcal{G}(1,2,6) = \mathcal{G}(1,10,14) = \mathcal{G}(1,18,22) = \cdots = \mathcal{G}(1,2+8i,6+8i) = 5;$
$\mathcal{G}(1,5,6) = \mathcal{G}(1,21,22) = \mathcal{G}(2,47,48) = \cdots = \mathcal{G}(1,5+16i,6+16i) = 11;$
$\mathcal{G}(2,5,5) = \mathcal{G}(2,21,21) = \mathcal{G}(1,37,37) = \cdots = \mathcal{G}(2,5+16i,5+16i) = 11.$

To illustrate the second claim of the lemma, let us consider $k = 1$, $x = (1,2,3)$, $x + \Delta^k = (1,5,6)$, and note that $\mathcal{G}(1,2,3) = 6$ while $\mathcal{G}(1,4,5) = 2$.

Lemma 7 results immediately the following claim.

Corollary 2. *For any $k \in \mathbb{Z}_\ge$, if $\mathcal{G}(x) < 2^k$ and $x \ge \Delta^k$ then $\mathcal{G}(x - \Delta^k) = \mathcal{G}(x)$.* □

We shall also need the following arithmetic statement, which is elementary.

Lemma 8. *Let a be an integer with $2^{k-1} \leq a < 2^k$ for some positive integer k. Then for an integer $b < 2^k$, we have $a \oplus b > b$ if $0 \leq b < 2^{k-1}$, and $a \oplus b < b$ if $2^{k-1} \leq b < 2^k$.*

Proof. The claim results immediately from the definition of the NIM-sum, since the binary representation of a includes 2^{k-1}. □

Notation 1. For a nonnegative integer v, let $k(v)$ denote a unique nonnegative integer such that $2^{k(v)-1} \leq v < 2^{k(v)}$.

The following consequence of Lemma 7 provides a characterization of the SG values.

Theorem 2. *Let $x = (x_0, x_1, x_2)$ be a position with $\mathcal{G}(x) = v$ and $x_1 \leq x_2$. Then there exists a position $\hat{x}$ such that $\hat{x} \leq (v, 2^{k(v)-1} - 1, 2^{k(v)} - 1)$, furthermore, $\mathcal{G}(\hat{x}) = \mathcal{G}(x)$, and $x = \hat{x} + \lambda\Delta^{k(v)}$ for some $\lambda \in \mathbb{Z}_{\geq}$.*

Proof. By Corollary 2, there exists a position $\hat{x} = (\hat{x}_0, \hat{x}_1, \hat{x}_2) = x - \lambda\Delta^{k(v)}$ for a nonnegative integer λ such that $\mathcal{G}(\hat{x}) = \mathcal{G}(x) = v$ and $\hat{x}_1 < 2^{k(v)}$. We show that $\hat{x}_0 \leq v$, $\hat{x}_1 < 2^{k(v)-1}$, and $\hat{x}_2 < 2^{k(v)}$, which will imply the statement. Since

$$v = \mathcal{G}(\hat{x}) \geq \ell(\hat{x}) = \hat{x}_0 + (\hat{x}_1 \oplus \hat{x}_2), \tag{9}$$

$\hat{x}_0 \leq v$ holds. Moreover, if $\hat{x}_2 \geq 2^{k(v)}$, then $\hat{x}_1 \oplus \hat{x}_2 \geq 2^{k(v)}$ by $\hat{x}_1 < 2^{k(v)}$, which again contradicts (9), since $v < 2^{k(v)}$ by definition of $k(v)$. Thus we have $\hat{x}_2 < 2^{k(v)}$. Suppose that $2^{k(v)-1} \leq \hat{x}_1 \leq \hat{x}_2$. Then $\hat{x} \to (0, \hat{x}_1, \hat{x}_1 \oplus v)$ is a legal move, since $\hat{x}_1 \oplus v < \hat{x}_1 \leq \hat{x}_2$ by Lemma 8. This together with $\mathcal{G}(0, \hat{x}_1, \hat{x}_1 \oplus v) = v$ contradicts that $\mathcal{G}(\hat{x}) = v$. □

For any nonnegative integer v, let us define

$$Core(v) \;=\; \{x = (x_0, x_1, x_2) \mid \mathcal{G}(x) = v, x_0 \leq v, x_1 < 2^{k(v)-1}, x_2 < 2^{k(v)}\}.$$

Then, Theorem 2 shows that every position x with $\mathcal{G}(x) = v$ and $x_1 \leq x_2$ has a position $\hat{x} \in Core(v)$ such that $x = \hat{x} + \lambda\Delta^{k(v)}$ for some nonnegative integer λ.

Note that $Core(v)$ has at most $2v^3$ positions, and by the definition of the SG function, we can compute their SG value in $O(v^5)$ time. This implies the following corollary.

Corollary 3. *For any $v \in \mathbb{Z}_{\geq}$ and for any position x we can compute the value $\mathcal{G}(x)$, if $\mathcal{G}(x) \leq v$, or prove that $\mathcal{G}(x) > v$ in $O(v^5)$ time.* □

4.3 Conjectures and partial results

4.3.1 If x_1 is a power of 2

Computations show that if x_1 is a power of 2 then $\mathcal{G}(x)$ equals either the lower or the upper bound in accordance with the following simple rule.

Conjecture 1. *Given* $x = (x_0, x_1, x_2)$ *such that* $x_0 \in \mathbb{Z}_\geq$ *and* $x_2 \geq x_1 = 2^k$ *for some nonnegative integer* k*, then* $\mathcal{G}(x) = \ell(x) = x_0 + (x_1 \oplus x_2)$ *for any*

$$x_2 = (2j+1)2^k + m \ \text{ such that } \ j \in \mathbb{Z}_\geq \ \text{ and } \ 0 \leq m < 2^k - x_0; \tag{10}$$

otherwise $\mathcal{G}(x) = u(x) = x_0 + x_1 + x_2$.

Note that (10) can be equivalently rewritten as $2^k \leq x_2 \pmod{2^{k+1}} < 2^{k+1} - x_0$. It is also convenient to equivalently reformulate this conjecture replacing $\mathcal{G}(x)$ by $\Delta(x) = u(x) - \mathcal{G}(x)$. For any nonnegative integer $a, b, c \in \mathbb{Z}_\geq$ let us introduce the function

$$f(a,b,c) \;=\; \begin{cases} 1, & \text{if } c \pmod{a} \geq b; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $f(a,b,c) = 0$ whenever $b \geq a$.

It is not difficult to verify that Conjecture 1 can be reformulated as follows:

Given $x = (x_0, x_1, x_2)$ such that $x_0 \in \mathbb{Z}_\geq$ and $x_2 \geq x_1 = 2^k$ for some $k \in \mathbb{Z}_\geq$, then

$$\Delta(x) = u(x) - \mathcal{G}(x) = 2^{k+1} f(2^{k+1}, x_0 + 2^k, x_0 + x_2) = 2x_1 f(2x_1, x_0 + x_1, x_0 + x_2). \tag{11}$$

In particular, the upper bound is achieved, $\mathcal{G}(x) = u(x)$, whenever $x_0 \geq x_1$.

Let us consider several simple examples.

If $x_0 = 0$ and $x_1 = 1$ then $\Delta(x)$ takes values $0, 2, 0, 2, \ldots$ for $x_2 = 2, 3, 4, 5, \ldots$

If $x_0 = 0$ and $x_1 = 2$ then $\Delta(x)$ takes values $0, 0, 4, 4, 0, 0, 4, 4, \ldots$ for $x_2 = 4, 5, 6, 7, 8, 9, 10, 11, \ldots$

If $x_0 = 0$ and $x_1 = 4$ then $\Delta(x)$ takes values $0, 0, 0, 0, 8, 8, 8, 8, \ldots$ for $x_2 = 8, 9, 10, 11, 12, 13, 14, 15, \ldots$

If $x_0 = 2$ and $x_1 \leq 2$ then $\Delta(x) \equiv 0$. If $x_0 = 2$ and $x_1 = 4$ then $\Delta(x)$ takes values $0, 0, 0, 0, 0, 0, 8, 8, 0, 0, 0, 0, 0, 0, 8, 8, \ldots$ for $x_2 = 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \ldots$

In all cases it is convenient to extend these periodic sequences with one more "initial" period corresponding to the semi-interval $x_2 \in [-x_0, 2x_1 - x_0)$.

Note that Conjecture 1, if true, would imply the following useful addition to Lemma 7:

$$x_0 + x_2 = u(x_0, 0, x_2) = \mathcal{G}(x_0, 0, x_2) = \mathcal{G}(x_0, 2^k, x_2 + 2^k) = \ell(x_0, 2^k, x_2 + 2^k),$$

whenever vector $x' = (x_0, 2^k, x_2 + 2^k)$ satisfies condition (10) of the lemma, otherwise

$$\mathcal{G}(x_0, 2^k, x_2 + 2^k) = u(x_0, 2^k, x_2 + 2^k) = x_0 + x_2 + 2^{k+1}.$$

The following examples illustrate the above statement:

$13 = \mathcal{G}(0,0,13) = \mathcal{G}(0,2,15) = \mathcal{G}(0,16,29) = \mathcal{G}(0,32,45) = \mathcal{G}(0,64,79) = \cdots,$
while $\mathcal{G}(0,0,13) = 13, \mathcal{G}(0,4,17) = 21$, and $\mathcal{G}(0,8,21) = 29$.

$17 = \mathcal{G}(1,0,16) = \mathcal{G}(1,2,18) = \mathcal{G}(1,4,20) = \mathcal{G}(1,8,24) = \mathcal{G}(1,32,48) = \mathcal{G}(1,64,80) = \cdots,$
while $\mathcal{G}(1,1,17) = 19$ and $\mathcal{G}(1,16,32) = 49$;

$19 = \mathcal{G}(2,0,17) = \mathcal{G}(2,4,21) = \mathcal{G}(2,8,25) = \mathcal{G}(1,32,49) = \mathcal{G}(1,64,81) = \cdots,$
while $\mathcal{G}(2,1,18) = 21, \mathcal{G}(2,2,19) = 23$, and $\mathcal{G}(2,16,33) = 51$.

4.3.2 If x_1 is close to a power of 2

Our computations demonstrate that for a position $x = (x_0, x_1, x_2)$ the upper bound is achieved whenever the semi-interval $(x_1, x_1 + x_0]$ contains a power of 2. Let us recall that $x_1 \leq x_2$ is assumed.

Conjecture 2. *If $x_1 < 2^k \leq x_0 + x_1$ for some $k \in \mathbb{Z}_{\geq}$, then $\mathcal{G}(x) = u(x)$.*

However, we are able to prove only the following special case of this conjecture.

Proposition 3. *If $x_0 \geq 2^{k-1}$ and $x_1 < 2^k$ for some $k \in \mathbb{Z}_{\geq}$, then $\mathcal{G}(x) = u(x)$.*

Proof. We show the claim by induction on $u(x)$. If $u(x) = 0$ (that is, $x = (0, 0, 0)$) then clearly $\mathcal{G}(x) = u(x) = 0$. Assuming that the claim holds for all x with $u(x) \leq p-1$, consider a position x with $u(x) = p$. We claim that for each integer v with $x_0 \leq v < u(x)$, there exists a legal move $x \to x'$ such that $\mathcal{G}(x') = v$. This will imply $\mathcal{G}(x) = u(x)$, since x_0 and $u(x)$ are lower and upper bounds for $\mathcal{G}(x)$, respectively.

Let $x' = (x'_0, x'_1, x'_2)$ be a position such that $x'_0 = x_0$, $x'_1 = x_1$, and $0 \leq x'_2 < x_2$. It is easily seen that $x \to x'$ is a legal move, x' satisfies $x'_0 \geq 2^{\alpha-1}$, and $\min\{x'_1, x'_2\} < 2^\alpha$ for some α. Thus, by inductive hypothesis, for any integer v satisfying $x_0 + x_1 \leq v < u(x)$ there exists a legal move $x \to x'$ such that $\mathcal{G}(x') = v$.

Then, let us consider moves $x \to x'$ such that $0 \leq x'_0 < x_0$, $x'_1 = x_1$, and $x'_2 = 0$. By definition, we have $\mathcal{G}(x') = u(x')$, which shows that for any integer v with $x_1 \leq v < x_0 + x_1$, there exists a legal move $x \to x'$ such that $\mathcal{G}(x') = v$. Hence, our claim is proven if $x_0 \geq x_1$.

If $x_0 < x_1$ then for each integer v with $2^{\alpha-1} \leq v < x_1$ consider a position x' such that $x'_0 = 0$, $x'_1 = x_1$, and $x'_2 = x_1 \oplus v$. It follows from Lemma 8 that $x_1 \oplus v < x_1 \leq x_2$, which implies that x' is legally reachable from x. Since $\mathcal{G}(x') = v$, the proof is completed. □

Corollary 4. *If $x_0 \geq x_1$, the upper bound is achieved, $\mathcal{G}(x) = u(x)$.* □

4.3.3 If x_2 is close to a multiple of a power of 2

Let us summarize the previous results and conjectures:

(a) If $x_0 = 0$ then the lower bound is achieved: $\mathcal{G}(x) = x_1 \oplus x_2$.

(b) If x_1 is a power of 2 then Conjecture 1 is applicable.

(c) If the semi-interval $(x_1, x_1 + x_0]$ contains a power of 2, that is, $x_1 < 2^k \leq x_0 + x_1$ for some $k \in \mathbb{Z}_{\geq}$, then Conjecture 2 is applicable.

Thus, without loss of generality, we can assume from now on that

$$2^{k-1} < x_1 < x_0 + x_1 < 2^k \text{ for some } k \in \mathbb{Z}_{\geq}. \tag{12}$$

In this case, our computations show that $\mathcal{G}(x) = u(x)$ whenever x_2 is distant from a multiple of 2^k by at most x_0.

Conjecture 3. *The upper bound is achieved,* $\mathcal{G}(x) = u(x) = x_0 + x_1 + x_2$*, whenever* $x = (x_0, x_1, x_2)$ *satisfy* (12)*,* $x_1 \leq x_2$*, and also*

$$x_0 + x_2 \geq j2^k \geq x_2 \ \text{ or } \ x_2 - x_0 \leq j2^k \leq x_2 \ \text{ hold for some } \ k, j \in \mathbb{Z}_{\geq}.$$

Note that assumption (12) is essential. For example, by Conjecture 1, we have $\mathcal{G}(1, 4, 20) = \ell(1, 4, 20) = 1 + (4 \oplus 20) = 17$, yet, $5 \times 4 = 20 \in [19; 21] = [x_2 - x_0; x_2 + x_0]$.

4.3.4 If x_2 is large

Although the SG function looks "chaotic" in general, it seems that the pattern becomes much more regular when x_2 is large enough. Unfortunately, we cannot predict how large should it be or prove the observed pattern.

Conjecture 4. *The upper bound is achieved,* $\mathcal{G}(x) = u(x)$*, whenever* (12) *and the following two conditions hold simultaneously:*

(*i*) ($x_0 > 1$) *or* ($x_0 = 1$ *and* x_1 *is odd*)*;*

(*ii*) x_2 *is sufficiently large.*

Let us start with several examples with $x_0 = 1$ and odd x_1.

$x_1 = 5$: $\mathcal{G}(x) = u(x)$ for $x_2 > 14$ while $\mathcal{G}(1, 5, 14) = 19 = u - 1$;
$x_1 = 9$: $\mathcal{G}(x) = u(x)$ for $x_2 > 94$ while $\mathcal{G}(1, 9, 94) = 103 = u - 1$;
$x_1 = 11$: $\mathcal{G}(x) = u(x)$ for $x_2 > 30$ while $\mathcal{G}(1, 11, 30) = 41 = u - 1$;
$x_1 = 13$: $\mathcal{G}(x) = u(x)$ for $x_2 > 30$ while $\mathcal{G}(1, 13, 30) = 43 = u - 1$;
$x_1 = 17$: $\mathcal{G}(x) = u(x)$ for $x_2 > 446$ while $\mathcal{G}(1, 17, 446) = 463 = u - 1$;
$x_1 = 19$: $\mathcal{G}(x) = u(x)$ for $x_2 > 158$ while $\mathcal{G}(1, 19, 158) = 177 = u - 1$;
$x_1 = 21$: $\mathcal{G}(x) = u(x)$ for $x_2 > 94$ while $\mathcal{G}(1, 21, 94) = 113 = u - 3$;
$x_1 = 23$: $\mathcal{G}(x) = u(x)$ for $x_2 > 62$ while $\mathcal{G}(1, 23, 62) = 85 = u - 1$;
$x_1 = 25$: $\mathcal{G}(x) = u(x)$ for $x_2 > 126$ while $\mathcal{G}(1, 25, 126) = 151 = u - 1$;
$x_1 = 27$: $\mathcal{G}(x) = u(x)$ for $x_2 > 62$ while $\mathcal{G}(1, 27, 62) = 87 = u - 3$;
$x_1 = 29$: $\mathcal{G}(x) = u(x)$ for $x_2 > 30$ while $\mathcal{G}(1, 29, 30) = 50 = u - 10$;

Note that we skip the values $x_1 = 2^k - 1$, that is, $x_1 = 1, 3, 7, 15, 31$ because, according to Conjecture 2, in this case $\mathcal{G}(x) = u(x)$ for all $x_2 \geq x_1$.

It seems that for $x_0 > 1$ the upper bound is achieved sooner (that is, for smaller x_2) and for both odd and even x_1. For $x_1 < 2^5 = 32$ and $x_0 = 2$ we obtain:

$x_1 = 5$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 5$ while $\mathcal{G}(2, 5, 5) = 11 = u - 1$;
$x_1 = 9$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 45$ while $\mathcal{G}(2, 9, 45) = 55 = u - 1$;
$x_1 = 10$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 44$ while $\mathcal{G}(2, 10, 44) = 55 = u - 1$;
$x_1 = 11$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 13$ while $\mathcal{G}(2, 11, 13) = 25 = u - 1$;
$x_1 = 12$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 13$ while $\mathcal{G}(2, 12, 13) = 3 = u - 24$;
$x_1 = 13$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 13$ while $\mathcal{G}(2, 13, 13) = 26 = u - 2$;

$x_1 = 17$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 125$ while $\mathcal{G}(2, 17, 125) = 143 = u - 1$;
$x_1 = 18$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 125$ while $\mathcal{G}(2, 18, 125) = 144 = u - 1$;
$x_1 = 19$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 61$, while $\mathcal{G}(2, 19, 61) = 81 = u - 1$;
$x_1 = 20$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 93$ while $\mathcal{G}(2, 20, 93) = 113 = u - 2$;
$x_1 = 21$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 61$ while $\mathcal{G}(2, 21, 61) = 83 = u - 1$;
$x_1 = 22$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 61$ while $\mathcal{G}(2, 22, 6) = 84 = u - 1$;
$x_1 = 23$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 29$ while $\mathcal{G}(2, 23, 29) = 52 = u - 2$;
$x_1 = 24$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 92$ while $\mathcal{G}(224, 92) = 114 = u - 4$;
$x_1 = 25$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 61$ while $\mathcal{G}(2, 25, 61) = 87 = u - 1$;
$x_1 = 26$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 61$ while $\mathcal{G}(2, 26, 61) = 88 = u - 2$;
$x_1 = 27$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 29$ while $\mathcal{G}(2, 27, 29) = 55 = u - 1$;
$x_1 = 28$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 29$ while $\mathcal{G}(2, 28, 29) = 3 = u - 56$;
$x_1 = 29$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 29$ while $\mathcal{G}(2, 29, 29) = 56 = u - 4$.

For $x_0 = 3$ the upper bound is achieved even faster; for $x_1 < 2^5 = 32$ we have:

$x_1 = 9$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 28$ while $\mathcal{G}(3, 9, 28) = 39 = u - 1$;
$x_1 = 10$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 20$ while $\mathcal{G}(3, 10, 28) = 40 = u - 1$;
$x_1 = 11$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 12$ while $\mathcal{G}(3, 11, 12) = 25 = u - 1$;
$x_1 = 12$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 12$ while $\mathcal{G}(3, 12, 12) = 3 = u - 24$;
$x_1 = 17$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 92$ while $\mathcal{G}(3, 17, 92) = 111 = u - 1$;
$x_1 = 18$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 92$ while $\mathcal{G}(3, 18, 92) = 112 = u - 1$;
$x_1 = 19$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 60$, while $\mathcal{G}(3, 19, 60) = 81 = u - 1$;
$x_1 = 20$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 60$ while $\mathcal{G}(3, 20, 60) = 82 = u - 1$;
$x_1 = 21$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 60$ while $\mathcal{G}(3, 21, 60) = 83 = u - 1$;
$x_1 = 22$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 28$ while $\mathcal{G}(3, 22, 28) = 51 = u - 2$;
$x_1 = 23$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 28$ while $\mathcal{G}(3, 23, 28) = 52 = u - 2$;
$x_1 = 24$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 56$ while $\mathcal{G}(3, 24, 56) = 36 = u - 47$;
$x_1 = 25$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 28$ while $\mathcal{G}(3, 25, 28) = 53 = u - 3$;
$x_1 = 26$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 28$ while $\mathcal{G}(3, 26, 28) = 54 = u - 3$;
$x_1 = 27$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 28$ while $\mathcal{G}(3, 27, 28) = 55 = u - 3$;
$x_1 = 28$: $u(x) = \mathcal{G}(x)$, for all $x_2 > 28$ while $\mathcal{G}(3, 28, 28) = 3 = u - 56$.

Remark 1. In all the above examples we skip the values of x_1 such that $x_1 < 2^k \leq x_1 + x_0$ for some $k \in \mathbb{Z}_\geq$, since in this case Conjecture 2 is applicable while (12) fails. We also skip values $x_1 = 2^k$ because Conjecture 1 is applicable in this case

Finally, let us consider the case when x_1 is even and $x_0 = 1$.

Conjecture 5. *Given $x_0^* = 1$ and an even x_1^* such that $2^{k-1} < x_1^* < 2^k$ for some $k \in \mathbb{Z}_\geq$, then $\Delta(x_0^*, x_1^*, x_2) = u(x_0^*, x_1^*, x_2) - \mathcal{G}(x_0^*, x_1^*, x_2)$ as a function of x_2 takes only even values and is periodic with the period 2^k when x_2 is large enough.*

The example for $k \leq 5$ are presented below. The corresponding 2^k values of $\Delta(x_0^*, x_1^*, x_2)$ are given in the order $x_2 \pmod{2^k} = -1, 0, 1, \ldots, 2^k - 2$. Let us also notice that the first three values always equal 0, in agreement with Conjecture 3.

$\Delta(1,6,x_2) = [00040202] = [0^3 4(02)^2]$; for $x_2 > 14$ while $\mathcal{G}(1,6,14) = 9 = u - 12$;

$\Delta(1,10,x_2) = [(0^3 4)^2\, 12\, (04)^3]$, for $x_2 > 109$ while $\mathcal{G}(1,10,109) = 118 = u - 2$, rather than $u - 4$;

$\Delta(1,12,x_2) = [0^5 6\, 8\, 2(0242)^2]$, for $x_2 > 109$ while $\mathcal{G}(1,12,109) = 120 = u - 2$, rather than $u - 4$; this is the only case when $u(x) \neq \mathcal{G}(x)$ for arbitrarily large odd x_2;

$\Delta(1,14,x_2) = [0^3 4(02)^6]$ for $x_2 > 30$ while $\mathcal{G}(1,14,30) = 17 = u - 28$;

$\Delta(1,18,x_2) = [(0^3 4)^4 (02)^8]$ for $x_2 > 446$ while $\mathcal{G}(1,18,446) = 464 = u - 1$;

$\Delta(1,20,x_2) = [(0^5 602)^2\, 022(0206)^3 02]$ for $x_2 > 400$ while $\mathcal{G}(1,20,400) = 418 = u - 3$;

$\Delta(1,22,x_2) = [(0^3 40202)^2 (02)^8]$ for $x_2 > 94$ while $\mathcal{G}(1,22,94) = 149 = u - 4$;

$\Delta(1,24,x_2) = [0^9\, 16\, 02020\, 16\, ((02)^3 0\, 10)^2]$ for $x_2 > 456$ while $\mathcal{G}(1,24,456) = 471 = u - 10$, rather than $u - 2$;

$\Delta(1,26,x_2) = [(0^3 4)^2 (02)^{12}]$ for $x_2 > 126$ while $\mathcal{G}(26,126,1) = 152 = u - 1$;

$\Delta(1,28,x_2) = [0^5 602014(0206)^5 02]$ for $x_2 > 104$ while $\mathcal{G}(28,104,1) = 130 = u - 3$;

$\Delta(1,30,x_2) = [0^3 4(02)^{14}]$, for $x_2 > 30$ while $\mathcal{G}(30,30,1) = 1 = u - 60$.

Interestingly, $\Delta = 0$ whenever x_1 or x_2 is odd, except for only one case: $x_1 = 12$, when $\Delta(1,12,x_2)$ takes non-zero values $8,4,4$ for $x_2 = 5,9,13 \mod 16$, respectively.

Let us also recall that for $x_1 = 2^k$ Conjecture 1 of Section 4.3.1 gives similar periodical sequences, which take only two values: $\Delta = 0$ and $\Delta = 2x_1$.

4.3.5 A plan of the proof and problems related to it

To prove that $\mathcal{G}(x) = u(x)$, we have for every $u' < u$ to find a legal move to a position with the SG value u'. For example, $\mathcal{G}(6,19,122) = u(6,19,122) = 147$, according to Conjecture 3: $122 + 6 = 128 = 2^7$. Let us try to prove Conjecture 3 by induction. We can easily find a legal move realizing any SG value in the interval $[128,146]$, simply reducing $x_1 = 19$. The induction hypothesis is applicable, since both $x_0 = 6$ and $x_2 = 122$ hold. After x_1 is reduced to 0, we can still arbitrarily reduce x_0, thus getting the SG values in the interval $[122,127]$. Yet, how to realize the values in the interval $[111,121]$? The induction hypothesis (Conjecture 3) does not help any more. It seems that we should extend the base of the induction7 trying to prove several (all) of our conjectures simultaneously. For example, we could realize the SG value 111 by a legal move from $x = (6,19,122)$ to $x' = (5,16,122)$. Indeed, according to Conjecture 1, $\mathcal{G}(5,16,122) = \ell(5,16,122) = 111$. (Thus, the "lower bound conjectures" may be needed to prove the "upper bound ones".) However, to get the SG values in the interval $[112,121]$ we will have to include in the induction process also Conjectures 4 and 5, which hold only for sufficiently large x_2. Unfortunately, at this moment we have no assumptions about the corresponding lower bounds x_2^*.

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Appendix. The alternative proof proof of Theorem 1

It will be convenient to denote the digits of the binary representation of $x \in \mathbb{Z}_{\geq}$ by $\beta_k(x)$ for $k = 0, 1, ...$, that is, we have

$$x = \beta_0(x) + 2\beta_1(x) + 2^2\beta_2(x) + \cdots .$$

Furthermore, let us introduce

$$x[k^-] \ = \ \sum_{j=0}^{k-1} \beta_j(x) \cdot 2^j \text{ and } x[k^+] \ = \ \sum_{j=k+1}^{\infty} \beta_j(x) \cdot 2^j,$$

that is, we have $x = x[k^-] + x[k^+] + \beta_k(x)2^k$ for every integer x and integer k.

To simplify notation, in this proof, we assume that $x_0 = c$, and $x_i = m + p_i$, where $0 = p_1 \leq p_2 \leq \cdots \leq p_n$. We shall also view parameter $y = y(x)$ as a function of x, and parameter $z = z(y)$ as a function of y. Note that with our notation, we have $m(x) = m$ and $y(x) = u(x) - nm = c + p_1 + p_2 + \cdots + p_n$.

As before, we denote by $\mathbb{Z}_{\geq}$ the set of nonnegative integers. Let us introduce

$$V(y) = [z(y) - 1, z(y) + y - 1] = \{l \in \mathbb{Z}_{\geq} | z(y) - 1 \leq l \leq z(y) + y - 1\},$$

and note that parameters y and hence z do not depend on m. Let us also note that these intervals partition set $\mathbb{Z}_{\geq}$ of nonnegative integers as shown in the following lemma.

Lemma 9. *If y and y' are distinct nonnegative integers, then $V(y) \cap V(y') = \emptyset$. Furthermore we have*

$$\cup_{y \in \mathbb{Z}_{\geq}} V(y) \; = \; \mathbb{Z}_{\geq}.$$

Proof. Both of the above claims will follow from the observation that

$$(z(y)+y-1)+1 \; = \; \frac{1}{2}(y^2+y+2)+y \; = \; \frac{1}{2}(y^2+3y+2) \; = \; \frac{1}{2}((y+1)^2+(y+1)+2)-1 \; = \; z(y+1)-1,$$

and from the fact that $z(0) = 1$. □

The above lemma implies that for any positive integer v, there exists unique pair $(\eta(v), \rho(v))$ of integers such that

$$v = z(\eta(v)) - 1 + \rho(v) \quad \text{and} \quad 0 \leq \rho(v) \leq \eta(v). \tag{13}$$

Accordingly, we have $v \in V(\eta(v))$.

For the rest of the proof let us introduce the function γ as follow

$$\gamma(y, m) = (z(y) - 1) + ((m - z(y)) \bmod (y + 1)).$$

Let us call a position x type I if $m(x) < z$ and call it type II otherwise.

The following simple lemma will be useful in our proof.

Lemma 10. *Let v be a nonegative integer. Then a position x is of type I and has $u(x) = v$ if and only if*

$$m(x) \; < \; z(y(x)) \tag{14a}$$

and

$$v \; = \; y(x) + n\, m(x). \tag{14b}$$

Proof. It follows immediate from the definitions. □

Lemma 11. *Given $v \in \mathbb{Z}_{\geq}$, a position x is of type II and*

$$\gamma(y(x), m(x)) = v$$

if and only if

$$y(x) \; = \; \eta(v), \tag{15a}$$

and

$$m(x) \; = \; z(\eta(v)) + \rho(v) + l(\eta(v) + 1) \tag{15b}$$

for some positive integer l.

Proof. The first equation follows from Lemma 9. The second equation follows from the definition of γ while $l \geq 0$ is derived from the fact that for a type II position x we must have $z(\eta(v)) = z(y(x)) \leq m(x)$.

For the converse, assume that x satisfies equations (15). Then $m(x) \geq z(\eta(v)) = z(y(x))$, implying that x is of type II. Furthermore,

$$\begin{aligned}
\gamma(y(x), m(x)) &= z(y(x)) - 1 + ((m(x) - z(y(x))) \bmod (y(x) + 1)) \\
&= z(\eta(v)) - 1 + ((m(x) - z(\eta(v))) \bmod (\eta(v) + 1)) \\
&= z(\eta(v)) - 1 + \rho(v) && \text{(by (15b))} \\
&= v. && \text{(by (13))}
\end{aligned}$$

□

To prove Theorem 1, we need to show that for type I positions x we have $\mathcal{G}(x) = u(x)$ while for type II positions x we have $\mathcal{G}(x) = \gamma(y(x), m(x))$. We shall do this by induction on the value $\mathcal{G}(x)$.

For $\mathcal{G}(x) = 0$ the claim follows from Proposition 1. Indeed, x is a nonterminal $\mathcal{P}$-position if and only if $m(x) > 0$ and $y(x) = 0$, from which $z(y(x)) = 1 \leq m(x)$ follows. In other word, all non-zero $\mathcal{P}$-positions are of type II with $y(x) = 0$, and we have $\gamma(0, m(x)) = 0 = \mathcal{G}(x)$.

Let us next assume for some $k > 0$ that we have already proved the claim for all type I positions x' with $\mathcal{G}(x') = u(x') < k$ and for all type II positions x'' with $\mathcal{G}(x'') = \gamma(y(x''), m(x'')) < k$.

To complete our proof, we shall consider type I positions x with $u(x) = k$ and type II positions x with $\gamma(y(x), m(x)) = k$, and show, using our inductive hypothesis, that $\mathcal{G}(x) = k$. We shall do our proof separately for type I and type II positions. In both cases we show first that for all positions x' reachable form x we have $\mathcal{G}(x') < k$ by our inductive hypothesis. Next we show that for each v such that $0 \leq v < k$, there is a position $x(v)$ reachable form x and such that $\mathcal{G}(x(v)) = v$ by our inductive hypothesis.

Let us assume first that x is a type I position with $u(x) = k$.

Lemma 12. *If x' is reachable from x, then*

(i) either it is of type I with $u(x') < k$,

(ii) or it is of type II with $\gamma(y(x'), m(x')) < k$.

Thus, by the inductive hypothesis, we have $\mathcal{G}(x') < k$ in both cases.

Proof. If x' is reachable form x, then we must have $u(x') < u(x)$, since we must take at least one token in a legal move. Thus, if x' is of type I, then $u(x') < k = u(x)$. If x' is of type II and $\gamma(y(x'), m(x')) = w$, then we have $y(x') = \eta(w)$ by (15a), and $m(x') \geq z(\eta(x)) + \rho(w)$ by (15b), and so

$$\begin{aligned} k = u(x) > u(x') = y(x') + n\,m(x') &\geq \eta(w) + n(z(\eta(w)) + \rho(w)) \\ &\geq \eta(w) + z(\eta(w)) + \rho(w) && \text{(when } n = 1\text{)} \\ &> z(\eta(w)) - 1 + \rho(w) \\ &= w, && \text{(by (13))} \end{aligned}$$

completing our proof. □

To complete our proof for type I positions, we shall construct for each v such that $0 \leq v < k$ a position $x(v) = (x(v)_0, x(v)_1, \ldots, x(v)_n)$ which is reachable form x and for which $\mathcal{G}(x(v)) = v$ by our inductive hypothesis. To simplify notation, we shall use δ_0 instead of $x(v)_0$ and δ_i instead of $x(v)_i$ for $i = 1, \ldots, n$.

Recall that we represent $x = (x_0, x_2, \ldots, x_n)$ with $c = x_0$, $0 = p_1 \leq p_2 \cdots \leq p_n$, $x_i = m(x) + p_i$. The construction of $x(v)$ will depend on the ranges of value v, which are listed below in decreasing order:

Range $k - m(x) \leq v < k = u(x)$: we set $\delta_0 = c$, $\delta_1 = m(x) - (k - v)$, and $\delta_i = m(x) + p_i$ for all $i \geq 2$.

Range $m(x) + p_2 \leq v \leq k - m(x)$: we set $\delta_1 = 0$, $\delta_2 = m(x) + p_2$, and choose the other components δ_i such that $0 \leq \delta_0 \leq c$, $0 \leq \delta_i \leq m(x) + p_i$ for $i = 3, \ldots, n$, and

$$\delta_0 + \sum_{i=3}^{n} \delta_i \;=\; v - m(x) - p_2.$$

This is possible, since

$$\begin{aligned} c + \sum_{i=3}^{n}(m(x) + p_i) &= u(x) - x_1 - x_2 = u(x) - 2m(x) - p_1 - p_2 \\ &= u(x) - 2m(x) - p_2 && (p_1 = 0) \\ &= k - 2m(x) - p_2 \geq v - m(x) - p_2 \end{aligned}$$

in this range of values.

Range $m(x) \leq v < m(x) + p_2$: we set $\delta_1 = m(x)$, $\delta_2 = v - m(x)$, and $\delta_i = 0$ for all indices $i \neq 1, 2$.

Note that all of the above positions are of type I, and satisfy $u(x(v)) = v$.

Note next that since any legal move keeps one of the piles intact, $u(x') \geq m(x)$ for all positions x' reachable from x. Thus, positions $x(v)$ for $0 \leq v < m(x)$ must be of type II.

For the remaining values of v, let us observe that

Lemma 13. *If $0 \leq v < m(x)$, then $\eta(v) < y(x)$.*

Proof. By the definition of η, we have $\eta(v') \leq \eta(v)$ whenever $v' < v$. Since x is of type I, we must have $m(x) < z(y(x))$, implying $m(x) - 1 \in \cup_{y' < y(x)} V(y')$. Thus, $\eta(m(x) - 1) < y(x)$, which together with the observed monotonicity of η implies the claim. □

Range $v = m(x) - 1$**:** we define $x(v)$ by setting $\delta_1 = m(x)$, and choosing other δ_i such that $0 \leq \delta_0 \leq c$, $m(x) \leq \delta_i \leq m(x) + p_i$ for $i \geq 2$, and

$$\delta_0 + \sum_{i=2}^{n} (\delta_i - m(x)) \;=\; \eta(v). \tag{16}$$

This is possible since the smallest possible value on the left hand side is zero while the largest possible value is $c + p_2 + \cdots + p_n = y(x) > \eta(v)$ by Lemma 13.

By the definition of η in (13), we have $v \geq z(\eta(v)) - 1$ and so $m(x) - 1 \geq z(\eta(v)) - 1$, implying that $m(x(v)) \geq z(y(x(v)))$, or that $x(v)$ is indeed of type II.

Furthermore, note that the left hand side of (16) is exactly $y(x(v))$, and so $y(x(v)) = \eta(v)$. Also note that construction in this range also implies that $m(x(v)) = m(x)$. By (13), we have $v + 1 = z(\eta(v)) + \rho(v)$, and so $m(x(v)) = m(x) = v + 1 = z(\eta(v)) + \rho(v)$. Therefore, equalities (15) hold.

Range $0 \leq v < m(x) - 1$**:** set $\alpha(v) = (m(x) - v - 1) \bmod (\eta(v) + 1)$. Note that $0 \leq \alpha(v) \leq \eta(v)$.

Subcase $\eta(v) \geq p_2 + \alpha(v)$**:** set $\delta_1 = m(x) - \alpha(v)$, $\delta_2 = m(x) + p_2$, and choose other δ_i such that $0 \leq \delta_0 \leq c$, $m(x) - \alpha(v) \leq \delta_i \leq m(x) + p_i$ for $i = 3, \ldots, n$, and

$$\delta_0 + \alpha(v) + p_2 + \sum_{i=3}^{n} (\delta_i - m(x) + \alpha(v)) \;=\; \eta(v). \tag{17}$$

This is possible, since the smallest possible value on the left hand side is $\alpha(v) + p_2 \leq \eta(v)$ in this subcase while the largest possible value is $c + \alpha(v) + p_2 + \sum_{i=3}^{n} (\alpha(v) + p_i) \geq y(x) > \eta(v)$ by Lemma 13.

Subcase $\eta(v) < \alpha(v) + p_2$**:** we set $\delta_1 = m(x)$, $\delta_2 = m(x) - \alpha(v)$, and choose other δ_i such that $0 \leq \delta_0 \leq c$, $m(x) - \alpha(v) \leq \delta_i \leq m(x) + p_i$ for $i = 3, \ldots, n$, and

$$\delta_0 + \alpha(v) + \sum_{i=3}^{n} (\delta_i - m(x) + \alpha(v)) \;=\; \eta(v). \tag{18}$$

This is now possible because the smallest possible value on the left hand side is $\alpha(v) \leq \eta(v)$, by the definition of $\alpha(v)$ while the largest possible value one is $c + \alpha(v) + \sum_{i=3}^{n} (p_i + \alpha(v)) \geq p_3 + \alpha(v) \geq p_2 + \alpha(v) \geq \eta(v)$ in this subcase.

Note that in both subcases, we have $m(x(v)) = m(x) - \alpha(v)$ by our choices of δ_i, and so one can check that the left hand side of each of (17) and (18) is equal to $y(x(v))$, implying $y(x(v)) = \eta(v)$. Also note that since $m(x) > v+1$ in these cases, the definition of $\alpha(v)$ implies that $m(x) - \alpha(v) \geq v + 1 \geq z(\eta(v))$, proving that $x(v)$ is indeed of type II in these cases. Equalities (15a) and (15a) hold, since $y(x(v)) = \eta(v)$ and since, by the definition of α, we have $m(x(v)) = m(x) - \alpha(v) \equiv (v + 1) \mod (\eta(v) + 1) = v + 1 + \ell(\eta(v) + 1)$ for some $\ell \geq 0$, and thus (13) implies (15b).

Thus Lemma 11 implies that $\gamma(y(x(v)), m(x(v))) = v$ in the above two ranges. This completes the proof of the inductive step for type I positions.

Note that for range $m(x) \leq v < m(x) + p_2$ we need $m(x(v)) = 0$ for otherwise our proof is not going through. Thus we need $n \geq 3$ in this case. Similarly, the last subcase above also requires $n \geq 3$.

Let us assume next that x is of type II with $\gamma(y(x), m(x)) = k$.

As for the first case, we have to construct for every value $v < k$ a position $x(v)$ reachable from x such that $\mathcal{G}(x(v)) = v$. We shall see that in this case all considered positions will be of type II.

Let us start this case by observing some inequalities. Since x is of type II, $\eta(k) = y(x)$ by Lemma 11. Note that $\binom{y(x)+1}{2} = \frac{1}{2}(y + 1)y = z(y(x)) - 1$. From the definition of γ, we have $z(y(x)) - 1 \leq \gamma(y(x), m(x)) = k$. Therefore,

$$\eta(k) = y(x) \leq \binom{y(x) + 1}{2} = z(y(x)) - 1 \leq k \leq m(x) - 1. \tag{19}$$

Lemma 14. *If x' is reachable from x, then*

(i) either it is of type I and $u(x') > k$,

(ii) or it is of type II and $\gamma(y(x'), m(x')) \neq k$.

Proof. (i) If x' is of type I reachable form x, then by (19) and by the fact that x' must have at least one component at least as large as $m(x)$, we can conclude that $u(x') \geq k + 1$.

(ii) Assume that x' is of type II and reachable from x. We show that $\gamma(y(x'), m(x')) \neq k$. Assume to the contradiction that $\gamma(y(x'), m(x')) = k$. Then, by Lemma 11, we have $y(x') = \eta(k) = y(x)$ and $m(x') \equiv m(x) \mod (y(x) + 1)$. Also note that $m(x') \leq m(x)$ by the definition of m. If $m(x') < m(x)$ then $m(x') \leq m(x) - y(x) - 1$, being equivalent to $m(x) - m(x') \geq y(x) + 1$. Since x' must have at least one pile x_{i_0} whose size does not change after the move from x, we have $y(x') \geq x_{i_0} \geq m(x) \geq m(x) - m(x')$ and so $y(x') \geq y(x) + 1$, giving a contradiction. Therefore, $m(x') = m(x)$. By the definitions of u, m, and y, we have $u(x') = u(x)$, giving a contradiction since every move from x to x' requires removing some tokens and so $u(x') < u(x)$. Thus, $\gamma(y(x'), m(x')) \neq k$. ∎

To complete our inductive proof, we must show that for every value $v < k$ there exists a type II position $x(v)$ reachable form x such that $\gamma(y(x(v)), m(x(v))) = v$. Note that by Lemma 14 we cannot cover values $v < k$ by type I positions. To achieve this goal, let us consider the set

$$M = \{(\mu, \alpha) \mid 0 \leq \mu \leq \eta(k-1),\ 0 \leq \alpha \leq \mu\} \tag{20}$$

of pairs of integers and note that by (13) we have a unique pair $(\mu, \alpha) \in M$ for every value $v < k$ (and, depending on k maybe even for $v = k$, $v = k + 1$, etc.) such that $\mu = \eta(v)$ and $\alpha = \rho(v)$.

In the rest of the proof we construct type II positions $x(\mu, \alpha)$ reachable from x for all $(\mu, \alpha) \in M$, such that

$$y(x(\mu, \alpha)) = \mu \quad \text{and} \quad m(x(\mu, \alpha)) = m(x) - \alpha \geq z(\mu), \tag{21}$$

To simplify our notation, let us again use δ_0 instead of $x(\mu, \alpha)_0$ and δ_i instead of $x(\mu, \alpha)_i$ for $i = 1, \ldots, n$. Let us also introduce $P_i = p_1 + \cdots + p_i$ for $i = 1, \ldots, n+1$, where $p_{n+1} = c$.

Case $\mu = 0 = \alpha$: We set $\delta_0 = 0$, and $\delta_\beta = m(x)$ for $\beta = 1, \ldots, n$. Then, we have $y(x(0, 0)) = 0$, $m(x(0, 0)) = m(x)$ and $\gamma(0, m(x)) = 0$, as claimed in (21). Note that $y(x) > 0$ and so $x(0, 0)$ is reachable from x.

Note that for any $(\mu, \alpha) \in M$, $\mu > 0$ we have a unique index $i = i(\mu) \in [1, n]$ such that $P_i < \mu \leq P_{i+1}$.

Case $\alpha = 0$ and $i(\mu) < n$: We define $x(\mu, 0)$ by setting

$$\begin{aligned} \delta_0 &= 0, \\ \delta_\beta &= m(x) + p_\beta \text{ for } \beta = 1, .., i, \\ \delta_{i+1} &= m(x) + \mu - P_i, \text{ and} \\ \delta_\beta &= m(x) \text{ for } \beta > i + 1. \end{aligned}$$

In this case $x(\mu, 0)$ is reachable form x. It follows from the above construction that $m(x(\mu, 0)) = m(x)$. By (19), $m(x) \geq z(y(x))$ and by (15b), $z(y(x)) = z(\eta(k))$. Note that $\eta(k) = y(x)$ by (15a) and $y(x) \geq \mu$ by the definitions of y and μ, so $z(\eta(k)) \geq z(\mu)$ as the function z is increasing. Therefore, $m(x(\mu, 0)) \geq z(\mu)$. This construction implies also that $y(x(\mu, 0)) = \mu$.

Case $\alpha = 0$ and $i(\mu) = n$: We define $x(\mu, 0)$ by setting

$$\begin{aligned} \delta_0 &= \mu - P_n \text{ and} \\ \delta_\beta &= m(x) + p_\beta \text{ for } \beta = 1, \ldots, n. \end{aligned}$$

It can be check that $x(\mu, 0)$ is reachable from x, and is of type II satisfying equalities (21), except for the case $\mu = \eta(k-1) = \eta(k)$, in which we have $x(\mu, 0) = x$.

Case $\mu > 0$, $\alpha > 0$ and $i(\mu) < n$: Let us also consider the unique index $j = j(\alpha)$ defined by $\mu - P_{j+1} < \alpha \le \mu - P_j$, $1 \le j \le i$, and define $x(\mu, \alpha)$ by setting

$$\begin{aligned}
\delta_0 &= 0, \\
\delta_\beta &= m(x) + p_\beta - \alpha \text{ for } \beta = 1, \dots, j-1, \\
\delta_j &= m(x) + p_j, \\
\delta_{j+1} &= m(x) + \mu - P_j - 2\alpha, \text{ and} \\
\delta_\beta &= m(x) - \alpha \text{ for } \beta > j+1.
\end{aligned}$$

The inequality $\delta_\beta \ge 0$ for $\beta \ne j+1$ follows from (19) since $\alpha \le \mu \le \eta(k-1) \le \eta(k)$. For $\beta = j+1$ we can write

$$m(x) + \mu - P_j - 2\alpha = m(x) - \alpha + (\mu - P_j - \alpha) \ge m(x) - \alpha \ge 0, \tag{22}$$

where the second inequality follows from the choice of $j = j(\alpha)$ and the last inequality follows from (19) with the notice that $\alpha \le \eta(k)$ as observed.

It is simple to verify that $\delta_\beta \le m(x) + p_\beta$ for every index $\beta \le j$, and particularly $\delta_j = m(x) + p_j$.

We now show that $\delta_{j+1} < m(x) + p_{j+1}$. First consider the case $j < i$. Then, we can write

$$\begin{aligned}
m(x) + \mu - P_j - 2\alpha &< m(x) + \mu - P_j - 2(\mu - P_{j+1}) \\
&= m(x) - \mu + p_{j+1} + P_{j+1} \\
&\le m(x) - (P_i + 1) + p_{j+1} + P_{j+1} \\
&= m + p_{j+1} - 1 - (P_i - P_{j+1}) \\
&\le m(x) + p_{j+1}.
\end{aligned}$$

The last inequality follows from the fact that $P_i \ge P_{j+1}$ as $i > j$, and the previous inequalities follow from the choices of indices i and j.

If $i = j$, then we have $\mu \le P_{j+1}$ and $\alpha \ge 1$ (since $\mu - P_{j+1} \le 0$ due to $i = j$, and since we considered $\alpha = 0$ separately), and so

$$\delta_{j+1} \le m(x) + P_{j+1} - P_j = m(x) + p_{j+1}.$$

Thus, in this case, the position $x(\mu, \alpha)$ is reachable form x. The equality $y(x(\mu, \alpha)) = \mu$ follows from the construction of δ and the definition of y, u, and m while the equality $m(x(\mu, \alpha)) = m(x) - \alpha$ is derived from the construction of δ and (22). Thus, we proved that $x(\mu, \alpha)$ is a type II position and reachable form x and satisfies (21).

Case $\mu > 0$, $\alpha > 0$ and $i(\mu) = n$: Note that $0 \le \alpha \le \mu \le y(x) = \eta(k)$. For each pair (α, μ), either there exists an unique index $\ell = \ell(\alpha) \in [1, n-1]$ such that

$$P_n - P_{\ell+1} \; < \; \alpha \; \le \; P_n - P_\ell,$$

or we have

$$\alpha > P_n$$

with the notice that $P_1 = 0$ by our notation.

In the first case we define $x(\mu, \alpha)$ by

$$\begin{aligned} \delta_0 &= \mu - P_n \\ \delta_\beta &= m(x) + p_\beta - \alpha \text{ for } \beta = 1, \ldots, \ell - 1 \\ \delta_l &= m(x) + p_\ell \\ \delta_{\ell+1} &= m(x) + P_n - P_\ell - 2\alpha \text{ and} \\ \delta_\beta &= m(x) - \alpha \text{ for } \beta > \ell + 1, \end{aligned}$$

while in case of $\alpha > P_n$ we define $x(\mu, \alpha)$ by setting

$$\begin{aligned} \delta_0 &= \mu - \alpha \\ \delta_1 &= m(x) + p_1 = m(x) \\ \delta_\beta &= m(x) - \alpha \text{ for } 2 \leq \beta \leq n. \end{aligned}$$

By using the definition of $y(x)$, one can verify that $y(x(\mu, \alpha)) = \mu$ in both cases. For both constructions, we have $m(x(\mu, \alpha)) = m(x) - \alpha$, since either $\delta_1 = m(x) - \alpha$ or $\delta_n = m(x) - \alpha$, and all other components are at least as large. The only non-trivial component here is $\delta_{\ell+1}$ in the first case in which the inequality

$$m(x) + P_n - P_\ell - 2\alpha \geq m(x) - \alpha$$

follows from the fact that $P_n - P_\ell \geq \alpha$ by the definition of $\ell(\alpha)$. One of the components in $x(\mu, \alpha)$ is the same as in x. In the first case this is δ_ℓ while in the second case it is δ_1. Finally, none of the components exceeds the corresponding component of x. For $\delta_{\ell+1}$ in the first case this follows from the inequality $P_n - P_{\ell+1} + 1 \leq \alpha$, from which

$$\begin{aligned} \delta_{\ell+1} &= m(x) + P_n - P_\ell - 2\alpha \\ &\leq m(x) + P_n - P_\ell - 2(P_n - P_{\ell+1} + 1) \\ &= m(x) - P_n + P_{\ell+1} + p_{\ell+1} - 2 \\ &\leq m(x) + p_{\ell+1}, \end{aligned}$$

since $\ell \leq n - 1$ implies $P_{\ell+1} \leq P_n$.

Thus we proved that $x(\mu, \alpha)$ defined above are type II positions, reachable from x, and satisfy (21).

To complete our proof we have to show that

$$\{\gamma(\mu, m(x) - \alpha) \mid (\mu, \alpha) \in M\} \supseteq [0, k-1]. \tag{23}$$

To this end observe first that, by the definition of $\gamma(y, m)$, if $V(\mu) \subseteq [0, k-1]$ holds, then we have

$$\begin{aligned} V(\mu) &= \{\gamma(\mu, m(x) - \alpha) \mid 0 \leq \alpha \leq \mu\} \\ &= \{\gamma(y(x(\mu, \alpha)), m(x(\mu, \alpha))) \mid 0 \leq \alpha \leq \mu\} \\ &= \{\mathcal{G}(x(\mu, \alpha)) \mid 0 \leq \alpha \leq \mu\}, \end{aligned} \tag{24}$$

where the last equation follows from our inductive hypothesis.

For $\mu < \eta(k) = y(x)$, we have $V(\mu) \subseteq [0, k-1]$ by Lemma 9. It remains to consider the case of $\mu = \eta(k-1) = \eta(k)$ and the range of values $V(\eta(k-1)) \cap [0, k-1]$.

In this case we have $y(x) = \eta(k) = \eta(k-1)$ and thus $z(\eta(k)) = z(\eta(k-1))$. Therefore, it follows from (13) and (15b) that

$$\begin{aligned} k &= z(y(x)) - 1 + ((m(x) - z(y(x))) \bmod (y(x)+1)) \\ &= z(\eta(k-1)) - 1 + ((m(x) - z(\eta(k-1))) \bmod (\eta(k-1)+1)), \end{aligned}$$

implying

$$\begin{aligned} m(x) - z(\eta(k-1)) &\equiv (k - z(\eta(k-1)) + 1) \bmod (\eta(k-1)+1)) \\ &\equiv (k - 1 - z(\eta(k-1)) + 2) \bmod (\eta(k-1)+1)) \\ &\equiv (\rho - 1 + 2) \bmod (\eta(k-1)+1)) && \text{(by (13))} \\ &\equiv (\rho + 1) \bmod (\eta(k-1)+1). \end{aligned}$$

Thus, we have

$$V(\eta(k-1)) \cap [0, k-1] \;=\; \{\gamma(\eta(k-1), x(\eta(k-1), \alpha)) \mid 1 \le \alpha \le \rho + 1\},$$

completing the proof of the theorem. □