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RRR 2-2015, July 2015

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On the Sprague-Grundy function of exact k-Nim

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Abstract. Moore's generalization of the game of Nim is played as follows. Given two integer parameters n, k such that $1 \leq k \leq n$, and n piles of tokens. Two players take turns. By one move a player reduces at least one and at most k piles. The player who makes the last move wins. The $\mathcal{P}$-positions of this game were characterized by Moore in 1910 and an explicit formula for its Sprague-Grundy function was given by Jenkyns and Mayberry in 1980, for the case $n = k+1$ only. We modify Moore's game and introduce Exact k-Nim in which each move reduces exactly k piles. We give a simple polynomial algorithm computing the Sprague-Grundy function of Exact k-Nim in case $2k > n$ and an explicit formula for it in case $2k = n$. The last case shows a surprising similarity to Jenkyns and Mayberry's solution even though the meaning of some of the expressions are quite different.
Key words: Moore's Nim, exact Nim, impartial games, $\mathcal{P}$-position, Sprague-Grundy function.

Acknowledgements: Part of this research was done at the Mathematisches Forschungsinstitut Oberwolfach during a stay within the *Research in Pairs* Program in 2015. This research was partially supported by the Scientific Grant-in-Aid from Ministry of Education, Science, Sports and Culture of Japan. The first author also thanks the National Science Foundation (Grant IIS-1161476).

1 The Sprague-Grundy theory for the impartial games

We assume that the reader is familiar with basics of the Sprague-Grundy (SG) theory of the impartial games [6, 7, 10, 11]. In this paper we will need only two concepts: of a $\mathcal{P}$-position and of the SG function. A more detailed introduction to the SG theory (and much more) can be found, for example, in [2, 5].

1.1 Modeling impartial games by directed graphs

An impartial game is a two-player game with perfect information, in which the legal moves from each position are the same for both players and there are no moves of chance. Such a game is modeled by a directed graph (digraph) $\Gamma = (X, E)$; in which a vertex $x \in X$ is a *position*, while a directed edge $(x, x') \in E$ is a *legal move* (or simply a *move*, for short) from x to x'. We will also use notation $x \to x'$ and say in this case that x' is *legally and immediately reachable* (or simply *reachable*, for short) from x.

Digraph Γ may be infinite, but we will always assume that any sequence of successive moves from any fixed initial position, (a *play*) $x \to x', x' \to x'', \ldots,$ is finite. In particular, this implies that Γ has no directed cycles. Imagine that a token is initially placed in a position and two players alternate turns moving this token from its current vertex x to x' so that $x \to x'$ is a move. The game ends when the token reaches a *terminal*, that is, a vertex with no outgoing edges. Then, the player who makes the last move wins.

The simplest example for an impartial game is the ancient game of NIM played as follows. There are n piles containing $x_1, \ldots, x_n$ tokens. Two players alternate turns. By one move, a player chooses a pile $i \in \{1, \ldots, n\}$ and takes from it δ tokens; $0 < \delta \leq x_i$. The player who takes the last token wins.

1.2 Winning positions and moves

It is not difficult to characterize the winning strategies of an impartial game. The subset $\mathcal{P} \subseteq X$ is called the set of $\mathcal{P}$-positions if the following two properties hold:

(i) $\mathcal{P}$ is *independent*, that is, for any $x \in \mathcal{P}$ and move $x \to x'$ we have $x' \notin \mathcal{P}$;

(a) $\mathcal{P}$ is *absorbing*, that is, for any $x \notin \mathcal{P}$ there is a move $x \to x'$ such that $x' \in \mathcal{P}$.

It is easily seen that the set $\mathcal{P}$ can be obtained by the following simple recursive algorithm: include in $\mathcal{P}$ all terminal positions of Γ; delete from Γ all positions from which there is a move to a terminal position, together with all terminal positions, and repeat the above.

It is also clear that any move $x \to x'$ of a player to a $\mathcal{P}$*-position* $x' \in \mathcal{P}$ is a winning move. Indeed, by (i), the opponent must leave $\mathcal{P}$ by the next move, and then, by (a), the player can reenter $\mathcal{P}$. Since, by definition, all plays of Γ are finite and all terminals are in $\mathcal{P}$, sooner or later the opponent will be out of moves.

1.3 Sum of impartial games and the NIM-sum

Given two games Γ_1 and Γ_2, their sum $\Gamma_1 + \Gamma_2$ is played as follows: On each turn, a player chooses either Γ_1 or Γ_2 and plays in it, leaving the other game unchanged. The game ends when no move is possible, neither in Γ_1 nor in Γ_2. Obviously, this operation is commutative and associative and, hence, it allows us to define the sum $\Gamma_1 + \ldots + \Gamma_n$ of n summand games for any integer $n \geq 2$.

By these definitions, NIM can be viewed as the sum of n games, each of which (a one-pile-NIM) is trivial. Yet, NIM itself is not. It was solved by Charles Bouton in his seminal paper [3] as follows. The NIM-sum $x_1 \oplus \ldots \oplus x_n$ is defined as the bitwise binary sum. For example,

$$3 \oplus 5 = 011_2 \oplus 101_2 = 110_2 = 6, 3 \oplus 6 = 5, 5 \oplus 6 = 3, \text{ and } 3 \oplus 5 \oplus 6 = 0.$$

It was shown in [3] that $x = (x_1, \ldots, x_n)$ is a $\mathcal{P}$-position of NIM if and only if $x_1 \oplus \ldots \oplus x_n = 0$.

1.4 The Sprague-Grundy function and its basic properties

To play the sum $\Gamma = \Gamma_1 + \Gamma_2$, it is not sufficient to know $\mathcal{P}$-positions of Γ_1 and Γ_2 since $x = (x^1, x^2)$ may be a $\mathcal{P}$-position of Γ even when x^1 is not a $\mathcal{P}$-position of Γ_1 and x^2 is not a $\mathcal{P}$-position of Γ_2. For example, $x = (x_1, x_2)$ is a $\mathcal{P}$-position of the two-pile NIM if and only if $x_1 = x_2$, while only $x_1 = 0$ and $x_2 = 0$ are the unique $\mathcal{P}$-positions of the corresponding one-pile games. To play the sums one needs the concept of Sprague-Grundy (SG) function, which is a refinement of the concept of $\mathcal{P}$-positions.

Standardly, we denote by $\mathbb{Z}_\geq$ the set of nonnegative integers. Given a finite subset $S \subseteq \mathbb{Z}_\geq$, let $\text{mex}(S)$ (the minimum excluded value) be the smallest nonnegative integer that is not in S. In particular, $\text{mex}(\emptyset) = 0$, by the definition.

Given an impartial game $\Gamma = (X, E)$, the SG function $\mathcal{G} : X \to \mathbb{Z}_\geq$ is defined recursively, as follows: $\mathcal{G}(x) = 0$ for any terminal x and, in general, $\mathcal{G}(x) = \text{mex}(\{g(x') | x \to x'\})$.

This definition implies the following important properties of the SG function:

- No move keeps the SG value, that is, $\mathcal{G}(x) \neq \mathcal{G}(x')$ for any move $x \to x'$.
- The SG value can be arbitrarily (but strictly) reduced by a move, that is, for any integer v such that $0 \leq v < \mathcal{G}(x)$ there is a move $x \to x'$ such that $\mathcal{G}(x') = v$. In particular, the SG value can be reduced to 0 whenever it is not 0.
- The $\mathcal{P}$-positions are exactly the zeros of the SG function: $\mathcal{G}(x) = 0$ if and only if x is a $\mathcal{P}$-position.
- The SG function of the sum of n games is the NIM-sum of the n SG functions of the summands. More precisely, let $\Gamma = \Gamma_1 + \cdots + \Gamma_n$ be the sum of n games and $x = (x^1, \ldots, x^n)$ be a position of Γ, where x^i is the corresponding position of Γ_i for $i = 1, \ldots, n$, then $\mathcal{G}(x) = \mathcal{G}(x^1) \oplus \cdots \oplus \mathcal{G}(x^n)$.

A position x with $\mathcal{G}(x) = t$ will be called a *t-position*.

For example, the SG function of NIM is the NIM-sum of the cardinalities of its piles, that is, $\mathcal{G}(x) = x_1 \oplus \cdots \oplus x_n$ for $x = (x_1, \ldots, x_n)$.

1.5 Moore's k-NIM

In 1901, Charles Bouton published his seminal paper solving NIM with n piles. Soon after, in 1910, in the same journal, Eliakim Hastings Moore [9] suggested the following natural generalization for any $k \in \{1, \ldots, n\}$. In Moore's NIM a player by one move reduces at least one and at most k piles. More precisely, in a position $x^0 = (x_1, \ldots x_n)$ the player can choose any vector $x' = (x'_1, \ldots x'_n)$ such that $0 \leq x'_i \leq x^0_i$ for all i and $0 < x'_i$ for at most k and at least one $i \in \{1, \ldots, n\}$. Such a move results in the position $x^1 = (x^0_1 - x'_1, \ldots, x^0_n - x'_n)$. Two player alternate turns and one who has no moves loses, or in other words, one who takes the last token wins. Moore denoted this game NIM$_k$. In this paper, we will use NIM$^{\leq}_{n,k}$ for it.

Moore introduced the operation $\oplus_\ell$ and the function $M(x)$ generalizing the NIM-sum $\oplus$ and the SG function $\mathcal{G}(x)$ of NIM respectively as follows. We represent x_is as binary numbers and take their bitwise sum modulo $k+1$. The obtained sequence represents the number $M(x)$ in the base $k+1$ numeral system:

$$M(x) = x_1 \oplus_{k+1} \cdots \oplus_{k+1} x_n.$$

It is announced in [9] that x is a $\mathcal{P}$-position of NIM$^{\leq}_{n,k}$ if and only if $M(x) = 0$. This is a generalization of Bouton's result, which represents the case $k = 1$. Indeed, in this case the Moore function $M(x)$ turns into the SG function $\mathcal{G}(x)$ and the game NIM$^{\leq}_{n,1}$ becomes the standard n-pile NIM. Moore did not include the proof possibly because he considered it too simple for the Annals of Mathematics. Yet, computing the SG function (which was introduced only in 25 years later) is not simple at all. In his book [1], Claude Berge claimed that it is simply equal to $M(x)$ (Theorem 3, page 55 of [1]). However, Jenkyns and Mayberry [8] pointed out that this is an overstatement, although the claim holds when $M(x) \leq 1$ (see Theorems 11 and 12 on page 61 and also page 53 of [8]). In Section 5 we give an alternative proof for convenience of the reader.

The following examples show that the restriction $M(x) \leq 1$ is essential:

$2 = \mathcal{G}(x) < M(x) = 3$ for $k = 2, n = 3, x = (0, 0, 2)$;
$8 = \mathcal{G}(x) > M(x) = 2$ for $k = 2, n = 3, x = (2, 3, 3)$;
$2 = \mathcal{G}(x) < M(x) = 17$ for $k = 3, n = 4,\ x = (0, 3, 3, 4)$;
$2 = \mathcal{G}(x) < M(x) = 26$ for $k = 3, n = 4,\ x = (2, 2, 3, 3)$.

No explicit formula for the SG function of the game NIM$^{\leq}_{n,k}$ is known if $1 < k < n-1$, e.g., for NIM$^{\leq}_{4,2}$.

Obviously, NIM$^{\leq}_{n,k}$ is the standard n-pile NIM in case $k = 1$, while in case $k = n$ it turns into the 1-pile NIM, which is trivial. Another tractable case is $k = n-1$. If $n = 2$, we obtain the standard 2-pile NIM. For $n \geq 3$, the following explicit formula was obtained in [8] for the SG function. Given a position $x = (x_1, \ldots, x_n)$ of $\text{NIM}^{\leq}_{n,n-1}$, let us define

$$u(x) = \sum_{i=1}^{n} x_i, \quad m(x) = \min_{1 \leq i \leq n} x_i, \quad y(x) = u(x) - n\, m(x), \quad z(x) = \frac{1}{2}(y^2(x) + y(x) + 2). \quad (1)$$

The SG function of $\text{NIM}^{\leq}_{n,n-1}$ is given by the formula:

$$\mathcal{G}(x) \;=\; \begin{cases} u(x), & \text{if } m(x) < z(x); \\ v(x) = \big(z(x) - 1\big) + \Big(\big(m(x) - z(x)\big) \mod \big(y(x) + 1\big)\Big), & \text{otherwise.} \end{cases} \quad (2)$$

A slightly more general case was solved in [4].

2 Main results

2.1 Tetris functions

Given positive integers n and k, such that $n \geq k \geq 1$, and n-vector $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n_{\geq}$, the *tetris* function $T_{n,k}(x)$ is the largest cardinality of a set of binary n-vectors, with k 1's and $n - k$ 0's, such that the sum of these vectors is at most x (componentwise).

The following upper bound is immediate $T_{n,k}(x) \leq \lfloor \frac{1}{k} \sum_{i=1}^{n} x_i \rfloor$.

Equivalently, $T_{n,k}(x)$ can be defined as follows. Set $x^0 = x$, subtract 1 from the k largest coordinates of x^0 getting the next n-vector x^1, do the same for x^1 getting x^2, etc., as long as x^i remains non-negative. Let x^t be the last such n-vector, which contains more than $n - k$ zero coordinates. Then $T_{n,k}(x) = t$.

For example, $T_{3,2}(1, 2, 3) = 3$. In fact, the following subtraction sequence has length 3: $(1, 2, 3) \to (1, 1, 2) \to (1, 0, 1) \to (0, 0, 0)$ and moreover, the sum of entries in $(1, 2, 3)$ is exactly 6 which shows by the above trivial upper bound that $T_{3,2}(1, 2, 3)$ equals 3.

In Section 3 we will study the tetris function in more details. In particular, we will prove that the above two definitions are indeed equivalent and give a polynomial algorithm computing $T_{n,k}$. Note that both above definitions, if applied directly, provide only a pseudo-polynomial algorithm.

2.2 EXACT k-NIM

Given positive integers n and k such that $n \geq k \geq 1$, we study a restriction of Moore's k-NIM that we call EXACT k-NIM and denote by $\text{NIM}^{=}_{n,k}$ as follows. By one move a player chooses k piles and removes any positive number of tokens from each of them. The game terminates when more than $n - k$ piles are empty, and the player who made the last move is the winner. $\text{NIM}^{=}_{n,k}$ turns into the standard NIM when $k = 1$, and is the trivial one-pile NIM when $k = n$.

We are able to characterize the SG functions when $n \leq 2k$. We have different expressions for the cases $n < 2k$ (Theorem 1) and $n = 2k$ (Theorem 2).

Theorem 1 *If $n < 2k$, then the SG function of* $\text{NIM}^{=}_{n,k}$ *is given by* $\mathcal{G}(x) = T_{n,k}(x)$.

Let us note that the inequality $\mathcal{G}(x) \leq T_{n,k}(x)$ is obvious since by the definition, $T_{n,k}(x)$ is the length (the number of moves) of the longest play beginning in x. The inverse inequality will be proven in Section 3.

Theorem 2 *Let us associate to each position* $x = (x_1, \ldots, x_n)$ *of* $\text{NIM}^{=}_{2k,k}$ *the following quantities:*

$$u(x) = T_{2k,k}(x), \tag{3}$$
$$m(x) = \min_{1 \leq i \leq n} x_i, \tag{4}$$
$$y(x) = T_{2k,k}(x_1 - m(x), \ldots, x_n - m(x)), \tag{5}$$
$$z(x) = 1 + \binom{y(x)+1}{2}, \text{ and} \tag{6}$$
$$v(x) = (z(x) - 1) + ((m(x) - z(x)) \mod (y(x)+1)). \tag{7}$$

Then the SG function of $\text{NIM}^{=}_{2k,k}$ *is given by*

$$\mathcal{G}(x) \;=\; \begin{cases} u(x), & \text{if } \; m(x) < z(x); \\ v(x), & \text{if } \; m(x) \geq z(x). \end{cases} \tag{8}$$

The proof will be given in Section 4.

Now we can characterize the 0- and 1-positions of $\text{NIM}^{=}_{n,k}$. A position $x = (x_1, \ldots, x_n)$ is said to be *non-decreasing* if $x_1 \leq \cdots \leq x_n$.

Corollary 1 *Given a non-decreasing position* $x = (x_1, \ldots, x_{2k})$ *of the game* $\text{NIM}^{=}_{2k,k}$,

(i) *x is a $\mathcal{P}$-position if and only if more than half of its smallest coordinates are equal, that is,* $x_1 = \cdots = x_k = x_{k+1}$.

(ii) *x is a 1-position if and only if* $x_1 = \cdots = x_{k-\ell} = 2c$ *and* $x_{k-\ell+1} = \cdots = x_{k+\ell+1} = 2c+1$ *for some integer* $c \in \mathbb{Z}_{\geq}$ *and* $\ell \in \{0, 1, \ldots, k-1\}$.

Both statements (i) and (ii) follow from Theorem 2, but can also be derived much simpler, directly by the definitions.

So far we have not been able to characterize $\mathcal{P}$-positions of EXACT k-NIM for $1 < k < n/2$, e.g., for $\text{NIM}^{=}_{5,2}$.

The paper is organized as follows. In Section 3 we collect results about Tetris function. In Section 4 we prove the main theorem. In Section 5 we prove Theorem 3. In the last section we outline simple cases when the SG theory works for k-sums of impartial games.

3 More on tetris functions

Let us observe first some basic properties of tetris functions following directly from the definition.

Lemma 1 *Consider two integer vectors* $x, x' \in \mathbb{Z}^n_{\geq}$*. If* $x' \leq x$ *then* $T_{n,k}(x') \leq T_{n,k}(x)$*, and if in addition* $\sum_{i=1}^n (x_i - x'_i) = 1$*, then* $T_{n,k}(x) - 1 \leq T_{n,k}(x') \leq T_{n,k}(x)$*. Furthermore, if* $x \to x'$ *then* $T_{n,k}(x') < T_{n,k}(x)$*.* □

3.1 Two definitions of the tetris function are equivalent

Consider the game $\text{NIM}^{=}_{n,k}$. A move is called *slow* if exactly one token from each of the k chosen piles is taken. In Section 2.1 the tetris function $T_{n,k}(x)$ was defined as the maximum number of successive slow moves that can be made from x.

The following observation will be used several times.

Lemma 2 *Given a position* $x = (x_1, \ldots, x_n)$ *and* $i, j \in N$ *such that* $x_i < x_j$*, then for the position* $x' = (x'_1, \ldots, x'_n)$ *defined by*

$$x'_l = \begin{cases} x_i + 1, & \text{if } l = i, \\ x_j - 1, & \text{if } l = j, \\ x_l, & \text{otherwise,} \end{cases} \tag{9}$$

we have $T_{n,k}(x) \leq T_{n,k}(x')$*. In other words, the tetris function is not increasing when we move a token from a larger pile to a smaller one.*

Proof: Consider any sequence of slow moves from x resulting in x''. If $x''_j > 0$ then the same sequence of slow moves can be made from x' since $x'_l \geq x_l$ for $l \neq j$. If $x''_j = 0$ then since $x_j > x_i$, this sequence contains a slow move reducing x_j but not x_i. Let us modify this move reducing x_i rather than x_j and keeping all other moves of the sequence unchanged. The obtained sequence has the same length and consists of slow moves from x'. □

Notice that we can generalize Lemma 2 replacing ± 1 in (9) by $\pm\Delta$ for any integral $\Delta \in [0, x_j - x_i]$.

Lemma 3 *The slow move that reduces the* k *largest piles of* x *reduces the tetris value* $T_{n,k}(x)$ *by exactly one.*

Proof: Let x' be the position obtained from x by such a slow move, and let x'' be another position obtained by some slow move. By applying (9) repeatedly, we can obtain x' from x'' with $T_{n,k}(x'') \leq T_{n,k}(x')$ by Lemma 2. This implies that x' has the highest tetris value among all positions each reachable from x by a slow move. By Lemma 1, each slow move reduces the tetris value by at least one and there exists a slow move reducing it by exactly one. Hence, $T_{n,k}(x') = T_{n,k}(x) - 1$. □

This lemma immediately implies that the two definitions of the tetris function given in Section 2.1 are indeed equivalent. It also provides a pseudo-polynomial algorithm of calculating the tetris value by repeatedly decreasing by 1 each of the k currently largest piles until one of them becomes empty. The number of such reductions is the tetris value of x.

3.2 Proof of Theorem 1

By the definition, the tetris value is the largest number of moves one can take from a position x, implying that the SG value at x is at most $T_{n,k}(x)$. Therefore, it is enough to show that for all integral v such that $0 \leq \delta < T_{n,k}(x)$ there exists a move $x \to x'$ such that $T_{n,k}(x') = \delta$.

Consider the move $x \to x'$ that reduces the largest k piles to 0. For the resulting position we have $T_{n,k}(x') = 0$ because $2k > n$. Let us also consider the move $x \to x''$ that reduces the k largest piles each by only 1. Then we have $T_{n,k}(x'') = T_{n,k}(x) - 1$ according to Lemma 3. Any position between x' and x'' is reachable form x. Thus, by Lemma 1 the claim follows. □

3.3 Computing the tetris function in polynomial time

Given a non-decreasing position x, let us construct $\bar{x}$ from x by emptying the first $n-k$ piles and adding these $\sum_{i=1}^{n-k} x_i$ tokens, one by one, to the last k piles as follows. By each step add one token to the smallest of these k piles. If there are several such piles, break the tie by adding this token to the pile of the largest index, to keep the resulting x vector nondecreasing. It is easy to see that $T_{n,k}(\bar{x}) = \min(\bar{x}_i | n-k+1 \leq i \leq n) = \bar{x}_{n-k+1}$.

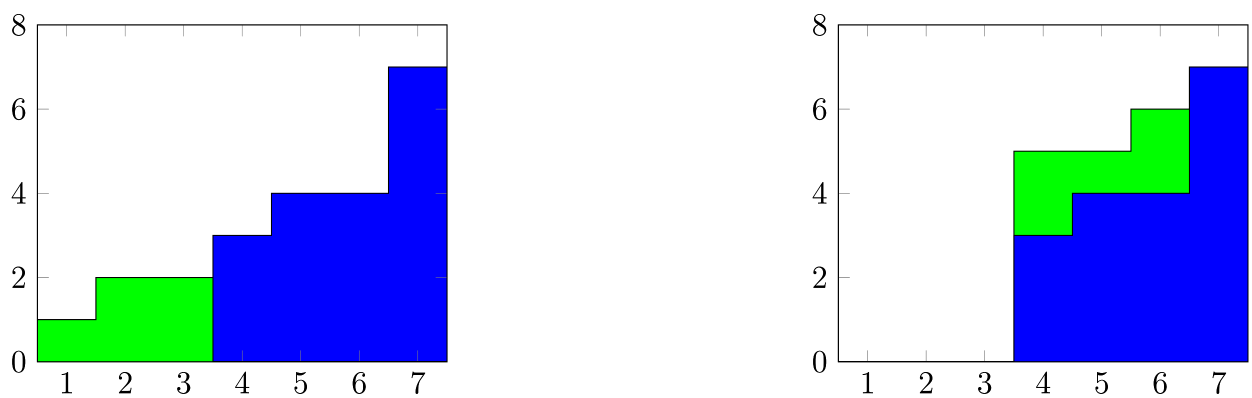

Figure 1: $k = 4, n = 7$, $x = (1, 2, 2, 3, 4, 4, 7)$, and $\bar{x} = (0, 0, 0, 5, 5, 6, 7)$.

Proposition 1 *The above construction keeps the tetris value, $T_{n,k}(\bar{x}) = T_{n,k}(x)$.*

Proof: Let us note that $T_{n,k}(\bar{x}) \leq T_{n,k}(x)$ by Lemma 2. By the definition of $\bar{x}$, none of tokens from the smallest $n-k$ piles of x is moved to any pile of size larger than $T_{n,k}(\bar{x})+1$ and so we have

$$V(x) := \sum_{i=1}^{n} \min(x_i, T_{n,k}(\bar{x})+1) = \sum_{i=n-k+1}^{n} \min(\bar{x}_i, T_{n,k}(\bar{x})+1). \tag{10}$$

Since $\bar{x}_{n-k+1} = T_{n,k}(\bar{x})$, we get by (10) that

$$V(x) \leq k - 1 + kT_{n,k}(\bar{x}) < k(T_{n,k}(\bar{x})+1).$$

Assume now indirectly that $T_{n,k}(\bar{x}) < T_{n,k}(x)$. Then it is possible to construct a sequence of $T_{n,k}(\bar{x})+1$ slow moves from x. By such sequence any pile would be reduced at most $T_{n,k}(\bar{x})+1$ times, and therefore the total number of the removed tokens is at most $V(x)$, implying $k(T_{n,k}(\bar{x})+1) \leq V(x)$, contradicting the above inequality. The obtained contradiction implies that $T_{n,k}(x) = T_{n,k}(\bar{x})$. □

Let us note that position $\bar{x}$ is defined above by a non-polynomial algorithm. However $\bar{x}$ and consequently $T_{n,k}(x) = \bar{x}_{n-k+1}$ can be computed in a more efficient way.

Proposition 2 *Given a nondecreasing position x, the corresponding $\bar{x}$ can be obtained in linear time in n.*

Proof: Let $s = \sum_{i=1}^{n-k} x_i$ be the number of tokens we shift on top of the largest k piles; see Figure 1. By the definition of $\bar{x}$ we know that for some $\ell < k$ the first $\ell+1$ columns of $\bar{x}$ will have almost the same number of tokens (at most one difference.) To determine this index ℓ and the height of the resulting piles, we use simple volume based arguments. We need to compute first the following parameters.

For each $i = 1, \ldots, k-1$, we denote by $y_i = x_{n-k+i+1} - x_{n-k+i}$ the difference of the sizes of consecutive piles. Set $s_0 = 0$, $s_k = \infty$, and for $i = 1, ..., k-1$, set $s_i = s_{i-1} + i \cdot y_i$ (i.e., the number of tokens we need to shift on top of the first i piles $(n-k+1), \ldots, (n-k+i)$ to make them all equal to $x_{n-k+1+i}$.) We define a unique ℓ by $s_\ell \leq s < s_{\ell+1}$. We define $a = s - s_\ell$, $\alpha = \lfloor \frac{a}{\ell+1} \rfloor$, and $\beta = a \mod (\ell+1)$. We fill up the first $\ell+1$ columns to level $x_{n-k+\ell+1}$ using s_ℓ tokens. Then, we place the remaining a tokens by increasing each of the first $\ell+1$ columns (indexed $n-k+1, ..., n-k+\ell+1$) by α and the last β of these by one more, as in the following expression.

$$\bar{x}_i = \begin{cases} 0, & \text{if } i = 1, \ldots, n-k; \\ x_{n-k+\ell+1} + \alpha, & \text{if } i = n-k+1, \ldots, n-k+1+\ell; \\ x_{n-k+\ell+1} + \alpha + 1, & \text{if } i = n-k+2+\ell-\beta, \ldots, n-k+1+\ell; \\ x_i, & \text{if } i = n-k+2+\ell, \ldots, n. \end{cases}$$

It is easy to see that this defines $\bar{x}$ correctly, and that all these parameters can be computed in $O(n)$ time, if x is a nondecreasing vector. □

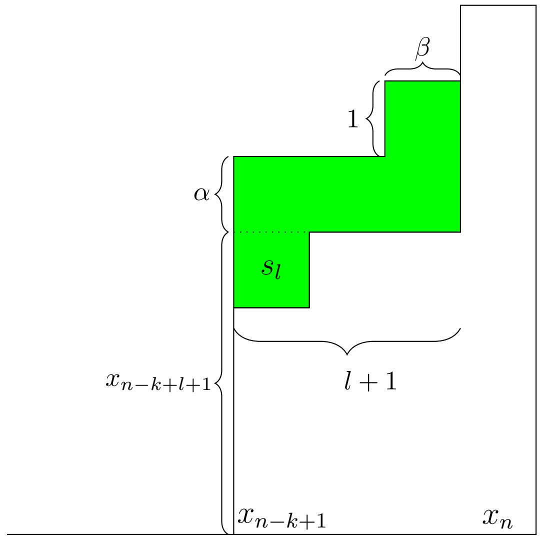

Figure 2: An example of calculating $\bar{x}$ for $x = (1, 2, 2, 3, 4, 4, 7)$ with $k = 4$.

Remark 1 *Technically, only the computation of $s = \sum_{i=1}^{n-k} x_i$ depends on n. All other computations in the previous proof can be done in $O(k)$ time.*

Proposition 1 has the following consequence that we need to use in the sequel several times.

Corollary 2 *Any position x such that*

$$k\delta \leq \sum_{i=1}^{n} \min(x_i, \delta) \quad \text{and} \quad \sum_{i=1}^{n} \min(x_i, \delta+1) < k(\delta+1) \tag{11}$$

has the tetris value of $T_{n,k}(x) = \delta$.

Proof: The sequence $g(\delta) = \frac{1}{\delta}\sum_{i=1}^{n} \min(x_i, \delta)$ is monotone non-increasing for $\delta \in \mathbb{Z}_{\geq}$, and hence, the inqualities (11) can hold for only one δ. Proposition 1 implies that (11) holds for $\delta = T_{n,k}(x) = T_{n,k}(\bar{x})$. □

3.4 Polynomial computation of a move to a given tetris value

Let us now consider moves from a given position x in the game $\text{Nim}^{=}_{n,k}$. Given an integer value δ, we are interested to find efficiently a move $x \to x'$ such that $T_{n,k}(x') = \delta$. First, let us note that this may not be possible for certain values of δ. By Lemma 2 the position x' that has the smallest such δ value is the one in which the largest k piles of x are reduced to zero. By Lemma 1 we can conclude that for every value $T_{n,k}(x') \leq \delta < T_{n,k}(x)$ there exists a move $x \to x''$ such that $x \geq x'' \geq x'$ and $T_{n,k}(x'') = \delta$ (see the proof of Theorem 1.)

Proposition 3 *Such an x'' can be determined in $O(n\log(\sum_{i=n-k+1}^n x_i))$ time.*

Proof: Let us start with $x^\ell = x'$ and x^u obtained from x by decreasing the largest k piles by one unit each. Then we have $T_{n,k}(x^\ell) \leq \delta \leq T_{n,k}(x^u)$. Using the monotonicity of the tetris function we perform a binary search in the space of positions between x^ℓ and x^u. In a general step we compute $L = \sum_{i=n-k+1}^n x_i^\ell$ and $U = \sum_{i=n-k+1}^n x_i^u$, set $M = \lfloor \frac{L+U}{2} \rfloor$ and compute $y_i = int(\frac{x_i^\ell + x_i^u}{2})$ for $i = n-k+1$, where $int(\cdot)$ is a rounding to a nearest integer value in such a way that $\sum_{i=n-k+1}^n y_i = M$. Finally, we set $y_i = x_i$ for $i < n-k+1$. If $T_{n,k}(y) < \delta$ then we replace x^ℓ by y, otherwise we replace x^u by y.

Clearly these computations can be done in each step in $O(n)$ time, and computing the tetris value of y can also be done in $O(n)$ time by Proposition 2. □

Remark 2 *Similarly to the proof of Proposition 2, we could improve the complexity of the above algorithm to $O(n)$.*

3.5 Tetris functions and degree sequences of graphs and hypergraphs

A related problem is the hypergraph realization of a given degree sequence. Let us fix $V = \{1, 2, ..., n\}$ as the set of vertices. A multi-hypergraph $\mathcal{H} = \{H_1, H_2, ..., H_m\}$ is a family of subsets (called hyperedges) of V, i.e., $H_j \subseteq V$ for all $j = 1, ..., m$. It is called k-uniform if $|H_j| = k$ for all $j = 1, ..., m$. The degree $d_{\mathcal{H}}(i)$ of a vertex $i \in V$ is the number of hyperedges H_j of $\mathcal{H}$ that contain i. We allow the same subset to appear multiple times in $\mathcal{H}$.

Given an integer vector $x \in \mathbb{Z}^n_{\geq}$, one can ask if there exists a k-uniform multi-hypergraph $\mathcal{H}$ on the vertex set V such that $d_{\mathcal{H}}(i) = x_i$ for all $i \in V$. Equivalently, we ask the existence of a bipartite graph $G = (X, Y, E)$ such that $|X| = n$, $|Y| = m$, $d_G(i) = x_i$ for all $i \in X$, and $d_G(j) = k$ for all $j \in Y$. For the latter we can apply the classical Gale-Ryser theorem claiming that the answer is yes if and only if

(*) $\sum_{i=1}^n x_i = km$

(**) $\sum_{i=1}^n \min(x_i, \delta) \geq k\delta$ for all $\delta = 1, ..., m$.

Let us note that checking these conditions may not be polynomial in x and k, since $m = \sum_{i=1}^n x_i/k$ according to (*). Let us also note that following a sequence of slow moves starting from position x, each time the set of columns that we decrease by 1 can be considered as a hyperedge of $\mathcal{H}$. Thus a maximal sequence of slow moves will construct $\mathcal{H}$ if the tetris function achieves its trivial bound $kT_{n,k}(x) = \sum_{i=1}^n x_i$. This equality is in fact equivalent with (*), since we must have $m = T_{n,k}(x)$ in this case. Our results in this section thus prove that for the above degree sequence realization problems the most efficient answer is to compute the tetris function value in linear time, and then compare it to its trivial upper bound. If these are the same then the answer is yes.

Havel (1955) and Hakimi (1962) provided a simple greedy algorithm based characterization for the recognition of degree sequences of bipartite graphs. For the above case their criterion states that x is a degree sequence of a k-uniform multi-hypergraph if and only if the position x' is, where x' is obtained from x by decreasing the k largest components of x by 1. Note that this implies a recursive process that is one of the definitions we used for the tetris function.

Let us remark finally that in general x is not the degree sequence of a k-uniform multi-hypergraph. In this case however a move $x \to x'$ such that $T_{n,k}(x') = 0$ provides us with a minimal modification such that $x'' = x - x'$ becomes the degree sequence of such a hypergraph.

4 SG function in case of $n = 2k$

Let us consider the game $\text{NIM}^{=}_{2k,k}$, where $k \geq 2$, and let x be a position in this game. Recall that to x we associated several parameters in (3)–(7).

Let us call a position x of type I, if $m(x) < z(x)$, and type II otherwise.

We shall need a few simple lemmas in our proof. In the sequel all positions considered will be positions of $\text{NIM}^{=}_{2k,k}$, even if not stated.

Lemma 4 *Consider a position x and two moves $x \to x'$ and $x \to x''$ such that $x' \geq x''$ (componentwise) and $y(x') \geq y(x'')$. Then, for every integer $y(x') \geq \eta \geq y(x'')$ there exists a move $x \to x'''$ such that $y(x''') = \eta$. Furthermore, we have $x' \geq x''' \geq x''$ componentwise.*

Proof: Note that $x' \geq x''$ implies that in the two moves $x \to x'$ and $x \to x''$ the same k components are decreased. Let us denote by $K \subseteq \{1, 2, ..., 2k\}$ these components. Let us now start decreasing the components x'_j, $j \in K$ one by one, keeping their values always above the corresponding x''_j values. In $\sum_{j \in K} x'_j - x''_j$ steps we arrive to x''. In each of these steps the corresponding y-value can only decrease, and by at most 1. Hence there will be at least such position $x' \geq x''' \geq x''$ with $y(x''') = \delta$. It also follows that $x \to x'''$ is a move, completing the proof. □

Let us also note that in fact z defined in (6) depends uniquely on $y(x)$ and hence can be considered as a function of y. We shall also need the following easy arithmetical facts.

Lemma 5 *Any nonnegative integer δ belongs to exactly one of the intervals*

$$[z(y) - 1, z(y) + y - 1] = \left[\binom{y+1}{2}, \binom{y+2}{2} - 1 \right] \text{ for } y \in \{0, 1, 2, ...\}.$$

□

Corollary 3 *For every nonnegative integer δ there exist unique $\nu(\delta)$ and $\epsilon(\delta)$ integer values such that*

$$\delta \ = \ \binom{\nu(\delta)+1}{2} + \epsilon(\delta) \quad \textit{and} \quad 0 \leq \epsilon(\delta) \leq \nu(\delta).$$

Proof of Theorem 2. Our main result claims that the SG function $\mathcal{G}(x)$ of $\text{Nim}^{=}_{2k,k}$ for $k \geq 2$ is equal to the function

$$g(x) = \begin{cases} u(x), & \text{if } m(x) < z(x); \\ v(x), & \text{if } m(x) \geq z(x); \end{cases}$$

where u, m, y, v, and z are defined in (3)–(7). To prove this theorem, it is enough to show the following two properties of $g(x)$:

(I) for any move $x \to x'$ we have $g(x) \neq g(x')$ and

(II) for any $0 \leq \delta < g(x)$ there exists a move $x \to x'$ with $g(x') = \delta$.

4.1 Proof of (I)

We now prove property (I). Let us first consider the case $z(x) > m(x)$. Obviously, any move $x \to x'$ reduces the tetris value by at least 1, implying $u(x) > u(x')$. Using this and the definitions, we get $g(x) = u(x) > u(x') \geq 2m(x')$, where the last inequality follows by the fact that x' has an $m(x') \times 2k$ block. Thus, we get both $g(x) > u(x')$ and $g(x) \geq m(x') - 1 \geq v(x')$, implying the claim since $g(x')$ is in the set $\{u(x'), v(x')\}$.

It remains to consider the case $z(x) \leq m(x)$, in which case $v(x) \leq m(x) - 1$ follows by the definitions.

(1) Suppose $z(x') > m(x')$. We can estimate $u(x') \geq m(x) + m(x')$ since $2k \leq n$. Furthermore, $g(x) = v(x) \leq m(x) - 1$ and thus $g(x') = u(x') \geq m(x) + m(x') > m(x) - 1 \geq g(x)$.

(2) Suppose $z(x') \leq m(x')$. Then $g(x') = v(x')$. Note that $z(x) \leq m(x)$ and so $g(x) = v(x)$. Also note that $m(x') \leq m(x)$. We examine the last inequality.

  (a) Suppose $m(x') = m(x)$. Then $y(x) > y(x')$ and $m(x') = m(x) \geq z(x) > z(x')$, where the last inequality follows by the fact that if $m(x) = m(x')$, then $x - m(x) \to x' - m(x')$ is a move, and hence decreases the tetris value implying $y(x) > y(x')$. By Lemma 5, we have $z(x') + y(x') - 1 < z(x) - 1$ and so $z(x) > z(x') + y(x')$ implying $v(x) \geq z(x) - 1 > z(x') - 1 + y(x') \geq y(x') \geq v(x')$.

  (b) Suppose $m(x') < m(x)$. We compare $y(x)$ with $y(x')$.

    (i). If $y(x) = y(x')$ then $z(x) = z(x')$. By the definition of a legal move, x' has at least k piles not smaller than $m(x)$. Therefore, $1 \leq m(x) - m(x') \leq y(x') = y(x)$, which implies that $m(x) \mod (y(x) + 1) \neq m(x') \mod (y(x) + 1)$ and thus $v(x) \neq v(x')$.

    (ii). If $y(x) \neq y(x')$, since the y values are different, by Lemma 5 we have that $v(x)$ and $v(x')$ are in different intervals, therefore $v(x) \neq v(x')$.

4.2 Proof of (II)

We prove property (II) by considering type I and type II positions, separately.

4.2.1 Type I positions: $m(x) < z(x)$

First, let us consider the case $m(x) = 0$. Then there are at most $2k-1$ nonempty piles. By Lemma 3 we can reduce the tetris value $T_{2k,k}(x)$ to 0 by emptying the k largest piles of x. Therefore for any $0 \le \delta < T_{2k,k}(x)$, there exists a move $x \to x'$ with $T_{2k,k}(x') = \delta$. All such moves also have $m(x') = 0$, therefore $m(x') < z(x')$.

From now on we can assume that $m(x) \ge 1$. Let us consider the following four subcases, depending on the value δ.

(1) $0 \le \delta < m(x)$. To simplify our proof, let us use simply m instead of $m(x)$ in this section. Let us observe first that since x is type I we have $m < z(x)$ implying by (6) that

$$y(x) > \nu(m-1) \ge 0. \tag{12}$$

Let us next define a set $Q = Q(x)$ of pairs of integers by setting

$$Q^1(x) = \left\{ (\mu, \eta) \;\middle|\; \begin{array}{c} m - \eta \le \mu \le m \\ \eta \le \nu(m-1) - 1 \end{array} \right\}$$

$$Q^2(x) = \left\{ (\mu, \eta) \;\middle|\; \begin{array}{c} m - \epsilon(m-1) \le \mu \le m \\ \eta = \nu(m-1) \end{array} \right\},$$

and defining $Q = Q^1(x) \cup Q^2(x)$.

We show that if for a position x^* we have $(m(x^*), y(x^*)) \in Q$, then x^* is of type II. To see this consider a pair $(\mu, \eta) \in Q^1(x)$. Then we have by the definition of $Q^1(x)$ that $\mu \ge m - \eta$ and that $\nu(m-1) - 1 \ge \eta$ from which $z(\nu(m-1)-1) \ge z(\eta)$ follows by the definition of z in (6). We also have the inequality $m - \nu(m-1) + 1 \ge z(\nu(m-1)-1)$ since $m \ge \binom{\nu(m-1)+1}{2}$ by the definition of ν in Corollary 3. Putting these together, we obtain $\mu \ge z(\eta)$ as stated. For $(\mu, \eta) \in Q^2(x)$ we have $\mu \ge m - \epsilon(m-1)$ and $\eta = \nu(m-1)$ by the definition of $Q^2(x)$. Since $m - 1 = \binom{\nu(m-1)+1}{2} + \epsilon(m-1)$ by Corollary 3, the inequality $\mu \ge m - \epsilon(m-1) = z(\eta)$ follows again.

We show next that for all pairs $(\mu, \eta) \in Q$ there exists a move $x \to x^*$ such that $\mu = m(x^*)$ and $\eta = y(x^*)$ (and x^* is of type II, as we argued in the previous paragraph.) For this let us consider first $\mu \le m - 1$ and note that $(\mu, \eta) \in Q$ if and only if $m - \mu \le \eta \le \nu(m-1)$. Our plan is to use Lemma 4 and to cover this range of η values by two constructions.

Let us define a pair of positions $x' \ge x''$ by

$$x'_i = \begin{cases} x_i & \text{for } i = 1 \text{ and } i \ge k+2, \\ \mu & \text{for } i = 2, \\ x_i - 1 & \text{for } i = 3, ..., k+1 \end{cases} \quad \text{and } x''_i = \begin{cases} x_i & \text{for } i = 1 \text{ and } i \ge k+2, \\ \mu & \text{for } i = 2, ..., k+1. \end{cases}$$

Note that since $\mu < m = x_1 \leq x_2 \leq x_i$ for $i \geq 3$, both x' and x'' are reachable form x. We claim next that $y(x'') = m - \mu$ and $y(x') \geq \nu(m-1)$. The first claim follows easily, since in x'' we have exactly k positions larger than μ, and $x_1 = m = \mu + (m - \mu)$. For the second let us note that since $x_1 - \mu = m - \mu \geq 1$ we have $y(x') \geq x_{k+1} - \mu$. We can now apply Lemma 4 for x' and x'', and conclude that for all values $m - \mu \leq \eta \leq x_{k+1} - \mu \leq y(x')$ there exists a move $x \to x^*$ such that $m(x^*) = \mu$ and $y(x^*) = \eta$. For larger values of η we need a modified construction:

$$x'_i = \begin{cases} \mu & \text{for } i = 1, \\ x_i - 1 & \text{for } i = 2, ..., k, \\ x_i & \text{for } i = k+1, ..., 2k, \end{cases} \quad \text{and } x''_i = \begin{cases} \mu & \text{for } i = 1, ..., k, \\ x_i & \text{for } i = k+1, ..., 2k. \end{cases}$$

Since $\mu < m$, we have $y(x') \geq y(x) \geq \nu(m) \geq \nu(m-1)$, while $y(x'') = x_{k+1} - \mu$. We have again $m(x') = m(x'') = \mu$. Thus by applying Lemma 4 for x' and x'' we can conclude that for all values $x_{k+1} - \mu \leq \eta \leq \nu(m-1)$ there exists a move $x \to x^*$ such that $m(x^*) = \mu$ and $y(x^*) = \eta$.

Finally, for $\mu = m$ we proceed analogously, but with a third construction. Note first that $(m, \eta) \in Q$ if and only if $0 \leq \eta \leq \nu(m-1)$. Let us now proceed with constructing two positions reachable form x:

$$x'_i = \begin{cases} x_i & \text{for } i = 1, ..., k, \\ x_i - 1 & \text{for } i = k+1, ..., 2k \end{cases} \quad \text{and } x''_i = \begin{cases} x_i & \text{for } i = 1, ..., k, \\ m & \text{for } i = k+1, ..., 2k. \end{cases}$$

It is easy to see that both are reachable form x, and that $m(x') = m(x'') = m$, $y(x') \geq y(x) - 1 \geq \nu(m-1)$ by (12) and $y(x'') = 0$. Thus, the existence of an x^* reachable from x with $m(x^*) = m$, $y(x^*) = \eta$ follows by Lemma 4 for all $0 \leq \eta \leq \nu(m-1)$.

By the above arguments we have a move $x \to x^*$ to a type II position x^* with $m(x^*) = \mu$ and $y(x^*) = \eta$ for all $(\mu, \eta) \in Q$. To conclude the proof for this case we claim that the corresponding $v(x^*)$ values include all integers in the interval $[0, m-1]0$. Note that $v(x^*)$ depends only on $m(x^*)$ and $y(x^*)$ for type II positions by (7), and that for a fixed value of $\eta \leq \nu(m-1) - 1$ we have exactly $\eta + 1$ consecutive integer values for μ such that $(\mu, \eta) \in Q$, implying that the corresponding values $v(\mu, \eta) = (z(\eta) - 1 + ((\mu - z(\eta)) \bmod (\eta + 1))$ is exactly the set of integers in the interval $[z(\eta) - 1, z(\eta) + \nu(\eta) - 1]$. Thus our claim follows by the construction of Q and by Lemma 5.

(2) $m(x) \leq \delta < x_2$ (Figure 3). Set $x'_i = 0$ for $i = k+1, \ldots, n-1$ and $x'_n = \delta - x_1$.

By definition $x_n \geq x_2 > \delta \geq x_1$ which implies $x'_n = \delta - x_1 < x_n - x_1 \leq x_n$. Furthermore, we have $x_i \geq x_2 > 0$ for $i = k+1, ..., n-1$, and therefore we indeed decrease exactly k piles of x to obtain x'. Thus x' is reachable from x.

For x' we have

$$\sum_{i=1}^{2k} \min(x'_i, \delta) \geq x'_1 + (k-1)\delta + (\delta - x'_1) = k\delta \text{ and}$$

$$\sum_{i=1}^{2k} \min(x'_i, \delta+1) \leq x'_1 + (k-1)(\delta+1) + (\delta - x_1) \ = \ \delta + (k-1)(\delta+1) < k(\delta+1).$$

Therefore, by Corollary 2, we have $T_{2k,k}(x') = \delta$.

Since $k \geq 2$, $x'_{k+1} = 0$ and therefore $m(x') = 0$. Thus, $m(x') = 0 < 1 \leq z(x')$ implying that x' is of type I, from which $g(x') = u(x') = \delta$ follows by the above.

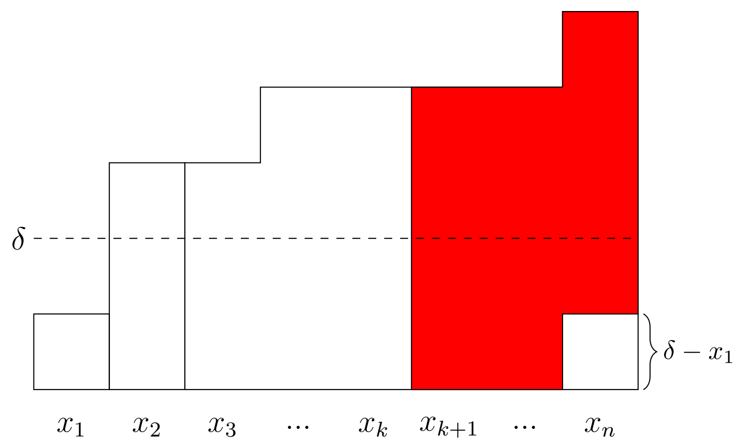

Figure 3: x' is obtained by removing the red area.

(3) $x_2 \leq \delta < u(x) - m(x)$ (Figure 4). Let $A = \sum_{i=2}^{k+1} \max(0, \delta - x_i)$. For $i = k+2, \ldots, n$ choose a_i such that $0 \leq a_i \leq \min(x_i - 1, \delta)$ and $\sum_{i=k+2}^{n} a_i = A$. First let us prove that this is possible, or equivalently that $A \leq \sum_{i=k+2}^{n} \min(\delta, x_i - 1)$. To see this let us consider two cases. If $x_{k+1} > \delta$ then $\sum_{i=k+2}^{n} \min(\delta, x_i - 1) \geq (k-1)\delta \geq A$. If $x_{k+1} \leq \delta$ then let us define

$$B = \sum_{i=2}^{k+1} x_i,$$

and observe that for any integer $\tau \geq \delta > x_{k+1}$ we have

$$\sum_{i=1}^{n} \min(\tau, x_i) = m + B + \sum_{i=k+2}^{n} \min(\tau, x_i).$$

Consequently, since we have $T_{2k,k}(x) = u(x)$, by Corollary 2 we can write for $\tau = u(x)$ that

$$k\tau \leq \sum_{i=1}^{n} \min(\tau, x_i) = m(x) + B + \sum_{i=k+2}^{n} \min(\tau, x_i).$$

Note that if we decrease $\tau = u(x)$ by 1, then the left hand side decreases by k while the right hand side decreases by at most $k-1$, hence the inequality remains valid. Let us repeat this $m(x)$ times, obtaining the inequality

$$k(u(x) - m(x)) + km(x) \leq m(x) + B + \sum_{i=k+2}^{n} \min(u(x) - m(x), x_i) + (k-1)m(x)$$

from which

$$k(u(x) - m(x)) \leq B + \sum_{i=k+2}^{n} \min(u(x) - m(x), x_i)$$

follows. Let us now decrease $\tau = u(x) - m(x)$ further by 1, as well as replace x_i by $x_i - 1$. Then the left hand side decreases by exactly k, while the right hand side decreases by at most k, yielding the valid inequality

$$k(u(x) - m(x) - 1) \leq B + \sum_{i=k+2}^{n} \min(u(x) - m(x) - 1, x_i - 1).$$

Finally, we can decrease $\tau = u(x) - m(x) - 1$ further on both sides to $\tau = \delta$ and similarly to the above argument obtain

$$k\delta \leq B + \sum_{i=k+2}^{n} \min(\delta, x_i - 1).$$

Noting that $B = k\delta - A$, we obtain the claimed inequality, and hence the proof for the existence of the a_i values for $k = k+2, ..., n$ that satisfy the desired inequalities.

Let us now consider the position x' defined by

$$x'_i = \begin{cases} 0, & \text{for } i = 1; \\ x_i, & \text{for } i = 2, \ldots, k+1; \\ a_i, & \text{for } i = k+2, \ldots, n. \end{cases}$$

By the above arguments $x \to x'$ is a move in the game. The equality $g(x') = u(x') = \delta$ now follows by the above analysis and Corollary 2, completing our proof in this case.

(4) $u(x) - m(x) \leq \delta < u(x)$ (Figure 5). Let us define position x' as follow

$$x'_i = \begin{cases} x_i - u(x) + \delta, & \text{for } i \in I_1 = \{i \mid i \leq k, x_{i+k} \leq \delta\}; \\ x_i - u(x) + \delta, & \text{for } i \in I_2 = \{i \mid i > k, \delta < x_i \leq u(x)\}; \\ \delta, & \text{for } i \in I_3 = \{i \mid i > k, x_i > u(x)\}; \\ x_i, & \text{otherwise.} \end{cases}$$

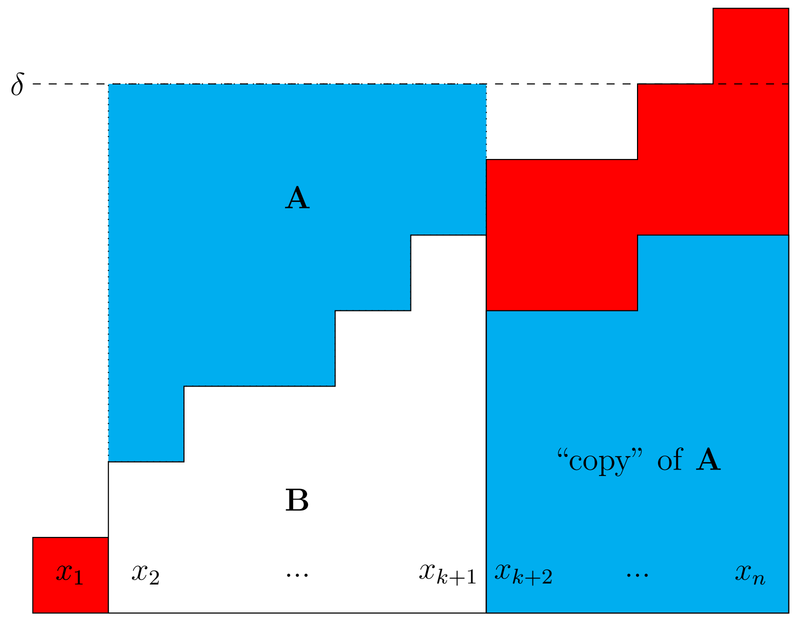

Figure 4: x' is obtained by removing the red area.

Note that $x_i \geq m(x) \geq u(x) - \delta$, therefore x'_i are all non-negative. It is easy to see that I_1+k, I_2, I_3 form a partition of $\{k+1, \ldots, n\}$. We have reduced x_i for all $i \in I_1 \cup I_2 \cup I_3$, therefore $x \to x'$ is a move.

Next we note that the above construction implies that

$$\sum_{i=1}^{n} \min(x'_i, \delta) \geq \sum_{i=1}^{n} \min(x_i, u(x)) - k(u(x) - \delta) \geq k\delta$$

where the last inequality follows by the fact that $u(x) = T_{2k,k}(x)$, and hence $\sum_{i=1}^{n} \min(x_i, u(x)) \geq ku(x)$ by Corollary 2. Similarly, we get

$$\sum_{i=1}^{n} \min(x'_i, \delta + 1) \leq \sum_{i=1}^{n} \min(x_i, u(x) + 1) - k(u(x) - \delta) < k(\delta + 1)$$

since $\sum_{i=1}^{n} \min(x_i, u(x) + 1) < k(u(x) + 1)$. Therefore $T_{n,k}(x') = u(x') = \delta$ follows by Corollary 2.

Note that $x_{k+1} \leq \delta$, since otherwise $u(x) \geq \delta + m(x)$ would follow, contradicting our choice of δ. It follows that $x_1 \in I_1$ and thus $m(x') = m(x) - u(x) + \delta < m(x)$.

By Lemma 2, we have $y(x) = y(\min(x, u(x)))$ and by construction of x' we have $\min(x, u(x)) - m(x)\mathbf{1} \leq x' - m(x')\mathbf{1}$, implying $y(x) = y(\min(x, u(x))) \leq y(x')$ by Lemma 1.

Finally, $m(x') = m(x) - u(x) + \delta < m(x) < z(x) \leq z(x')$, which implies $g(x') = u(x') = \delta$. $\square$

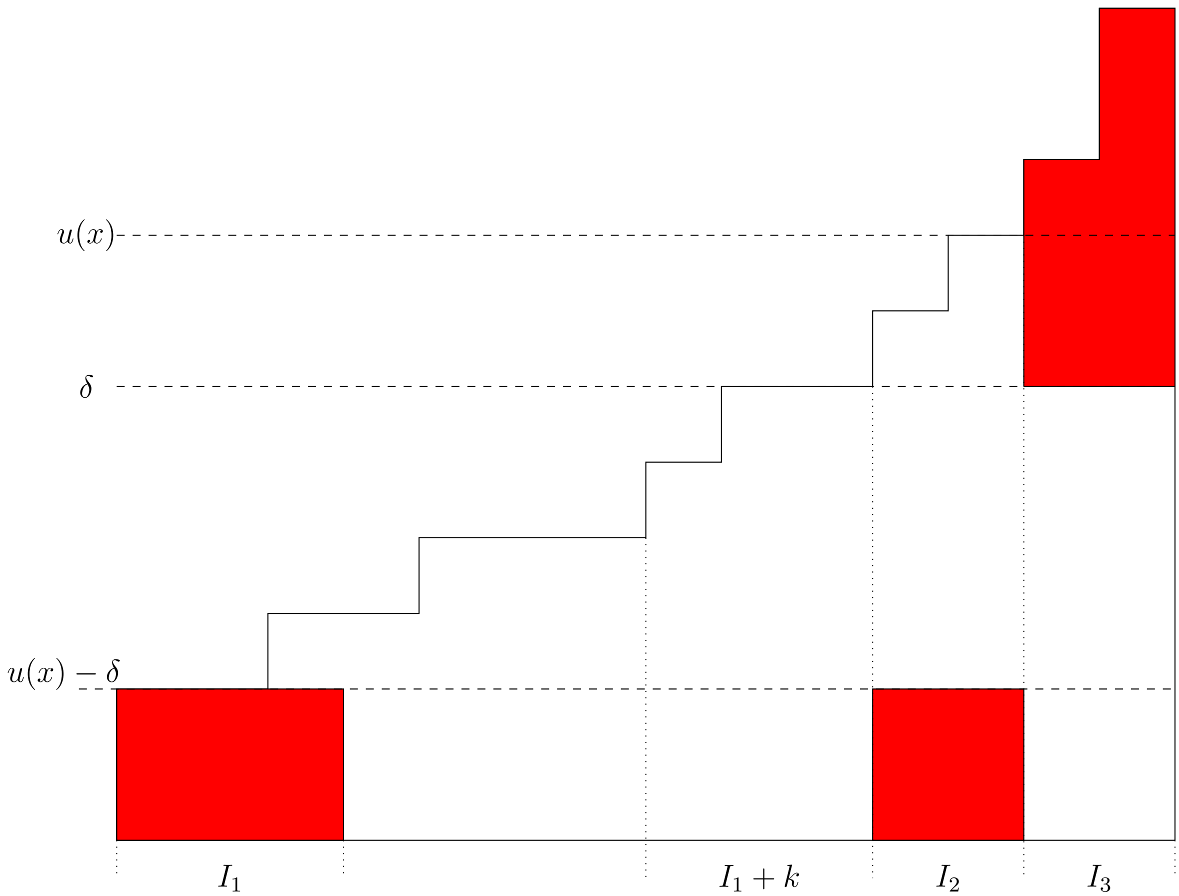

Figure 5: x' is obtained by removing the red area.

4.2.2 Type II positions: $m(x) \geq z(x)$

For a position x let

$$D_1(x) = \{(m, y) \mid 0 \leq m \leq m(x), m(x) - m \leq y < y(x)\}, \tag{13}$$
$$D_2(x) = \{(m, y) \mid m(x) - v(x) + z(x) - 1 \leq m < m(x), y = y(x)\}, \tag{14}$$
$$D(x) = D_1(x) \cup D_2(x). \tag{15}$$

Note that $D_2(x) = \emptyset$ whenever $(m(x) - z(x)) \mod (y(x) + 1)$ is zero.

Lemma 6 *For every x and for every $(m, y) \in D(x)$ there is a position x' reachable from x such that $m(x') = m$ and $y(x') = y$.*

Proof: For $(m, y) \in D(x)$ let us consider the following three cases.

Case 1: $0 \le m < m(x)$ and $x_2 - m \le y \le y(x)$. We consider two positions $\bar{x}$ and $\hat{x}$ reachable from x defined as follows

$$\bar{x}_i = \begin{cases} m, & \text{if either } i = 1 \text{ or } i \ge k+2; \\ x_i, & \text{otherwise;} \end{cases} \qquad \hat{x}_i = \begin{cases} m, & \text{if } i = 1; \\ x_i, & \text{if } i = 2, \dots, k+1; \\ \min(x_i, m(x) + y(x)) - 1, & \text{if } i \ge k+2; \end{cases}$$

Since we have $y(\bar{x}) = x_2 - m$ and $y(\hat{x}) \ge y(x) - 1 + m(x) - m \ge y(x)$, we can apply Lemma 4 and obtain that for all values $x_2 - m \le y \le y(x)$ there exists a position x' such that $\bar{x} \le x' \le \hat{x}$ and $y(x') = y$. All these positions have $m(x') = m$.

Let us also note that all these positions x' are reachable from x, because exactly k components of x are decreased.

Case 2: $0 \le m < m(x)$ and $m(x) - m \le y < x_2 - m$.

We consider two positions $\bar{x}$ and $\hat{x}$ reachable from x defined as

$$\bar{x}_i = \begin{cases} m, & \text{if } i = 2 \text{ or } i \ge k+2; \\ x_i, & \text{otherwise;} \end{cases} \qquad \hat{x}_i = \begin{cases} m, & \text{if } i = 2; \\ x_i, & \text{if } i = 1 \text{ or } i = 3, 4, \dots, k+1; \\ \min(x_i, m(x) + y(x)) - 1, & \text{if } i \ge k+2. \end{cases}$$

Since we have $y(\bar{x}) = m(x) - m$ and $y(\hat{x}) \ge x_{k+1} - m - 1 \ge x_2 - m - 1$, we can apply Lemma 4, and obtain that for all $y \in [m(x) - m, x_2 - m - 1]$ there exists a position $x' \in [\bar{x}, \hat{x}]$ such that $y = y(x')$.

Case 3: $m = m(x)$ and $0 \le y < y(x)$. Let us note that the last inequalitiy implies $y(x) \ge 1$ and therefore $x_{k+1} > m(x)$. We consider two positions $\bar{x}$ and $\hat{x}$ reachable from x defined as

$$\bar{x}_i = \begin{cases} x_i, & \text{if } i = 1, \dots, k; \\ m(x), & \text{if } i \ge k+1; \end{cases} \qquad \hat{x}_i = \begin{cases} x_i, & \text{if } i = 1, \dots, k; \\ x_i - 1, & \text{if } i \ge k+1. \end{cases}$$

Similarly to the previous cases, we have $y(\bar{x}) = 0$ and $y(\hat{x}) = y(x) - 1$, and thus by Lemma 4 it follows that for all $y \in [0, y(x) - 1]$ there exists an $x' \in [\bar{x}, \hat{x}]$ such that $y = y(x')$.

If we put all three cases together we cover all values $(m, y) \in D$. $\square$

Let us set

$$v(m, y) = \binom{y+1}{2} + \left[\left(m - 1 - \binom{y+1}{2}\right) \mod (y+1)\right]. \tag{16}$$

Note that if $m = m(x)$ and $y = y(x)$ then $v(m, y) = v(x)$. Furthermore, we have

$$V(y) := \{v(m, y) | m \in \mathbb{Z}_{\ge}\} = \left[\binom{y+1}{2}, \binom{y+2}{2}\right).$$

Therefore the sets $V(y)$, $y \in \mathbb{Z}_{\ge}$ form a partition of $\mathbb{Z}_{\ge}$, as shown in Lemma 5.

Lemma 7 *If $m(x) \geq z(x)$, then every $(m, y) \in D(x)$ satisfies the following relations*

$$m \geq \binom{y+1}{2} + 1 \quad \text{and} \quad \{v(m,y) \mid (m,y) \in D(x)\} = [0, v(x)).$$

Proof: Let us first consider $(m, y) \in D_1(x)$. By the definition of D_1 and the assumption of $m(x) \geq z(x) = \binom{y(x)+1}{2} + 1$ we get

$$m \geq m(x) - y \geq \binom{y(x)+1}{2} + 1 - y \geq \binom{y+2}{2} + 1 - y \geq \binom{y+1}{2} + 1.$$

By the definition of $D_1(x)$, for all $y \in [0, y(x))$ we have $(m, y) \in D_1(x)$ for all $m \in [m(x) - y, m(x)]$. Hence, by (16) we have $\{v(m, y) \mid m \in [m(x) - y, m(x)]\} = V(y)$. Thus, by Lemma 5 we get

$$\{v(m,y) \mid (m,y) \in D_1(x)\} = \bigcup_{y \in [0, y(x))} V(y) = [0, z(x) - 1).$$

Let us next consider $(m, y) \in D_2(x)$. By the definition of $v(x)$ we can write $v(x) = z(x) - 1 + r$, where $r = m(x) - z(x) - \lambda(y(x)+1)$ for some $\lambda \in \mathbb{Z}_{\geq}$. This implies $m(x) - r \geq z(x)$. By the definition of $D_2(x)$ we have $m \geq m(x) - r$. These two inequalities imply $m \geq \binom{y+1}{2} + 1$. Since $m \in [m(x) - r, m(x))$ takes r consequtive values, we have $\{v(m, y) \mid (m, y) \in D_2(x)\} = [z(x) - 1, v(x))$. □

The above lemma implies that any position x' with $m(x') = m$, $y(x') = y$ for some $(m, y) \in D$ is a type II position. Hence $g(x') = v(x') = v(m, y)$. Thus the second claim in the above lemma together with Lemma 6 implies that for any $0 \leq \delta < v(x)$ there exists a move $x \to x'$ such that $g(x') = \delta$.

Since we proved this for both type I and type II positions we concluded the proof of (II).

Properties (I) and (II) together now imply that $\mathcal{G} = g$. This concludes the proof of Theorem 2. □

Remark 3 *Jenkyns and Mayberry [8] introduced the concept of skeleton of an impartial game, and applied it to determine the SG function of Moore's game* $\text{NIM}^{\leq}_{n,n-1}$. *Their proof is based on characterizing the skeleton in terms of two integer parameters $m(x)$ and $y(x)$ as defined in (2). Our proof may suggest that a similar skeleton based approach may work in terms of $u(x)$, $m(x)$ and $y(x)$ as defined in Theorem 2. The following example however shows that this is not possible.*

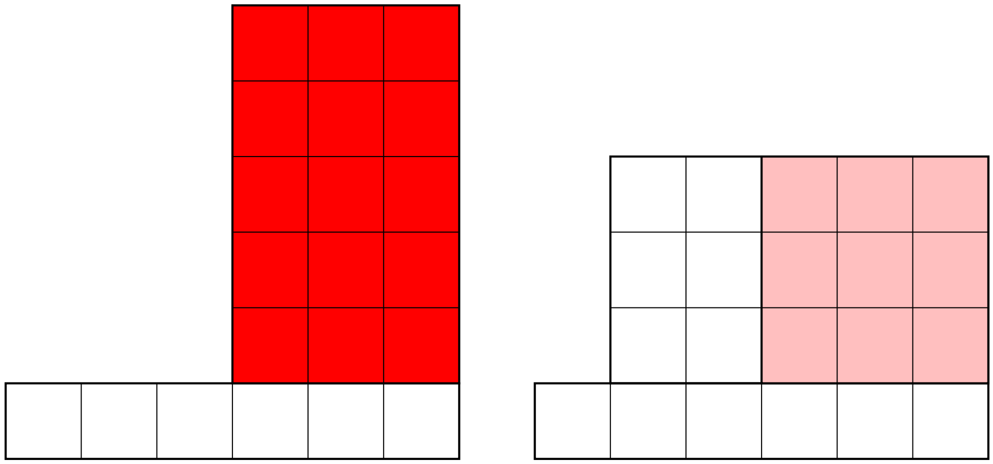

Figure 6: Case $n = 6, k = 3$. Both above positions have $m = 1, u = 7, y = 5$; but the position obtained by removing the red area has $m = 1, u = 2, y = 0$ and such a position cannot be obtained by the picture on the right since $m = 1$ implies that $u \geq 4$.

5 Characterization of the 0- and 1-positions in Moore's game

The following result was shown in [8]. We provide here a different proof for the convenience of the reader.

Theorem 3 (*see [8]*). *For any $k \in \{1, \dots, n\}$ and $m \in \{0, 1\}$, a position x is an m-position of the Moore game* $\text{NIM}^{\leq}_{n,k}$ *if and only if $M(x) = m$.*

Let us represent the components of x as binary sequences

$$x_i = \sum_{j=0}^{\infty} x_{ij} 2^j \text{ for } i = 1, ..., n,$$

and define

$$y_j = \sum_{i=1}^{n} x_{ij} \mod (k+1) \text{ for } j = 0, 1, ...$$

Then we have $M(x) = \sum_{j=0}^{\infty} y_j (k+1)^j$.

Lemma 8 *Let $x \to x'$ be a move, and let j be the highest index such that $x_{ij} \neq x'_{ij}$ for some i. Then we must have $x'_{ij} \leq x_{ij}$ for all $i = 1, ..., n$.*

Proof: In a move we can only decrease the components of x. Therefore, if $x'_{ij} > x_{ij}$, then we must have a $j' > j$ such that $x'_{ij'} < x_{ij'}$. $\square$

5.1 Case $m = 0$

First, let us prove Moore's result: $\mathcal{G}(x) = 0$ if and only if $M(x) = 0$.

By the properties of $\mathcal{P}$-positions it is enough to show that

(i0) for any position x with $M(x) = 0$, there exists no move $x \to x'$ such that $M(x') = 0$;

(a0) for any position x with $M(x) > 0$, there exists a move $x \to x'$ such that $M(x') = 0$.

To show (i0), let us consider a move $x \to x'$ from a position x with $M(x) = 0$.

Let j be the highest binary bit such that x_{ij} and x'_{ij} differ for some i. Such a j must exist since in a move we must change at least one components. By Lemma 8 we have $x'_{ij} \le x_{ij}$ for all $i = 1, ..., n$, implying $1 \le \sum_{i=1}^{n}(x_{ij} - x'_{ij}) \le k$ because in a move we can change at most k components. Therefore, $(\sum_i x'_{ij} \mod (k+1)) \ne 0$ and, thus, $M(x') > 0$.

To show (a0), let us consider a position x with $M(x) > 0$. We will construct a move $x \to x'$ such that $M(x') = 0$.

Let $T = \{t_1, \dots, t_p\}$ denote the set of bits j such that $y_j \ne 0$, assuming $t_1 > \dots > t_p$. Set $N = \{1, 2, \dots, n\}$. The following algorithm maintains the index set $I_j \subseteq N$ for $j = 0, 1, \dots, p$ to compute a move $x \to x'$. The set I_j represents the components of x that we decrease when creating x'. We will use the notation $\alpha_j = |I_j|$ and $\beta_j = |\{i \in I_j \mid x_{it_{j+1}} = 1\}|$.

Step 0. Initialize $I_0 = \emptyset$ and hence, $\alpha_0 = \beta_0 = 0$, and set $x'_i := x_i$ for all $i \in N$.

Step 1. For $j = 1, \dots, p$, construct I_j and update x' as follows.

Case 1. If $y_{t_j} \le \beta_{j-1}$, then let $I_j := I_{j-1}$, choose y_{t_j} many indices i from I_j such that $x'_{it_j} = 1$, and update $x'_{it_j} := 0$ for such indices.

Case 2. If $y_{t_j} > \beta_{j-1}$ and $(k+1) - y_{t_j} \le \alpha_{j-1} - \beta_{j-1}$, then let $I_j := I_{j-1}$, choose $(k+1) - y_{t_j}$ many indices i from I_j such that $x'_{it_j} = 0$, and update $x'_{it_j} := 1$ for such indices.

Case 3. If $y_{t_j} > \beta_{j-1}$ and $y_{t_j} - \beta_{j-1} \le k - \alpha_{j-1}$, then let I_j be an index set obtained from I_{j-1} by adding $y_{t_j} - \beta_{j-1}$ many indices i from $N \setminus I_{j-1}$ such that $x_{it_j} = 1$, and update $x'_{it_j} := 0$ for all $i \in I_j$ with $x'_{it_j} = 1$. □

We first note that three cases above are exclusive and cover all possible y_{t_j}, α_{j-1} and β_{j-1}. Moreover, it is easily seen that a position x' after the execution of the algorithm satisfies $M(x') = 0$, and no x'_i for $i \notin I_p$ was updated in Step 1, that is, $x'_i = x_i$ holds for $i \notin I_p$. Since $|I_p| \le k$, it remains to show that $x' < x$.

Assume that i is an index such that $x'_i \ne x_i$. Then some j satisfies $i \notin I_{j-1}$ and $i \in I_j$. This implies that x'_i was first updated during the jth iteration of Step 1. Namely, the t_jth bit of x'_i are modified from 1 to 0. Since $x'_{it} = x_{it}$ holds for all t with $t > t_j$, we have $x'_i < x_i$, which completes the proof. □

5.2 Case $m = 1$

Now, let us prove that $\mathcal{G}(x) = 1$ if and only if $M(x) = 1$.

The proof in the previous subsection implies that for a position x with $M(x) = 1$ there exists a move $x \to x'$ such that $M(x') = 0$. By the properties of the SG function, it remains to show that

(i1) for any position x with $M(x) = 1$, there exists no move $x \to x'$ such that $M(x') = 1$;

(a1) for any position x with $M(x) > 1$, there exists a move $x \to x'$ such that $M(x') = 1$;

We prove (i1) similarly to (i0). Let us assume that $M(x) = 1$ holds for a position x and consider a move $x \to x'$. Let j be the highest binary bit such that x_{ij} and x'_{ij} differ for some i. Then by Lemma 8 we have $x'_{ij} \leq x_{ij}$ for all i, and $1 \leq \sum_i (x_{ij} - x'_{ij}) \leq k$. Hence, $\sum_i x'_{ij} \neq \sum_i x_{ij} (\mod (k+1))$ and, thus, $M(x') \neq 1$.

To show (a1), let us consider a position x with $M(x) > 1$. Similarly to (a0), we will algorithmically construct a move $x \to x'$ such that $M(x') = 1$.

Let again $T = \{t_1, \ldots, t_p\}$ denote the set of bits j such that $y_j \neq 1_j$, where we assume that $t_1 > t_2 > \ldots, t_p$ and 1_j denotes the jth bit of 1, that is, $1_j = 0$ if $j > 0$, and $1_j = 1$ if $j = 0$. The algorithm remains the same, as for (a0), except for Step 1, when $t_p = 0$.

Step 1. Construct I_p and update x' as follows.

Case 1. If $y_0 > 1$ and $y_0 - 1 \leq \beta_{p-1}$, then let $I_p := I_{p-1}$, choose $y_0 - 1$ many indices i from I_p such that $x'_{i0} = 1$, and update $x'_{i0} := 0$ for such indices.

Case 2. If $y_0 > 1$ and $(k+2) - y_0 \leq \alpha_{p-1} - \beta_{p-1}$, let $I_p := I_{p-1}$, choose $(k+2) - y_0$ many indices i from I_p such that $x'_{i0} = 0$ and update $x'_{i0} := 1$ for such indices.

Case 3. If $y_0 > 1$ and $0 < (y_0 - 1) - \beta_{p-1} \leq k - \alpha_{p-1}$, then let I_p be an index set obtained from I_{p-1} by adding $(y_0 - 1) - \beta_{p-1}$ many indices i from $N \setminus I_{p-1}$ such that $x_{i0} = 1$, and update $x'_{i0} := 0$ for all $i \in I_p$ with $x'_{i0} = 1$.

Case 4. If $y_0 = 0$ and $\alpha_{p-1} > \beta_{p-1}$, then let $I_p := I_{p-1}$, choose an index i from I_p such that $x'_{i0} = 0$, and update $x'_{i0} := 1$.

Case 5. If $y_0 = 0$ and $\alpha_{p-1} = \beta_{p-1} = k$, then let $I_p := I_{p-1}$, and update $x'_{i0} := 0$ for all $i \in I_p$.

Case 6. If $y_0 = 0$ and $\alpha_{p-1} = \beta_{p-1} < k$, then let I_p be an index set obtained from I_{p-1} by adding $k - \alpha_{p-1}$ many indices i from $[n] \setminus I_{p-1}$ such that $x_{i0} = 1$, and update $x'_{i0} := 0$ for all $i \in I_p$. □

Note that the above six cases are exclusive and cover all possible y_0, α_{p-1}, and β_{p-1}. Note also that in Case 6, $\alpha_{p-1} > 0$ since otherwise $M(x) = 0$, giving a contradiction. Thus, in Case 6, we can choose $k - \alpha_{p-1}$ many indices i from $N \setminus I_{p-1}$ such that $x_{i0} = 1$. Let x' be a position obtained by the algorithm. Then, clearly $M(x') = 1$ and $x \to x'$ is a move. This completes the proof of (a1). □

6 Two cases when the k-sum of n impartial games is reduced to k-NIM with n-piles

The concept of the sum of n impartial games $\Gamma_1, \ldots, \Gamma_n$ was considered in Sections 1.3 and 1.4. We can naturally extend this concept and define, in two different ways. the k-sum of n games, for any integer k such that $0 < k < n$ as follows. Two players alternate turns; by one move a player has to choose at least one and at most k (respectively, exactly k) among the given n games and make a move in each of them. To start the game we fix an initial position v_i in Γ_i for $i = 1, \ldots, n$. The game terminates when a player is out of moves; (s)he lost, while the opponent, who made the last move, wins the game. Two versions defined above will be called the *Moore's* and, respectively, *exact* k-sum of the impartial games $\Gamma_1, \ldots, \Gamma_n$.

Obviously, both concepts turn into the standard sum when $k = 1$. In this case a remarkable property holds. Let us compute the SG function $x_i = \mathcal{G}_{\Gamma_i}(v_i)$ of each game Γ_i in its initial position v_i for each $i = 1, \ldots, n$ and replace the sum $\Gamma_1 \oplus \ldots \oplus \Gamma_n$ by the standard n-pile NIM with the initial position $x = (x_1, \ldots, x_n)$. The SG values are the same in the considered two positions [6, 7, 10, 11]

Of course, it is tempting to extend the SG theory from the case $k = 1$ to any k. Although it becomes not that easy to compute the SG function, but as we know, it can be done for Moore's k-sum when $n = k + 1$ and for the exact k-sum when $n \leq 2k$. However, such an attempt fails, together with the following key property: no move can keep the SG value. Indeed, whenever $k \geq 2$, there may exist moves in Γ_i and Γ_j that swap the SG values and thus the whole vector x remains the same after such moves. Obviously, this cannot happen when $k = 1$, since at least two summand games are required. This problem is not arising in the exact and Moore's k-NIM, since no move can make larger the number of tokens in a pile.

For example, $(0, 0, 0, 1)$ and $(0, 0, 1, 0)$ are $\mathcal{P}$-positions in $\text{NIM}^{=}_{4,2}$. Obviously, there is no move between these positions since it would increase either the third or the fourth pile. Yet, such moves can exist in impartial games Γ_3 and Γ_4. Thus, replacing $\text{NIM}^{=}_{4,2}$ by $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, and allowing to move in any two games each time, is not correct, because the SG functions of $\text{NIM}^{=}_{4,2}$ and Moore's 2-sum of $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 may differ. Similarly, for any $k > 1$, it is possible that in a k-sum of $\Gamma_1, \cdots, G_n$ there is a move permuting (for example, cyclically) some k and keeping the remaining $n - k$ SG values.

Yet, the property in the title of this subsection may hold if we restrict the digraphs $\Gamma_i, i = 1, \ldots, n$, of the summand-games. Recall that a digraph $\Gamma_i = (V_i, E_i)$ is called *transitive* if $(u, w) \in E_i$ whenever $(u, v), (v, w) \in E_i$ for some $v \in V_i$. Obviously, if Γ_i is transitive then the SG value cannot be increased by a move. It is also clear that an impartial game on a transitive graph Γ_i with an initial position v_i can be replaced just by one pile of cardinality equal to the corresponding SG value, $x_i = \mathcal{G}_{G_i}(v_i)$. Thus, both Moore's and the exact k-sum of n impartial games on transitive digraphs are reduced to $\text{NIM}^{\leq}_{n,k}$ and $\text{NIM}^{=}_{n,k}$, respectively.

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