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Slow k-Nim

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Abstract. Given n piles of tokens and a positive integer $k \leq n$, we study the following two impartial combinatorial games $\text{Nim}^1_{n,\leq k}$ and $\text{Nim}^1_{n,=k}$. In the first (resp. second) game, a player, by one move, chooses at least 1 and at most (resp. exactly) k non-empty piles and removes one token from each of these piles. For the normal and misère version of each game we compute the Sprague-Grundy function for the cases $n = k = 2$ and $n = k + 1 = 3$. For game $\text{Nim}^1_{n,\leq k}$ we also characterize its $\mathcal{P}$-positions for the cases $n \leq k + 2$ and $n = k + 3 \leq 6$.

Key words: impartial games, Nim, Moore's Nim, k-Nim, normal and misère versions, $\mathcal{P}$-positions, Sprague-Grundy function.

1 Introduction and previous results

We assume that the reader is familiar with basics of the Sprague-Grundy (SG) theory of impartial games [8, 9, 15, 16]. In this paper we will need only the concept of the SG function and $\mathcal{P}$-positions for the normal and misère versions, which are presented in almost every paper about impartial games; see, for example, [3, 4, 12], which are closely related to the present paper. An introduction to the SG theory can be found in [2, 6].

We denote by $\mathbb{Z}_\geq$ (resp. $\mathbb{Z}_>$) the set of non-negative (resp. positive) integers. For $t \in \mathbb{Z}_\geq$, a position whose SG value is t will be called a t-position.

In this section we briefly survey several variants of the game NIM. The normal version is considered in Sections 1.1 - 1.4, and the misère one in Section 1.5.

1.1 Classic NIM

The ancient game of NIM is played as follows. There are n piles containing $x_1, \dots, x_n$ tokens. Two players alternate turns. By one move, a player chooses a non-empty pile x_i and removes an arbitrary (strictly positive) number of tokens from it. The game terminates when all piles are empty. The player who made the last move wins the normal version of the game and loses its misère version. Both versions were solved (all $\mathcal{P}$-positions found) by Charles Bouton in 1901 in his seminal paper [5].

By definition, an n-pile NIM is the sum of n one-pile NIMs and hence the SG function of position $x = (x_1, \dots x_n)$ is the binary bitwise sum (so-called Nim-sum) [5, 8, 15, 16]

$$\mathcal{G}(x) = \mathcal{G}(x_1) \oplus \cdots \oplus \mathcal{G}(x_n) = x_1 \oplus \cdots \oplus x_n.$$

$\mathcal{P}$-positions of NIM, as well as of any impartial game, are the zeros of its SG function, that is, x is a $\mathcal{P}$-position if and only if $\mathcal{G}(x) = 0$.

1.2 Moore's k-NIM

In 1910, Eliakim Hastings Moore [14] suggested the following generalization of NIM for any $k, n \in \mathbb{Z}_>$ such that $k \leq n$. In this game a player, by one move, reduces at least one and at most k piles. The game turns into the standard n-piles NIM when $k = 1$ and is trivial when $k = n$; in the latter case it turns into NIM with one pile of size $s = \sum_{i=1}^n x_i$. Moore denoted his game by NIM_k; we will call it Moore's k-NIM and denote by $\text{NIM}_{n,\leq k}$.

Moore gave a simple and elegant formula characterizing of the $\mathcal{P}$-positions of $\text{NIM}_{n,\leq k}$, thus, generalizing Bouton's results. Later, Berge [1], Jenkyns and Mayberry [13] tried to extend Moore's formula for the SG function of $\text{NIM}_{n,\leq k}$ as follows.

Present the cardinalities of n piles as the binary numbers, $x_i = x_{i_0} + x_{i_1}2 + x_{i_2}2^2 + \cdots + x_{i_m}2^m$, take their bitwise sum modulo $k+1$ for every bit j, that is, set $y_j = \sum_{i=1}^n x_{i_j} \mod (k+1)$, and denote the obtained number $y_0 y_1 \dots y_m$ in base $k+1$ by $M(x)$.

In [14] Moore claimed that x is a $\mathcal{P}$-position of $\text{NIM}_{n,\leq k}$ if and only if $M(x) = 0$. This is a generalization of Bouton's result [5] corresponding to the case $k = 1$. Indeed, in this case $M(x)$ turns into $\mathcal{G}(x)$ and $\text{NIM}_{n,\leq k}$ turns into the standard n-pile NIM.

Remark 1 *Moore published his results in 1910, while the SG function was introduced only quarter of century later, in 1935-9 [8, 15, 16]. It was shown in these papers that the case $k = 1$ has the following remarkable property. The SG function can be used to solve the sum of arbitrary n impartial games (in the normal version; not only the n-pile* NIM*, which is the sum of n one-pile* NIM *games). The concept of the sum can be generalized to that of the k-sum as follows: by one move a player chooses at least one and at most k game-summands and makes an arbitrary legal move in each of them. For example,* $\text{NIM}_{n,\leq k}$ *is the k-sum of n one-pile* NIM *games. By Moore's result, function $M_{n,k}(x)$ can be used to find all $\mathcal{P}$-positions of the latter, but this application works only for the standard sum, which corresponds to the case $k = 1$; see Section 2.4 of [4] for more details.*

Moore's result for $\mathcal{P}$-positions was extended to 1-positions by Jenkyns and Mayberry [13]; see also [3] for an alternative proof.

Proposition 1 ([13]) *For $t = 0$ and $t = 0$, vector x is a t-position of* $\text{NIM}_{n,\leq k}$ *if and only if $M(x) = t$.* □

In his book [1], Claude Berge claimed that Proposition 1 can be generalized for any t, or in other words, that $\mathcal{G}(x) = M(x)$; see [1, page 55, Theorem 3]. However, this is an overstatement, as it was pointed out in [13]. The claim only holds when $M(x) \leq 1$; see Theorems 11 and 12 on page 61 and also page 53 of [13]. For $t > 1$, the t-positions are no longer related to Moore's function M; moreover, it seems difficult to characterize them in general, for all k; see [3, 4] for more details. Yet, the case $n = k + 1$ is tractable: for $n = 2$ the game turns into the two-pile NIM; for $n > 2$ the SG function was first given in [13]; this result was rediscovered and generalized in [3] as follows.

1.3 Game EXCO-NIM, as a generalization of Moore's Nim with $n = k + 1$

The following game called EXCO-NIM (extended complementary NIM) was suggested in [3]. Given $n + 1$ piles containing $x_0, x_1, \dots, x_n$ tokens, two players move alternatingly. By one move, a player can reduce x_0 together with at most $n - 1$ of the remaining n numbers such that at least one token is removed.

Obviously, this game is a generalization of Moore's NIM with $k = n - 1$: the former turns into the latter when $x_0 = 0$. In case $n \geq 3$, it is not difficult to generalize Jenkyns and Mayberry's formula for the SG function of Moore's NIM as follows.

Theorem 1 ([3]) *Given a position $x = (x_1, \dots, x_n)$, let us define*

$$u(x) = \sum_{i=0}^{n} x_i, \quad m(x) = \min_{i=1}^{n} x_i, \quad y = u(x) - n\,m(x), \quad z = \frac{1}{2}(y^2 + y + 2). \qquad (1)$$

Then, for any $n \geq 3$ the SG function of $\text{NIM}_{n,\leq n-1}$ *is given by the formula:*

$$\mathcal{G}(x) \;=\; \begin{cases} u(x), & \text{if } m(x) < z; \\ (z-1) + (m-z) \bmod (y+1), & \text{otherwise.} \end{cases} \tag{2}$$

□

In case $n = 2$ Moore's NIM turns into the standard NIM with two piles, which is trivial. Somewhat surprisingly, EXCO-NIM in this case becomes very difficult; see [3] for partial results.

1.4 Exact k-NIM

In [4], one more version of NIM was introduced; it was called EXACT k-NIM and denoted $\text{NIM}_{n,=k}$. Given integer k, n such that $0 < k \leq n$ and n piles of $x_1, \ldots, x_n$ tokens, by one move a player chooses *exactly* k piles and removes an arbitrary (strictly positive) number of tokens from each of them. Clearly, both games, $\text{NIM}_{n,=k}$ and $\text{NIM}_{n,\leq k}$, turn into the standard n-pile NIM when $k = 1$ and become trivial when $k = n$. In [4], the SG function $\mathcal{G}(x)$ of $\text{NIM}_{n,=k}$ was efficiently computed for the case $n \leq 2k$ as follows.

The *tetris* function $T_{n,k}(x)$ was defined in [4] by the formula:

$$T_{n,k}(x) = \max\{m \mid \exists v_i = (v_1^i, \ldots, v_n^i), 1 \leq i \leq m,\; v_j^i \in \{0,1\},\; \sum_{i=1}^{m} v_j^i \leq x_j\}. \tag{3}$$

In other words, $T_{n,k}(x)$ is the largest number m of n-vectors $(v_1^i, \ldots, v_n^i)$ with exactly k coordinates 1 and $n-k$ entries 0 such that each sum $\sum_{i=1}^{m} v_j^i$ of the jth coordinates does not exceed x_j. For example, $T_{4,3}(1,1,2,3) = 2$ since $v_1 = (0,1,1,1)$ and $v_2 = (1,0,1,1)$ satisfy (3) and any three 4-vectors with exactly three 1 and one 0 result in $\sum_{i=1}^{3} v_j^i \geq 2$ for more than two coordinates.

In [4], an algorithm computing $T_{n,k}(x)$ in polynomial time was obtained and the following statement shown.

Proposition 2 ([4]) *The SG function $\mathcal{G}$ of the game $\text{NIM}_{n,=k}$ and the tetris function $T_{n,k}$ are equal whenever $2k > n$.* □

Another tractable (but more difficult) case is $n = 2k$. Somewhat surprisingly, the games $\text{NIM}_{n,=k}$ for $n = 2k$ and $\text{NIM}_{n,\leq k}$ for $n = k + 1$ appear to be very similar.

Theorem 2 ([4]) *For $n = 2k \geq 4$, the SG function $\mathcal{G}(x)$ of $\text{NIM}_{n,=k}$ is still given by formula (2), but the variables $u = u(x)$ and $y = y(x)$ in it are defined by means of the tetris function (rather than by (1) as follows:*
$u(x) = T_{n,k}(x)$, $y(x) = T_{n,k}(x')$ for $x' = (x_1 - m(x), \ldots, x_n - m(x))$. □

The game becomes even more difficult when $n > 2k$. No closed formula is known already for the $\mathcal{P}$-positions of $\text{NIM}_{5,=2}$.

1.5 Tame, pet, and miserable games

In Sections 1.1 - 1.4 above, we surveyed several variants of Nim, but considered only their normal versions. Now we will recall some results for the misère play. A position x from which there is no legal move is called *terminal*. The SG functions $\mathcal{G}$ and $\mathcal{G}^-$ of the normal and misère versions are defined by the same standard SG recursion, but are initialized differently: $\mathcal{G}(x) = 0$, while $\mathcal{G}^-(x) = 1$, for any terminal position x.

A position x is called an *i-position* (resp. (i, j)-*position*) if $\mathcal{G}(x) = i$ (resp. $\mathcal{G}(x) = i$ and $\mathcal{G}^-(x) = j$). We will denote by X_T, X_i, and $X_{i,j}$ the sets of all terminals, i-, and (i, j)-positions, respectively. By definition, every terminal position is a $(0, 1)$-positions, $X_T \subseteq X_{0,1}$.

In 1976 John Conway [6] introduced the so-called *tame* games, which contain only $(0, 1)$-, $(1, 0)$- and (t, t)-positions. The positions of the first two classes are called the *swap* positions.

A game is called *miserable* [10, 11] if from each $(1, 0)$-position there is a move to a $(0, 1)$-position, from each non-terminal $(0, 1)$-position there is a move to a $(1, 0)$-position, and finally, from every non-swap position, either there is a move to a $(0, 1)$-position and another one to a $(1, 0)$-position, or there is no move to a swap position at all.

Proposition 3 ([10, 11]) *Miserable games are tame.* □

In [10] this result was applied to the game Euclid. Many other applications are suggested in [12]. It appears to be also applicable to the Moore's Nim.

Proposition 4 ([12]) *Game* $\text{Nim}_{n,\leq k}$ *is miserable. Moreover, let* $x = (x_1, \ldots, x_n)$ *be a position in* $\text{Nim}_{n,\leq k}$ *such that for* $2 \leq k < n$ *and let* $l = l(x)$ *be the number of non-empty piles in* x*. Then*

(i) $x \in X_{0,1}$ *if and only if* $x_i \leq 1$ *for all* i *and* $l \equiv 0 \mod (k + 1)$*;*

(ii) $x \in X_{1,0}$ *if and only if* $x_i \leq 1$ *for all* i *and* $l \equiv 1 \mod (k + 1)$*.*

□

The normal and misère versions of the game Exact k-Nim are related in accordance with the following two statements.

Let $d(x)$ is the largest number of moves from x to the terminal position.

Proposition 5 ([4]) *The game of* $\text{Nim}_{2k,=k}$ *is miserable. Moreover,* $x = (x_1, x_2, \ldots, x_n)$ *is*

(i) *a* $(0, 1)$*-position if and only if* $x_1 = x_2 = \cdots = x_{k+1} \leq 1$*;*

(ii) *a* $(1, 0)$*-position if and only if Tetris function* $T_{2k,k}(x) = 1$*.*

Let us remark that $\text{NIM}_{n,=k}$ may be not tame (and, hence, not miserable) when $2k < n$, which indicates indirectly that the game becomes much more difficult in this case. For example, our computations show that $(1, 2, 3, 3, 3)$ is a $(0, 2)$-position of $\text{NIM}(5, = 2)$.

In contrast, the case $2k > n$ is much simpler; in this case game $\text{NIM}_{n,=k}$ satisfies even a stronger property.

In [11], an impartial game was called *pet* if it contains only $(0, 1)$-, $(1, 0)$-, and (t, t)-positions with $t \geq 2$. Recall that only $t \geq 0$ is required for the tame games. Hence, any pet game is tame. Several characterizations of the pet games were given in [11]. In particular, it was shown that the following properties of an impartial game are equivalent:

(i) the game is pet,
(ii) there are no $(0, 0)$-positions,
(iii) there are no $(1, 1)$-positions,
(iv) from every non-terminal 0-position there is a move to a 1-position.

The last property was considered already in 1974 by Thomas Ferguson [7], for a different purpose. He proved that (iv) holds for the so-called subtraction games; see [7] and also [11, 12] for the definition, proof, and more details.

Proposition 6 ([4, 12]) *The game* $\text{NIM}_{n,=k}$ *is pet whenever* $n < 2k$. □

Let us note that Moore's game $\text{NIM}_{n,\leq k}$ is pet only when $k = n$ [3, 4]. In this case the game is trivial and equivalent to the standard NIM with only one pile.

2 Two versions of SLOW k-NIM

In this paper, we introduce two games modifying $\text{NIM}_{n,\leq k}$ and $\text{NIM}_{n,=k}$, respectively. We keep all their rules, but add the following extra restriction: by one move any pile can be reduced by at most one token. We will call the obtained two games by SLOW MOORE'S $k-$NIM and SLOW EXACT $k-$NIM and denote them by $\text{NIM}^1_{n,\leq k}$ and $\text{NIM}^1_{n,=k}$, respectively. As usual, the player who makes the last move wins the normal and loses misère version.

A position is a non-negative n-vector $x = (x_1, \ldots, x_n)$, as before. Without loss of generality, we assume that the coordinates are not decreasing, that is, $0 \leq x_1 \leq \cdots \leq x_n$. Yet, after a move $x \to x'$ (from x to x') this condition may fail for x'. In this case we will reorder the coordinates of x' to reinforce it.

When $k = 1$, the games $\text{NIM}^1_{n,\leq k}$ and $\text{NIM}^1_{n,=k}$ are identical and trivial. The SG function depends only on the parity of the total number $s = \sum_{i=1}^n x_i$ of tokens: if s is even, $\mathcal{G}(x) = 0$, $\mathcal{G}^-(x) = 1$, and vice versa if s is odd.

Also $\text{NIM}^1_{n,=k}$ is trivial when $k = n$. In this case any play consists of $m = x_1$ moves and hence the SG function depends only on the parity of m: if m is even, $\mathcal{G}(x) = 0$, $\mathcal{G}^-(x) = 1$, and vice versa if m is odd.

Thus, in all trivial cases considered above, both games are pet; moreover, they have only $(0, 1)$- and $(1, 0)$-positions that alternate with every move.

In Section 3 we analyze the SG function of SLOW MOORE'S NIM for $n = 2$ and $n = 3$ and show that in these cases $\mathcal{P}$-positions of the game are characterized by the parities of its coordinates. Then, we discuss miserability. In Section 4 we study SLOW EXACT $k-$NIM for $n = 3$ and obtain closed formulas for the SG functions for both normal and misère version of $\text{Nim}^1_{3,\leq 2}$.

Given a non-negative integer n-vector $x = (x_1, \ldots, x_n)$ such that $x_1 \leq \cdots \leq x_n$, its *parity vector* $p(x)$ is defined as n-vector $p(x) = (p(x_1), \ldots, p(x_n))$ whose coordinates take values $\{e, o\}$ according to the natural rule: $p(x_i) = e$ if x_i is even and $p(x_i) = o$ if x_i is odd.

3 SLOW MOORE'S $k-$NIM

The status of a position x in $\text{NIM}^1_{n,\leq k}$ is often uniquely determined by its parity vector $p(x)$. In particular, this is the case for $n = k = 2$ and $n = k + 1 = 3$. First, we will prove this claim and then characterize the $\mathcal{P}$-positions for several other cases.

3.1 Sprague-Grundy values for $n = 2$ and $n = 3$

Proposition 7 *The SG values $\mathcal{G}(x)$ for the cases $n = k = 2$ and $n = 3, k = 2$ are uniquely defined by the parity vector $p(x)$ as follows:*

(i) *For $n = k = 2$,*

$$\mathcal{G}(x) = \begin{cases} 0, & \text{if } p(x) = (e, e); \\ 1, & \text{if } p(x) = (e, o); \\ 2, & \text{if } p(x) = (o, o); \\ 3, & \text{if } p(x) = (o, e). \end{cases}$$

(ii) *For $n = 3, k = 2$,*

$$\mathcal{G}(x) = \begin{cases} 0, & \text{if } p(x) \in \{(e, e, e), (o, o, o)\}; \\ 1, & \text{if } p(x) \in \{(e, e, o), (o, o, e)\}; \\ 2, & \text{if } p(x) \in \{(e, o, o), (o, e, e)\}; \\ 3, & \text{if } p(x) \in \{(e, o, e), (o, e, o)\}. \end{cases}$$

Proof: We will prove only (ii). The proof of (i) is similar, simpler, and we leave it to the reader.

Let S_0 (resp. S_1, S_2, S_3) be the set of positions whose parity vectors belong to $\{(e, e, e), (o, o, o)\}$ (resp. $\{(e, e, o), (o, o, e)\}$, $\{(e, o, o), (o, e, e)\}$, $\{(e, o, e), (o, e, o)\}$.

Obviously, sets $S_g, g \in \{0, 1, 2, 3\}$, partition all positions of the game, since there exist exactly $2^3 = 8$ parity vectors and all are listed above.

Recall that the set of g-position is defined by the following two claims:

(1) there exists no move between positions of S_g, for any fixed $g \in \{0, 1, 2, 3\}$;

(2) for any $i, g \in \{0, 1, 2, 3\}$ such that $i < g$ and $x \in S_g$, there is a move from x to S_i, in other words, S_i is reachable from S_g.

Let us begin with (1). It is easily seen that a move from x to y exists only if these two positions have either of the following two properties:

(a) the parities in one of their three coordinates are the same, while the parities in the two other are opposite; then, y might be reachable from x by reduction of these two "other" piles; conversely,

(b) the parities in one of their three coordinates are opposite, while the parities in the remaining one are the say; then, y might be reachable from x by reduction of these one "remaining" pile.

It is easy to verify that neither (a) nor (b) can hold for any two positions that belong to the same set S_g for any fixed $g \in \{0, 1, 2, 3\}$. Thus, (1) holds.

To show (2), we consider four cases assuming that $x = (x_1, x_2, x_3) \in S_g$ for $g \in \{0, 1, 2, 3\}$.

If $g = 1$, it is sufficient to reduce x_3 by one to get $y \in S_0$.

Let $g = 2$. If $p(x) = (e, o, o)$, reduce x_2 and x_3 to get $y \in S_0$ with $p(y) = (e, e, e)$ and reduce x_2 to get $y \in S_1$ with $p(y) = (e, e, o)$. If $p(x) = (o, e, e)$, reduce x_1 to get $y \in S_0$ with $p(x') = (e, e, e)$ and reduce x_2 to get $y \in S_1$ with $p(y) = (o, o, e)$. Recall that the coordinates of a vector are assumed to be not decreasing.

Finally, let $g = 3$. If $p(x) = (e, o, e)$, then reduce x_2 to get $y \in S_0$ with $p(y) = (e, e, e)$, reduce x_2 and x_3 to get $y \in S_1$ with $p(y) = (e, e, o)$, and reduce x_3 to get $y \in S_2$ with $p(y) = (e, o, o)$.

If $p(x) = (o, e, o)$, then reduce x_2 to get $y \in S_0$ with $p(y) = (o, o, o)$, reduce x_1 to get $y \in S_1$ with $p(x') = (e, e, o)$, and reduce x_1 and x_2 to get $y \in S_2$ with $p(y) = (e, o, o)$. □

3.2 $\mathcal{P}$-position for the cases $n \leq k + 2$ and $n = k + 3 \leq 6$

In both these cases the $\mathcal{P}$-positions again are simply characterized by the corresponding parity vectors.

Proposition 8 *The $\mathcal{P}$-positions of game* $\text{NIM}^1_{n,\leq k}$ *are uniquely defined by their parity vectors as follows:*

(1) *for $n = k$ we have: $x \in \mathcal{P}$ if and only if $p(x) = (e, e, \ldots, e)$;*

(2) *for $n = k + 1$ we have: $x \in \mathcal{P}$ if and only if $p(x) \in \{(e, e, \ldots, e), (o, o, \ldots, o)\}$;*

(3) *for $n = k + 2$ we have: $x \in \mathcal{P}$ if and only if $p(x) \in \{(e, e, \ldots, e), (e, o \ldots, o)\}$;*

(4) *for* $n = 5, k = 2$ *we have:* $x \in \mathcal{P}$ *if and only if*

$$p(x) \in \{(e,e,e,e,e),(e,e,o,o,o),(o,o,e,e,o),(o,o,o,o,e)\};$$

(5) *for* $n = 6, k = 3$ *we have:* $x \in \mathcal{P}$ *if and only if*

$$p(x) \in \{(e,e,e,e,e,e),(e,e,o,o,o,o),(o,o,e,e,o,o),(o,o,o,o,e,e)\}.$$

Proof: For each case, we have to verify the following two statements:

(i) there is no move from x to y if both positions x and y are in $\mathcal{P}$, and

(ii) for any $x \notin \mathcal{P}$, there is a move to a position $y \in \mathcal{P}$.

Property (i) is obvious for all five cases. Indeed, it is enough to notice that in each case, the Hamming distance between $p(x)$ and $p(y)$ is either 0 (if they coincide) or larger than the corresponding k (if they are distinct). In both cases, there is no move from x to y.

To verify (ii), consider five statements $(1)-(5)$ separately. In each case, for every position $x \notin X$, we will construct a move to a position $y \in \mathcal{P}$.

To show (1) notice that x has odd coordinates whenever $x \notin \mathcal{P}$. Then, to move from x to a position $y \in \mathcal{P}$, it is sufficient to reduce all odd coordinates of x.

To show (2) notice that x has at least one even and at least one odd coordinate whenever $x \notin \mathcal{P}$. If $x_1 = 0$, we reduce all odd coordinates to get a position y with $p(y) = (e, e, \dots, e)$. If $x_1 > 0$, we reduce each coordinate whose parity is different from $p(x_1)$, to get a position y whose all coordinates have the same parities as $p(x_1)$, that is, either $(e, e, \dots, e)$ or $(o, o, \dots, o)$.

Let us show (3). If all coordinates of x are odd, reduce x_1, to get a position y with $p(y) = (e, o, o, \dots, o)$. Otherwise, let us consider two cases. Let exactly one coordinate of x be even. It cannot be x_1, since $x \notin \mathcal{P}$. Reduce this even coordinate together with x_1 to get a position y with the parity vector $(e, o, o, \dots, o)$. If at least two coordinates of x are even, reduce all odd coordinates to get a position y with the parity vector $(e, e, \dots, e)$.

To show (4) and (5), let us consider the following cases.

If all coordinates of x are odd, reduce x_1 and x_2 to get a position y with $p(y) = (e, e, o, \dots, o)$.

If exactly one coordinate x_i in x is even then it is easily seen that $i \neq 5$ for the case (4) and $i \neq 6$ for the case (5).

- If i is odd, reduce x_{i+1} to get a position y with
  $p(y) \in \{(e,e,o,o,o),(o,o,e,e,o)\}$ for the case (4) and
  $p(y) \in \{(e,e,o,o,o,o),(o,o,e,e,o,o),(o,o,o,o,e,e)\}$ for the case (5).
  Recall that $x_{i+1} \geq x_i + 1$ and note that the order of these two coordinates does not change after the above move.

- Let i be even. If $i = 2$, reduce x_1 to get a position y with $p(y) = (e, e, o, o, o)$ for the case (4) and $p(y) = (e, e, o, o, o, o)$ for the case (5). If $i = 4$, reduce the last 2 (resp. 3) coordinates x_4, x_5 (resp. x_4, x_5, x_6) to get a position y with $p(y) = (o, o, o, o, e)$ (resp. $p(y) = (o, o, o, o, e, e)$) for case (4) (resp. (5)).

If at least 3 coordinates of x are even, reduce all odd coordinates to get a position y with $p(y) = (e, e, \dots, e)$.

It remains to consider the case when there are exactly two even coordinates x_i, x_j, $i \neq j$. Note that $(i, j) \notin \{(1, 2), (3, 4), (5, 6)\}$. If i is odd, reduce x_{i+1} and x_j to get a position y with $p(y) \in \{(e, e, o, o, o), (o, o, e, e, o)\}$ for the case (4) and $p(y) \in \{(e, e, o, o, o, o), (o, o, e, e, o, o), (o, o, o, o, e, e)\}$ for the case (5). If $i = 2$, reduce x_i and x_j to get a position y with $p(y) = (e, e, o, o, o)$ (resp. $p(y) = (e, e, o, o, o, o)$) for the case (4) (resp. (5)). If $i = 4$, reduce x_i for case (4) to get a position y with $p(y) = (o, o, o, o, e)$, while for case (5), reduce x_i and the unique odd coordinate of the last two, x_5 and x_6. (one of them is x_j which is even) to get a position y with $p(y) = (o, o, o, o, e, e)$. □

Our computations show that Proposition 8 cannot be extended to cover the case $n \leq 6$, because in the remaining subcases the $\mathcal{P}$-positions are not uniquely characterized in terms of their parity vectors. For example, if $n = 6$ and $k = 2$ then $(3, 3, 3, 4, 4, 4)$ is a $\mathcal{P}$-position, while $(1, 1, 1, 2, 2, 4)$ is a 2-position, although their parity vectors are the same; also $(1, 3, 3, 3, 3, 3)$, $(1, 3, 3, 3, 5, 5)$, and $(1, 3, 5, 5, 5, 5)$ are $\mathcal{P}$-positions, while $(1, 1, 1, 1, 1, 3)$, $(1, 1, 1, 1, 1, 5)$ are 2-positions, and $(1, 1, 3, 3, 5, 5)$ is a 3-position.

3.3 Miserability

Now let us consider the misère version of $\text{Nim}^1_{n,\leq k}$.

Proposition 9 *When $k \geq n - 1$, Slow Moore's $\text{Nim}^1_{n,\leq k}$ is miserable. Furthermore,*

$V_{0,1} = \{(0, 0, \dots, 0, 2j) | j \in \mathbb{Z}_{\geq}\}$ and $V_{1,0} = \{(0, 0, \dots, 0, 2j + 1) | j \in \mathbb{Z}_{\geq}\}$ for $k = n$;

$V_{0,1} = \{(i, i, \dots, i, i+2j) | i, j \in \mathbb{Z}_{\geq}\}$ and $V_{1,0} = \{(i, i, \dots, i, i+2j+1) | i, j \in \mathbb{Z}_{\geq}\}$ for $k = n-1$.

The proof uses the following lemma from [12] characterizing miserability.

Given a game G and two sets V', V'' of its positions, we say that V' is *movable* to V'' if for every position $x' \in V'$ there is a move to a position $x'' \in V''$.

Lemma 1 ([12]) *A game G is miserable if and only if there exist two disjoint sets $V'_{0,1}$ and $V'_{1,0}$ satisfying the following five conditions:*

(i) *Both sets $V'_{0,1}$ and $V'_{1,0}$ are independent, that is, there is no legal move between two positions of one set.*

(ii) *$V'_{0,1}$ contains all terminal positions, that is, $V_T \subseteq V'_{0,1}$.*

(iii) *$V'_{0,1} \setminus V_T$ is movable to $V'_{1,0}$.*

(iv) *$V'_{1,0}$ is movable to $V'_{0,1}$.*

(v) *From every position $x \notin V'_{0,1} \cup V'_{1,0}$, either there is a move to $V'_{0,1}$ and another one to $V'_{1,0}$ or there is no move to $V'_{0,1} \cup V'_{1,0}$.*

Moreover, $V'_{0,1} = V_{0,1}$ and $V'_{1,0} = V_{1,0}$ whenever the above five conditions hold. □

Proof of Proposition 9. For both cases $k = n$ and $k = n - 1$, we will show that the sets defined by this proposition satisfy all conditions of Lemma 1, thus, proving miserability and getting all swap positions.

We will consider only case $k = n - 1$, leaving the simpler one $k = n$ to the reader.

Set $V'_{0,1} = \{(i, i, \dots, i, i + 2j) | i, j \in \mathbb{Z}_{\geq}\}$
and $V'_{1,0} = \{(i, i, \dots, i, i + 2j + 1) | i, j \in \mathbb{Z}_{\geq}\}$.

Clearly, a move from a position $(i, i, \dots, i, i + 2j)$ reduces at most $n - 1$ first or last coordinates and, hence, cannot lead to a position $(i', i', \dots, i', i' + 2j')$.

Therefore, (i) holds for $V'_{0,1}$. Similarly, (i) holds for $V'_{1,0}$.

Clearly, $V'_{0,1}$ contains the (unique) terminal position, with no tokens at all, and hence, (ii) holds.

Conditions (iii) and (iv) hold, since moves $(i, i, \dots, i, i+2j) \to (i, i, \dots, i, i+2j-1)$ and $(i, i, \dots, i, i + 2j + 1) \to (i, i, \dots, i, i + 2j)$, reducing the last coordinate, are legal.

Let us check (v). Assume that the part "there is no move from x to $V'_{0,1} \cup V'_{1,0}$" fails and show that there is a move from x to $V'_{0,1}$ and another one to $V'_{1,0}$.

Let us first assume that there is a move $\mathfrak{m}$ from x to $V'_{0,1}$ and show that there is also a move from x to $V'_{1,0}$. Consider two cases:

(a) Case 1: move $\mathfrak{m}$ reduces $x_1, x_2, \dots, x_{n-1}$. In other words, $x = (i + k_1, i + k_2, \dots, i + k_{n-1}, i + 2j)$ for some i, j and $0 \leq k_l \leq 1$ with $k_1 + k_2 + \cdots + k_{n-1} > 0$. Note that $j \geq 1$, since there is a coordinate $i + k_j > i$. If $k_l = 1$ for all l then $x \in V'_{1,0}$, which is a contradiction. Therefore, without loss of generality, we can assume that $k_1 = 0$. In this case, move $x \to (i, i, \dots, i, i + 2j - 1)$ terminates in $V'_{1,0}$.

(b) Case 2: move $\mathfrak{m}$ reduces the last coordinate x_n. Without loss of generality, assume that this move reduces the last $n - 1$ coordinates $x_2, x_3, \dots, x_n$, that is, $x = (i, i + k_1, \dots, i + k_{n-2}, i + 2j + k_{n-1})$ for some i, j and $0 \leq k_l \leq 1$ with $k_1 + k_2 + \cdots + k_{n-1} > 0$. Note that $j \geq 1$ when $k_{n-1} = 0$ and also that there is $l_0 \leq n - 2$ such that $k_{l_0} = 1$, since otherwise $x \in V'_{0,1} \cup V'_{1,0}$. Let us now consider k_{n-1}. If $k_{n-1} = 0$ (resp. $k_{n-1} = 1$) then move $x \to (i, i, \dots, i, i + 2j - 1)$ terminates in $V'_{1,0}$ (resp. in $V'_{1,0}$). Note that the latter move is legal since $k_{l_0} = 1$.

It remains to assume that there is a move from x to $V'_{1,0}$. and to show that there is a move from x to $V'_{0,1}$ too. The arguments are essentially the same as above and we leave this case to the reader.

Thus, by Lemma 1, the game is miserable; moreover, $V'_{0,1}$ and $V'_{1,0}$ are its swap positions, that is, $V_{0,1} = V'_{0,1}$ and $V_{1,0} = V'_{1,0}$. □

Our computations show that $\text{NIM}^1_{4,\leq 2}$ is not tame: for example, $(1, 1, 2, 3)$ is a $(4, 0)$-position. Unlike the normal version, for the misère one we have no simple characterization even for the $\mathcal{P}$-positions.

4 Slow exact k-Nim ($\text{NIM}^1_{n,=k}$)

As we already mentioned, the game is trivial when $k = 1$ or $k = n$. In both cases there are only $(0, 1)$- and $(1, 0)$-positions, which alternate with every move. Whether x is a $(0, 1)$- or a $(1, 0)$-position depends on only the parity. More precisely, $\mathcal{G}(x) = q(x) \pmod 2$, where $q(x) = \sum_{i=1}^n x_i$ when $k = 1$ and $q(x) = m(x) = \min_{i=1}^n x_i = x_1$ when $k = n$; in both cases $\mathcal{G}^-(x) + \mathcal{G}(x) = 1$.

We will show that both the misère and normal versions of the game are tractable when $n = 3$ and $k = 2$. Again the parity vector plays an important role, although in this case it does not define the SG function uniquely.

4.1 On Sprague-Grundy function of $\text{NIM}^1_{3,=2}$

Proposition 10 *Let us set*

$$\begin{aligned}
A &= \{(2a, 2b-1, 2(b+i)) | 0 \leq a < b, 0 \leq i < a, (a+i) \bmod 2 = 1\},\\
B &= \{(2a, 2b, 2(b+i)+1) | 0 \leq a \leq b, 0 \leq i < a, (a+i) \bmod 2 = 1\},\\
C_0 &= \{(2a-1, 2b-1, 2(b+i)-1) | 0 \leq a \leq b, 0 \leq i < a, (a+i) \bmod 2 = 0\},\\
C_1 &= \{(2a-1, 2b-1, 2(b+i)-1) | 0 \leq a \leq b, 0 \leq i < a, (a+i) \bmod 2 = 1\},\\
D_0 &= \{(2a-1, 2b, 2(b+i)) | 0 \leq a < b, 0 \leq i < a, (a+i) \bmod 2 = 1\},\\
D_1 &= \{(2a-1, 2b, 2(b+i)) | 0 \leq a < b, 0 \leq i < a, (a+i) \bmod 2 = 0\},\\
C &= C_1 \cup C_2, D = D_1 \cup D_2,
\end{aligned}$$

The SG function of game $\text{NIM}^1_{3,=2}$ *takes only four values,* $0, 1, 2, 3$ *and the sets of its* i*-positions are as follows:*

$$\begin{aligned}
S_0 &= (\{(2a, 2b, c) \mid 2a \leq 2b \leq c\} \setminus B) \cup A \cup C_0 \cup D_0,\\
S_1 &= (\{(2a, 2b+1, c) \mid 2a \leq 2b+1 \leq c\} \setminus A) \cup B \cup C_1 \cup D_1,\\
S_2 &= \{(2a+1, 2b+1, c) \mid 2a+1 \leq 2b+1 \leq c\} \setminus C,\\
S_3 &= \{(2a+1, 2b, c) \mid 2a+1 \leq 2b \leq c\} \setminus D.
\end{aligned}$$

Proof: It is easily seen that these four sets partition the set of all positions of $\text{NIM}^1_{3,=2}$ In addition, we will verify the following four statements that immediately imply that S_i is the set of i-positions, by definition of SG function.

(1) each set S_i is independent, that is, there is no move between any two of its positions;

(2) if $p \notin S_0$ then p is movable to S_0;

(3) if $p \notin S_0 \cup S_1$ then p is movable to S_1;

(4) if $p \notin S_0 \cup S_1 \cup S_2$ then p is movable to S_2.

(1) Case $i = 0$. Let $p \in S_0$. Assume that $p \in (\{(2a, 2b, c) \mid 2a \leq 2b \leq c\} \setminus B)$. There are three types of moves from p.

(i) $p \to (2a-1, 2b-1, c) = q$. By the parity vector of q, we have $q \notin \{(2a, 2b, c) \mid 2a \leq 2b \leq c\} \cup A \cup D_0$. Since $p \notin B$, one can derive that $q \notin C_0$. Indeed, assume that $q \in C_0$. Then $c = 2(b+i) - 1$ for some i such that $0 \leq i < a$ and $(a+i) \bmod 2 = 0$. Note that $i \geq 1$, since $c \geq 2b$. We have $p = (2a, 2b, 2(b+i)-1) = (2a, 2b, 2(b+i-1)+1)$, where $0 \leq i-1 < a$ and $(a+i-1) \bmod 2 = 1$. The last formula for p implies that $p \in B$, resulting in a contradiction. Thus, $q \notin C_0$. It follows that $q \notin S_0$.

(ii) $p \to (2a-1, 2b, c-1) = q$. If $c-1 = 2b-1$, $q = (2a-1, 2b-1, 2b) \notin S_0$ as shown in the case (i). If $c-1 \geq 2b$ then $q \notin \{(2a, 2b, c) \mid 2a \leq 2b \leq c\} \cup A \cup C_0$. Since $p \notin B$, one can conclude that $q \notin D_0$. Therefore, $q \notin S_0$.

(iii) $p \to (2a, 2b-1, c-1) = q$. If $2a > c-1$ then $p = (2a, 2a, 2a)$ and, hence, $q \notin S_0$, as shown in case (i). Assume that $2a \leq c-1$. If $2a > 2b-1$ then $p = (2a, 2a, c)$ and, hence, $q \notin S_0$ as shown in case (ii). Thus, we can assume that $2a \leq 2b-1$ and, hence, $2a < 2b-1$. In other words, the first (smallest) coordinate of q is $2a$ and the second is $2b-1$. Therefore, $q \notin \{(2a, 2b, c) \mid 2a \leq 2b \leq c\} \cup C_0 \cup D_0$. Since $p \notin B$, one can verify that $q \notin A$. Thus, $q \notin S_0$.

Similarly, one can show that there is no move from p to S_0, when $p \in A \cup C_0 \cup D_0$.

The case $i = 1$ is similar to the case $i = 0$ and we leave to the reader.

If $i = 2$ then a move from a position $(2a+1, 2b+1, c) \in S_2$ results in a position with the first or the second even coordinate. Clearly, such a move cannot terminate in S_2.

If $i = 3$ then a move from $(2a+1, 2b, c) \in S_3$ will change the parity of either the first or the second coordinate without changing any order. Clearly, such a move cannot terminate in S_3.

(2) Let $p \notin S_0$. Consider the following four cases:

(i) $p = (2a, 2b, c)$ such that $2a \leq 2b \leq c$. Then, $p \in B$, that is, $p = (2a, 2b, 2(b+i)+1)$ for some i such that $0 \leq g < a$ and $(a+i) \bmod 2 = 1$. If $a = b$ then move $p \to (2a-1, 2b, 2(b+i))$ terminates in $D_0 \subset S_0$; if $a < b$ then move $p \to (2a, 2b-1, 2(b+i))$ terminates in $A \subset S_0$.

(ii) $p = (2a, 2b+1, c)$ with $2a \leq 2b+1 \leq c$. Consider the move $p \to (2a, 2b, c-1) = q$. Let us show that $p \notin A$ implies that $q \notin B$. Assume that $q \in B$. Then,

$c-1 = 2(b+i)+1$ for some i such that $0 \le i < a$ and $(a+i) \bmod 2 = 1$. We set $c = 2(b+1+i) = 2(b'+i)$ with $b' = b+1$ and conclude that $p = (2a, 2b'-1, 2(b'+i))$. By the restrictions for of i, we have $p \in A$, which is a contradiction. Thus, $q \notin B$, implying also that $q \in S_0$.

(iii) $p = (2a+1, 2b+1, c)$ with $2a+1 \le 2b+1 \le c$. Consider a move $p \to (2a, 2b, c) = q$. Since $p \notin C_0$, similarly to the argument of (ii), we prove that $q \notin B$ and, hence, $q \in S_0$.

(iv) $p = (2a+1, 2b, c)$ such that $2a+1 \le 2b \le c$. If $c = 2(b+i)+1$ then move $p \to (2a, 2b, 2(b+i))$ terminates in S_0. Let $c = 2(b+i)$. If $i \ge a+1$, consider move $p \to (2a, 2b, 2(b+i)-1) = (2a, 2b, 2(b+i')+1) = q$ such that $i' = i-1$. Since $i' \ge a$, we have $q \notin B$ and, hence, $q \in S_0$. If $i \le a$, set $p = (2a'-1, 2b, 2(b+i))$, where $0 \le i < a' = a+1$. We have $p \in D$. Furthermore, since $p \notin S_0$, we have $p \notin D_0$ and, hence, $p \in D_1$, implying that $(a'+i) \bmod 2 = 0$. Considering move $p \to (2a'-1, 2b-1, 2(b+i)-1)$, we conclude that $q \in C_0 \subset S_0$.

(3) If $p \notin S_0 \cup S_1$ then either $p \in S_2$ or $p \in S_3$. Consider both cases.

(i) $p \in S_2$. Then $p = (2a+1, 2b+1, c)$ for some a, b, c. We examine the parity of c. If c is even or, equivalently, $c = 2(b+i)$, we consider move $p \to (2a, 2b+1, 2(b+i)-1) = q$. Note that $q \notin A$ since its third coordinate is odd. Hence, $q \in S_1$. If c is odd, since $p \notin C$, we have $c = 2(b+i+1)-1$ for some $i \ge a+1$. In fact, if $c = 2(b+i+1)-1$ for some $i < a+1$, defining $a' = a+1, b' = b+1$, we get $p = (2a'-1, 2b'-1, 2(b'+i)-1) \in C$, which is a contradiction. Now consider move $p \to (2a, 2b+1, 2(b+i)) = (2a, 2(b+1)-1, 2(b+1+i-1)) = q$. Since $i-1 \ge a$, $q \notin A$ and, hence, $q \in S_1$.

(ii) $p \in S_3$. Then $p = (2a+1, 2b, c)$ for some a, b, c. Again, we examine c. If c is odd or, equivalently, $c = 2(b+i)+1$, we consider the move $p \to (2a, 2b-1, 2(b+i)+1) = q$. Note that $q \notin A$ since its third coordinate is odd. Hence, $q \in S_1$. If c is even then $c = 2(b+i)$ for some $i \ge a+1$. In fact, if $i < a+1$, by setting $p = (2(a+1)-1, 2b, 2(b+i))$, we get $p \in D$, which is a a contradiction. Now consider move $p \to (2a, 2b-1, 2(b+i)) = q$. Since $i > a$, we have $q \notin A$ and, hence, $q \in S_1$.

(4) Let $p \notin S_0 \cup S_1 \cup S_2$. Then $p \in S_3$ and, hence, $p = (2a+1, 2b, c)$ for some a, b, c; yet, $p \notin D$. Examine c again. If c is odd or, equivalently, $p = (2a+1, 2b, 2(b+i)+1)$, we consider move $p \to (2a+1, 2b-1, 2(b+i)) = q$. Since the third coordinate of q is even, $q \notin C$ and, hence, $q \in S_2$. If c is even, $c = 2(b+i)$ for some $i \ge a+1$. In fact, if $i < a+1 = a'$ then $p = (2a'-1, 2b, 2(b+i)) \in D$, which is a contradiction. Now consider move $p \to (2a+1, 2b-1, 2(b+i)-1) = (2a'-1, 2b-1, 2(b+i)-1) = q$ such that $i \ge a+1 = a'$. Note that $q \notin C$ and, hence, $q \in S_2$.

Remark 2 *For $n = k+1 = 4$, our computations indicate that the SG function still takes only four values $\{0, 1, 2, 3\}$, corresponding to the parity vectors $\{(e, e, *, *), (e, o, *, *), (o, o, *, *), (o, e, *, *)\}$,* *respectively. Yet, the structure of exceptions is more complicated than in case $n = k+1 = 3$ and can hardly be characterized by simple closed formulas, like in Proposition 10.*

Remark 3 *For $n = k+1 = 3$ the SG function takes only values $\{0, 1, 2, 3\}$, because in this case there are at most three moves from each position. The corresponding number is "typically" 4 when $n = k+1 = 4$, yet, no 4-position was found. However, larger SG values can be taken when $n = k+1 > 4$. For example, our computations show that $(1, 2, 3, 3, x_5)$ for $3 \le x_5 \le 7$ are 5-positions of the game $\text{NIM}^1_{5,=4}$.*

4.2 On the Sprague-Grundy function of misère $\text{NIM}^1_{3,=2}$

Proposition 11 *Let us set*

$$\begin{aligned}
A_1 &= \{(2a, 2b-1, 2b+4i) \mid a = 2a', 0 \le i < a'\},\\
A_2 &= \{(2a, 2b-1, 2b+4i+2) \mid a = 2a'+1, 0 \le i < a'\},\\
B_1 &= \{(2a, 2b, 2b+4i+1) \mid a = 2a', 0 \le i < a'\},\\
B_2 &= \{(2a, 2b, 2b+4i+3) \mid a = 2a'+1, 0 \le i < a'\},\\
C_0 &= \{(2a-1, 2b-1, 2(b+i)-1) \mid 0 \le i < a-1, (a+i) \bmod 2 = 0\},\\
C_1 &= \{(2a-1, 2b-1, 2(b+i)-1) \mid 0 \le i < a-1, (a+i) \bmod 2 = 1\},\\
D_0 &= \{(2a-1, 2b, 2(b+i)) \mid 0 \le i < a-1, (a+i) \bmod 2 = 1\},\\
D_1 &= \{(2a-1, 2b, 2(b+i)) \mid 0 \le i < a-1, (a+i) \bmod 2 = 0\},\\
E &= \{1, 2b-1, 2b-1 \mid b \ge 1\},\\
F &= \{1, 2b, 2b \mid b \ge 1\},\\
A &= A_1 \cup A_2, B = B_1 \cup B_2, C = C_1 \cup C_2, D = D_1 \cup D_2,
\end{aligned}$$

The SG function of the of misère version of $\text{NIM}^1_{3,=2}$ takes only four values, $0, 1, 2, 3$ and the sets of its i-positions are as follows:

$$\begin{aligned}
S_0 &= (\{(2a, 2b, c) \mid 2a \le 2b \le c\} \setminus B) \cup A \cup C_0 \cup D_0 \cup E,\\
S_1 &= (\{(2a, 2b+1, c) \mid 2a \le 2b+1 \le c\} \setminus A) \cup B \cup C_1 \cup D_1 \cup F,\\
S_2 &= \{(2a+1, 2b+1, c) \mid 2a+1 \le 2b+1 \le c\} \setminus (C \cup E),\\
S_3 &= \{(2a+1, 2b, c) \mid 2a+1 \le 2b \le c\} \setminus (D \cup F).
\end{aligned}$$

Proof: It is essentially similar to the proof of Proposition 10 and we leave to the reader.

4.3 Swap, non-swap, and non-tame positions

The relatively simple closed formulas obtained in Propositions 9 and 10, respectively, for the normal and misère versions of $\text{NIM}^1_{3,=2}$ allow us to characterize the swap positions of the game as follows.

Corollary 1 *Consider position $x = (x_1, x_2, x_3)$ with $0 \leq x_1 \leq x_2 \leq x_3$.*

(1) *If $x_1 = 0$ then x is a $(0, 1)$-position (respectively, $(1, 0)$-position) whenever x_2 is even (resp. odd).*

(2) *If $x_1 = 1$ then x is a $(0, 1)$-position (respectively, $(1, 0)$-position) whenever $x_2 = x_3$ is even (resp. $x_2 = x_3$ is odd).*

(3) *For $x_1 \geq 2$, let us set $x_1 = a, x_2 = b$, and $x_3 = b + i$. Then, $x = (a, b, b + i)$ is a swap position if and only if $i \pmod 2 \neq a \pmod 2$, where $i < a$ when a is even and $i < a - 2$ when a is odd.*

Furthermore, in this case $\mathcal{G}(x) = b + \lceil \frac{i}{2} \rceil$ (while $\mathcal{G}^-(x) = 1 - \mathcal{G}(x)$).

Proof: It follows directly from Propositions 10 and 11. □

A position x is called *tame* if $\mathcal{G}(x) = \mathcal{G}^-(x)$. Recall that a game is tame if and only if it has only tame and swap positions. A game is called *domestic* [12] if it has neither $(0, k)$ nor $(k, 0)$ position for $k \geq 2$. The following result shows that $\text{NIM}^1_{3,=2}$ is not domestic. Moreover, we can characterize the positions that are neither tame nor swap.

Corollary 2 *A position $x = (x_1, x_2, x_3)$ such that $0 \leq x_1 \leq x_2 \leq x_3$ is neither swap nor tame if and only if x_1 is odd, $x_1 \neq 1$, and $x_1 + x_2 = x_3 + 1$. Under these conditions, x is a $(0, 3)$-position (resp. $(1, 2)$-position) if and only if x_2 is even (resp. odd).*

Proof: It results directly from Propositions 9 and 10. □

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