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On tame, pet, domestic, and miserable impartial games

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RRR 4-2015, August 2015

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Vladimir Gurvich   Nhan Bao Ho

Abstract. We study impartial games whose normal and misère plays are closely related. The first such class, so-called *tame* games, was introduced by Conway in 1976. Here we will consider a proper subclass, called *pet* games, and introduce a proper superclass, called *domestic* games. For each of three classes we provide several equivalent characterizations; some of which can be efficiently verified. These characterizations are based on an important subfamily of the tame games introduced in 2007 by the first author and called *miserable* games. We show that "slight modifications" turn these concepts into tame, pet, and domestic games. We also show that the sum of miserable games is miserable and find several other classes that respect sums. The obtained techniques allow us to prove that very many well-known impartial games fall into the considered classes. Such examples include all subtraction games which are pet, game Euclid which is miserable and hence tame, as well as many versions of the Wythoff game and Nim that may be miserable, pet, or domestic.

Keywords: impartial games, normal and misère play, sums of impartial games, Sprague-Grundy function, tame, pet, domestic, miserable.

1 Sprague-Grundy theory of impartial games

Combinatorial games were analyzed in the comprehensive books [3] and earlier in [14]. The reader can also find an introductory theory in [1, 36]; one familiar with the subject can skip this section.

A game is *short* if it has only a finite number of positions and each position can be visited at most once. A game is *impartial* if both players have the same possible moves in each position. We will consider only two-person short impartial games, calling them simply *games*, for brevity.

1.1 Modeling games by directed graphs

Any such game can be represented by a directed graph (digraph) $G = (V, E)$; each vertex $z \in V$ is interpreted as a *position* of the game and each arc $(x, y) \in E$ as a (*legal*) *move* from position x to position y. We will also denote such a move by $x \to y$ (to avoid confusion with a pair of integers) and say that x is *movable* to y, or y is *reachable* (by one move) from x, or y is an (immediate) *follower* or *successor* of x.

Given two subsets of positions $U, W \subseteq V$, we will also say that U is *movable* to W or that W is *reachable* (by one move) from U if from every position $x \in U$ there is a move $x \to y$ such that $y \in W$.

Note that the digraph G may be infinite, but we will always assume that any sequence of successive moves (a *play*) $x \to y, y \to z, \cdots$ is finite. This assumption implies that G has no directed cycles and, in particular, no loops or oppositely directed arcs.

Imagine that two players alternate turns moving a token from a current vertex x to y such that $x \to y$ is a legal move. The game begins in a fixed initial position and ends when the token reaches a vertex with no outgoing arcs. It will be called a *terminal position* or just a *terminal*, for short. Let $V_T \subseteq V$ denote the set of all terminals. Without loss of generality, they all can be coalesced into one.

Two versions of each game will be considered: the *normal* one: the player who has to move from the terminal (in other words, one who is out of moves) loses and the *misère* version in which this player wins.

In this section we briefly survey basic concepts of the Sprague-Grundy (SG) theory [22, 23, 37, 38] for the normal and misère versions. The aim of this paper is to study both versions simultaneously and to outline cases when they are related.

1.2 Winning positions and moves

Since any short impartial game terminates, exactly one player wins. It is not difficult to characterize the winning strategies for the normal version.

A position is an $\mathcal{N}$-position (resp. a $\mathcal{P}$-position) if the next (resp. previous) player wins the normal version of the game. The corresponding two sets are denoted by V_N and V_P; by definition they partition V and $V_T \subseteq V_P$.

In graph theory, the set of $\mathcal{P}$-positions is called the *kernel*. The kernel of a finite acyclic digraph $G = (V, E)$ is unique and can be obtained by the following simple recursive algorithm: include in V_P each terminal of G; include in V_N every position x movable to V_T; delete from G all considered positions and repeat.

This procedure implies that the $\mathcal{P}$- and $\mathcal{N}$-position are uniquely characterized by the next two simple properties:

(I) there exists no move between two $\mathcal{P}$-positions, or in other words, V_P is an *independent* set of G, and

(A) there exists a move from each $\mathcal{N}$-position to a $\mathcal{P}$-position, or in other words, V_P is an *absorbing* set of G.

In their turn, properties (I) and (A) imply that in the normal version

(V) any move to a $\mathcal{P}$-position is winning, while

(L) any move from a $\mathcal{P}$-position is losing.

Indeed, if a player moves from V_P then, by (I), (s)he must leave it, and, by (A) the opponent can return to V_P, etc. Since all plays of G are finite and all terminals are in V_P, sooner or later the player moving to (resp. from) V_P will win (resp. lose).

1.3 Sum of games

Given two games G and H, their *sum* $G + H$ is defined as follows. A position of $G + H$ is a pairs (x, y), where x and y are positions of G and H, respectively. A move in (x, y) consists of choosing one of two games and making a move in it. In other words, the followers of (x, y) form the set

$$\{(x, y'), (x', y) \mid x' \in X, y' \in Y\},$$

where X and Y are the sets of all followers of x in G and y in H, respectively. Position (x, y) is a terminal if both x and y are terminal positions in G and H, respectively.

Figure 10 at the beginning of Section 5 provides an example, where all legal moves of the sum of two games are given explicitly.

Obviously, the sum is an associative and commutative binary operation. Thus, the sum of k games is well defined for any integer $k \geq 2$.

1.4 The Sprague-Grundy function

As we already know, to play the normal version of an individual game, it is sufficient to know its $\mathcal{P}$-positions. In contrast, to know all $\mathcal{P}$-positions of G and H may be not sufficient for playing the sum $G + H$. (Although the sum of two $\mathcal{P}$-positions is always a $\mathcal{P}$-position and the sum of a $\mathcal{P}$ position and an $\mathcal{N}$-position is always an $\mathcal{N}$-position, yet, the sum of two $\mathcal{N}$-positions may be a $\mathcal{P}$- or an $\mathcal{N}$-position; see Section 6 for examples.)

Let S be a set of non-negative integers. The *minimum excludant* of S is denoted by $\text{mex}(S)$ and is defined as the least non-negative integer which is not in S.

The *Sprague-Grundy (SG) function* of a game G, denoted by $\mathcal{G}$, is defined recursively as follows: $\mathcal{G}(x) = 0$ for any terminal position x; otherwise

$$\mathcal{G}(x) = mex\{\mathcal{G}(y) \mid \ y \text{ is an immediate follower of } x\}. \tag{1}$$

The value $\mathcal{G}(x)$ is called the *SG value*, or alternatively the *nim-value*, of position x. It is easily seen that $\mathcal{P}$-positions are exactly the positions of the SG value 0. Thus, the SG function is a refinement of the concept of $\mathcal{P}$-positions.

It follows easily from the above recursion that no move can keep the SG value, but it can be arbitrarily reduced by one move.

Lemma 1. *$\mathcal{G}(x) = n$ if and only if the next two conditions hold:*

(i) *$\mathcal{G}(y) \neq \mathcal{G}(x)$ if there is a move from x to y, in particular, $\mathcal{G}(y) \neq n$ if $\mathcal{G}(x) = n$;*

(ii) *for each integer k such that $0 \leq k < n$ there is a move $x \to y$ such that $\mathcal{G}(y) = k$.* $\square$

The *nim-sum* $\oplus$ of two nonnegative integers is the bitwise sum in the binary number system without carrying, that is, $1 \oplus 1 = 0$. For example, $2 \oplus 3 = 10_2 \oplus 11_2 = 01_2 = 1$.

The SG function of the sum is the nim-sum of the SG functions of its summands.

Theorem 2 ([22, 37, 38])**.** *The SG value of the position (x, y) in the sum $G + H$ is the nim-sum $\mathcal{G}(x) \oplus \mathcal{G}(y)$.* $\square$

Obviously, the nim-sum is an associative and commutative binary operation. Thus, the above theorem can be naturally extended to the sum of k games for any integer $k \geq 2$.

1.5 The misère play

The misère SG value $\mathcal{G}^-(x)$ of a position x in a game G is defined by the same recursion (1), but the initialization is different: for any terminal position $x \in V_T$ we set $\mathcal{G}^-(x) = 1$, rather than $\mathcal{G}^-(x) = 0$.

For an individual game, the misère version can be easily reduced to the normal one by the following simple transformation of the graph $G = (V, E)$. Add to V one new position x_T and an arc (x, x_T) from each former terminal position $x \in V_T$ to x_T. Thus, x_T becomes a unique terminal in the obtained graph G^-. It is easy to verify that for every position $x \in V$ its misère SG value $\mathcal{G}^-(x)$, in the original digraph G, equals the normal SG value $\mathcal{G}(x)$ the extended digraph G^-.

As before, in misère play, a position x is a $\mathcal{P}$-position if and only if $\mathcal{G}^-(x) = 0$. Moreover, after replacing $\mathcal{G}$ by $\mathcal{G}^-$, Lemma 1 still holds, but the corresponding modification of Theorem 2 fails. For this reason, the misère play of the sum is typically more complicated than its normal play.

2 Introduction: main concepts and results

In general, functions $\mathcal{G}$ and $\mathcal{G}^-$ may differ significantly. In this paper, we outline the cases when $\mathcal{G}$ and $\mathcal{G}^-$ are related.

A position x will be called an *i-position* (resp. an (i,j)-position) if $\mathcal{G}(x) = i$ (resp. if $\mathcal{G}(x) = i$ and $\mathcal{G}^-(x) = j$). Furthermore, we will denote by V_i (resp. $V_{i,j}$) the set of all i-positions (resp. (i,j)-positions). Be definition, every terminal position is a $(0,1)$-position. A position $x \in V_{0,1} \cup V_{0,1}$ will be called a *swap position*.

Definition 3. An impartial will be called

(i) *domestic* if it has no $(0,k)$- and no $(k,0)$-positions with $k \geq 2$;

(ii) *tame* if it has only $(0,1)$-, $(1,0)$-, and (k,k)-positions with $k \geq 0$;

(iii) *pet* if it has only $(0,1)$-, $(1,0)$-, and (k,k)-positions with $k \geq 2$.

The tame games were introduced in [14, Chapter 12], pet games were introduced recently in [25, 26], while domestic games are introduced in this paper. According to the above definitions, domestic, tame, and pet games form nested classes: a pet game is tame and a tame game is domestic. Moreover, both containments are strict. Figures 1, 2, 3, and 4 distinguish these three classes.

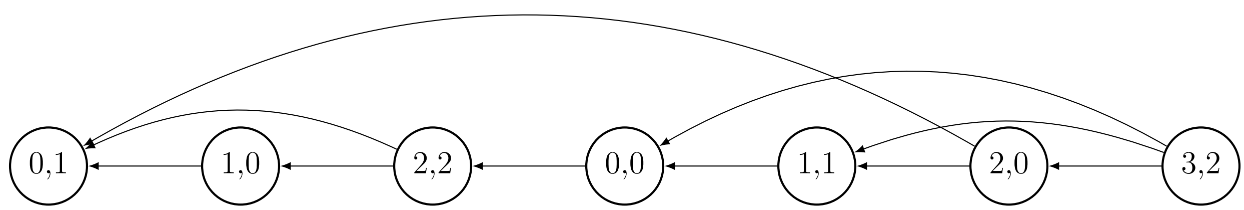

Figure 1: This game is not domestic since it contains a (2,0)-position.

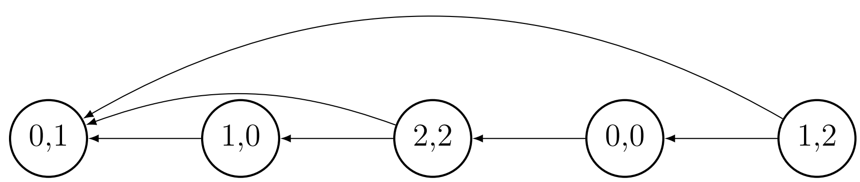

Figure 2: This game is domestic but not tame since it contains a (1,2)-position.

By definition, every non-swap position of a domestic game is an $\mathcal{N}$-position in both normal and misère versions. For the tame games, a stronger property holds: $\mathcal{G}(x) = \mathcal{G}^-(x)$

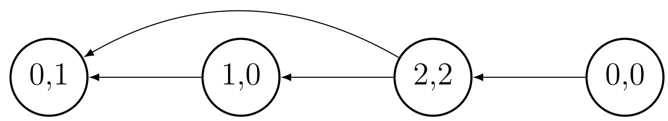

Figure 3: This game is tame but not pet since it contains a (0,0)-position; also it is miserable but not strongly miserable.

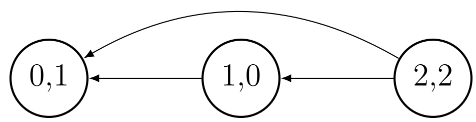

Figure 4: This game is pet.

for every non-swap position x. We will show that the above two properties allow us to reduce efficiently the misère play to the normal one, for any individual domestic game and for the sum of any tame games.

Furthermore, the sum of tame games is tame, as it was announced in [14]; the proof appears in [36]. In contrast, a sum of domestic games may be not domestic. An example will be given in Section 5; see Figure 10.

The following two groups of "technical" properties appear to be closely related to the above three classes. Later, in a slightly modified form, they will be applied as efficient membership tests.

Definition 4. A game G is said to be

(i) *forced* if each move from a $(0,1)$-position leads to a $(1,0)$-position and vice versa;

(ii) *returnable* if the following, weaker, implications hold: let x be a $(0,1)$-position (resp. a $(1,0)$-position) movable to a non-terminal position y, then y is movable to a $(0,1)$-position (resp. to a $(1,0)$-position).

Obviously, a forced game is returnable. Figures 5 and 6 give examples of a non-returnable game and a returnable game that is not forced, respectively. The games in Figures 1, 2, 3, and 4 are all forced.

Definition 5. For each position x let us consider the following properties:

($\mathfrak{a}$) x is a swap position, $x \in V_{0,1} \cup V_{1,0}$;

($\mathfrak{a}_\circ$) $x \in V_{0,1} \cup V_{1,0} \cup V_{0,0} \cup V_{1,1}$;

($\mathfrak{b}$) x is not movable to $V_{0,1} \cup V_{1,0}$;

($\mathfrak{c}$) x is movable to $V_{0,1}$ and to $V_{1,0}$ simultaneously;

($\mathfrak{c}_\circ$) x is movable to $V_{0,1}$ and to $V_{0,0}$ simultaneously;

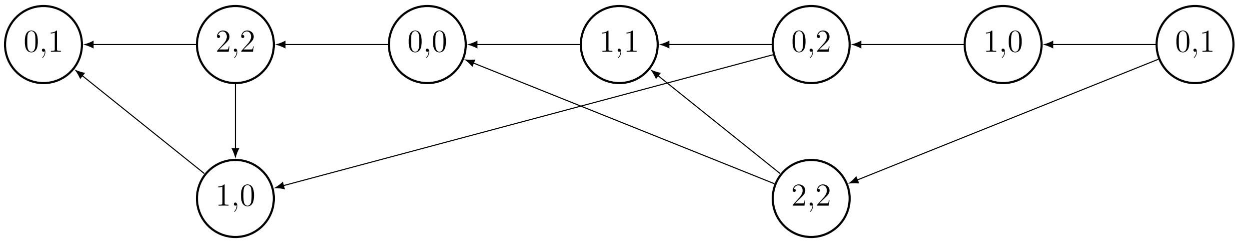

Figure 5: This game is not returnable.

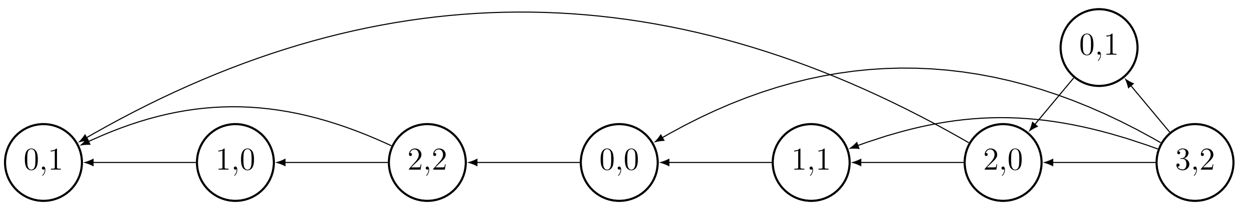

Figure 6: This game is returnable but not forced.

($\mathfrak{c}_1$) x is movable to $V_{1,0}$ and to $V_{0,0}$ simultaneously;

($\mathfrak{e}$) x is movable to $V_{0,0}$ and to $V_{1,1}$ simultaneously.

A game is called

(i) *strongly miserable* if either ($\mathfrak{a}$) or ($\mathfrak{c}$) holds for every position;

(ii) *miserable* if ($\mathfrak{a}$), or ($\mathfrak{b}$), or ($\mathfrak{c}$) holds for every position;

(iii) *t-miserable* if ($\mathfrak{a}_0$), or ($\mathfrak{c}$), or ($\mathfrak{e}$) holds for every position;

(iv) *weakly miserable* if ($\mathfrak{a}$), or ($\mathfrak{b}$), or ($\mathfrak{c}$), or ($\mathfrak{c}_0$), or ($\mathfrak{c}_1$) holds for every position.

The classes of miserable and strongly miserable games were introduced in [24] and [25, 26], respectively. It is immediate from the definitions that the four classes of Definition 5 are nested and the following three examples show that the containments are strict: Figure 2 presents a weakly miserable game that is not t-miserable, since none of the properties $\{(\mathfrak{a}), (\mathfrak{b}), (\mathfrak{c})\}$ holds for the initial position; the game in Figure 8 is t-miserable but not miserable; finally, the same Figure 3 presents also a miserable game that is not strongly miserable, since neither ($\mathfrak{a}$) nor ($\mathfrak{c}$) holds for the initial position.

It was shown in [24, 25] that a game is tame (resp. pet) if it is miserable (resp. strongly miserable). Several other equivalent characterizations of the pet games were also suggested in [26].

The paper is organized as follows. In Section 3 we strengthen the above statements and simplify their proofs; in particular, we show that a game is domestic (resp. tame, pet) if and only if it is weakly miserable (resp. t-miserable, strongly miserable).

Let us note, however, that these characterizations still do not provide straightforward membership tests, because verifying conditions of Definition 5 requires computing the sets $V_{0,1}$, $V_{1,0}$, $V_{0,0}$, and $V_{1,1}$, which are defined recursively. In Section 4 we reformulate the above conditions making them easy to verify.

We say that a class of games *respects the sum* if the sum of games from this class also belongs to it. In Section 5 we prove that tame games, miserable games, forced and miserable games, returnable and miserable games respect their sums, while neither pet nor domestic games do. For the tame games the result was stated in [14] and proved in [36]; we provide a simpler alternative proof.

In Section 6 we apply the results of Section 4 to attribute by the above properties for several well-known classes of games, including NIM, WYTHOFF, EUCLID, SUBTRACTION GAMES, as well as for several old and new variants of these games.

3 Containment relations between the considered classes

3.1 Summary

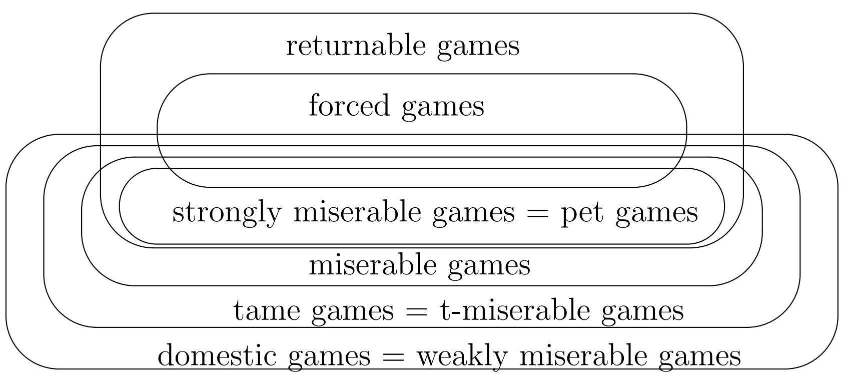

Figure 7: The diagram of containments.

The following classes of games appear to be identical:

- domestic games and weakly miserable games;
- tame games and t-miserable games;
- pet games and strongly miserable games.

Furthermore, the following strict containments hold:

- the pet (strongly miserable) games are miserable and the latter are tame.

We illustrate relations between the six considered classes by the diagram in Figure 7.

The following concept will be instrumental. Given a position x of a game G, let $d(x)$ denote the largest number of successive moves from x to the terminal position, or in other words, the length of the longest play from x. Let us denote by $G_{[x]}$ the subgame of G defined by the initial position x. Obviously, $G_{[x]}$ contains all positions that can be reached from x (by several moves; we assume that this set is finite) and all arcs between these positions.

3.2 Domestic games and weakly miserable games coincide

Lemma 6. *In a domestic game, from each* $(1,0)$*-position there is a move to a* $(0,1)$*-position and from each non-terminal* $(0,1)$*-position there is a move to a* $(1,0)$*-position.*

Proof. From each $(1,0)$-position (resp. non-terminal $(0,1)$-position) there is a move to a $(0,k)$-position (resp. $(k,0)$-position); obviously, $k \neq 0$ (resp. $k \neq 1$). Furthermore, $k \leq 1$ since the game is domestic. □

Theorem 7. *A game is weakly miserable if and only if it is domestic.*

Proof.

($\Rightarrow$) Assume that G is weakly miserable but not domestic. Let x be a $(0,k)$-position with $k \geq 2$ for which $d(x)$ takes the smallest possible value. Then, there is a move from x to a $(k',0)$-position x'. Since $d(x') < d(x)$, from our assumption we conclude that $G_{[x]}$ is domestic and, hence, $k' \leq 1$. Furthermore, $k' = \mathcal{G}(x') \neq 0$ since $\mathcal{G}(x) = 0$ and x is movable to x'; hence, $k' = 1$. Thus, x' is a $(1,0)$-position and ($\mathfrak{b}$) fails for x. Note that ($\mathfrak{a}$) does not hold for x either.

Similarly, x is movable to no position y with $\mathcal{G}(y) = 0$, because $\mathcal{G}(x) = 0$. Therefore, (c), (c_0), and (c_1) fail for x, resulting in a contradiction. Thus, G is domestic.

The case when x is a $(k,0)$-, rather than $(0,k)$-position, is similar.

($\Leftarrow$) Assume that G is domestic. If x is a swap position then ($\mathfrak{a}$) holds for x. If x is a $(0,0)$- or a $(1,1)$-position then ($\mathfrak{b}$) holds for x. If x is an (a,b)-position such that $\max(a,b) \geq 2$ then $\min(a,b) \geq 1$, because G is domestic.

Without loss of generality, assume that $a \leq b$. Since $a \geq 1$ and $b \geq 1$, there is a move from x to a $(0,i)$-position y and to a $(j,0)$-position z. Then, $i \leq 1$ and $j \leq 1$, because G is domestic. If $i = 1$ and $j = 1$ then ($\mathfrak{c}$) holds for x. Otherwise, x is movable to a $(0,0)$-position$^{(\star)}$.

If ($\mathfrak{b}$) fails for x then x is movable to either a $(0,1)$- or a $(1,0)$-position$^{(\star\star)}$. By $^{(\star)}$ and $^{(\star\star)}$, either ($\mathfrak{c}_0$) or ($\mathfrak{c}_1$) holds for x. Hence, the game is weakly miserable. □

3.3 Tame games and t-miserable games coincide

Theorem 8. *A game G is tame if and only if it is t-miserable.*

Proof.

($\Rightarrow$) Let us assume that G is tame and prove that for every position x at least one of three properties ($\mathfrak{a}_\circ$), ($\mathfrak{c}$), ($\mathfrak{e}$) holds.

Furthermore, ($\mathfrak{a}_\circ$) holds for x if x is either a swap, or a $(0,0)$- or a $(1,1)$-position. Assume that x is a (k,k)-position for some $k \geq 2$. By Lemma 1 and its misère version, there are moves from x:

to a $(0,i')$-position x', to a $(1,i'')$-position x'', to a $(i''',0)$-position x''', and to a $(i'''',1)$-position x''''.

Furthermore, $\max(i', i'', i''', i'''') \leq 1$, since the game is tame.

If $i' = 1$ and $i'' = 0$, ($\mathfrak{c}$) holds. If $i' = 0$ and $i'' = 1$, ($\mathfrak{e}$) holds. If $i' = 1$ and $i'' = 1$, we consider x'''. If $i''' = 1$, ($\mathfrak{c}$) holds; otherwise, ($\mathfrak{e}$) holds. If $i' = 0$ and $i'' = 0$, consider x''''. If $i'''' = 1$, ($\mathfrak{e}$) holds; otherwise, ($\mathfrak{c}$) holds.

($\Leftarrow$) Let us assume that ($\mathfrak{a}_\circ$), or ($\mathfrak{c}$), or ($\mathfrak{e}$) holds for every position and prove by induction on $d(x)$ that each x is either a (k,k)-position for some $k \geq 0$ or a swap position. Note that the claim holds when $d(x) \leq 1$. Indeed, $d(x) = 0$ if and only if position x is terminal; in this case x is a $(0,1)$-position. Furthermore, $d(x) = 1$ if and only if every move from x results in a terminal position; in this case x is a $(1,0)$-position.

Let us proceed by induction. Assume that the claim holds for every position x with $d(x) \leq n$, for some $n \geq 1$, and prove it for x with $d(x) = n+1$.

Assume that ($\mathfrak{a}_\circ$) fails for an (a,b)-position x. Then, obviously, $a \geq 2$ or $b \geq 2$. Without loss of generality, assume that $a \geq 2$ and consider two sets

$$M = \{\mathcal{G}(y) \mid \ y \text{ is a follower of } x\} \quad \text{ and } \quad M^- = \{\mathcal{G}^-(y) \mid \ y \text{ is a follower of } x\}.$$

If ($\mathfrak{c}$) or ($\mathfrak{e}$) holds for x, both M and M^- contain both 0 and 1. Furthermore, if y is a follower of x and $y \notin V_{0,1} \cup V_{1,0} \cup V_{0,0} \cup V_{1,1}$, then y is a (k,k)-position for some $k \geq 2$ by the inductive hypothesis. Therefore, $M = M^-$, implying that $\mathcal{G}(x) = \text{mex}(M) = \text{mex}(M^-) = \mathcal{G}^-(x)$ and, hence, x is a (k,k)-position for some $k \geq 0$. $\square$

3.4 Miserable games are tame

Theorem 9. *A miserable game is tame.*

This statement was announced in [24] and shown in [25, 26]. Here we provide simpler arguments.

Proof. Assume that G is miserable and prove by induction on $d(x)$ that every position x is either a swap position or a (k,k)-position for some $k \geq 0$.

The case $d(x) \leq 1$ was already considered in the proof of Theorem 8 above (if $d(x) = 0$ then x is a $(0,1)$-position; if $d(x) = 1$ then x is a $(1,0)$-position).

Let us assume that the claim holds for every position x with $d(x) \leq n$, for some $n \geq 1$, and prove that it holds for every position x with $d(x) = n+1$.

Since G is miserable, ($\mathfrak{a}$), or ($\mathfrak{b}$), or ($\mathfrak{c}$) holds for x.

(i) If ($\mathfrak{a}$) holds, x is a swap position and we are done.

(ii) If ($\mathfrak{b}$) holds, by the inductive hypothesis, each follower y of x is a (k_y, k_y)-position for some $k_y \geq 0$. Therefore, x is a (k, k)-position in which

$$k = \text{mex}\{k_y \mid \ y \text{ is a follower of } x\}.$$

(iii) If ($\mathfrak{c}$) holds, by the inductive hypothesis, each follower y of x is either a swap position or a (k_y, k_y)-position for some $k_y \geq 0$. Therefore, x is a (k, k)-position in which

$$k = \text{mex}\{0, 1, k_y \mid \ y \text{ is a follower of } x \text{ and } y \text{ is a } (k_y, k_y)\text{-position}\}.$$

Note that in this case, $k \geq 2$.

□

Figure 8 provides a tame game that is not miserable showing that the containment of Theorem 9 is strict.

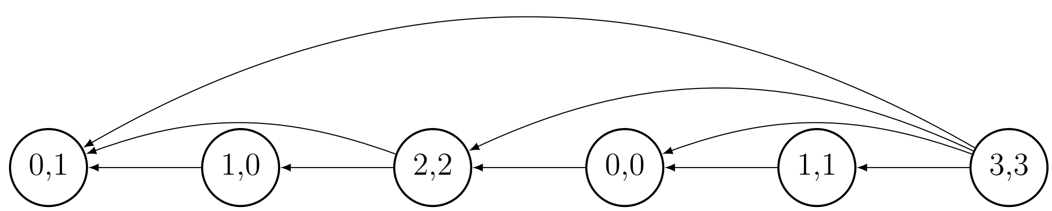

Figure 8: This game is tame but not miserable, since ($\mathfrak{a}$), ($\mathfrak{b}$), and ($\mathfrak{c}$) fail for the initial position.

3.5 Pet games and strongly miserable games coincide

The following list of equivalent characterizations of the pet games was suggested in [26]. Here we simplify proofs.

Theorem 10. *The following properties of a game G are equivalent.*

(i) *G is strongly miserable.*

(ii) *G is pet.*

(iii) *G has no $(0,0)$-position.*

(iv) *G has neither $(0,0)$-position nor $(1,1)$-position.*

(v) *If $\mathcal{G}(x) = 0$ and x is not terminal then x is movable to some x' with $\mathcal{G}(x') = 1$.*

(vi) *If $\mathcal{G}^-(x) = 0$ then x is movable to some x' with $\mathcal{G}^-(x') = 1$.*

Property (v), claiming that any non-terminal 0-position is movable to a 1-position, was introduced (for some other purposes) already in 1974 by Ferguson [16] who proved that it holds for the subtraction games; see Section 6.

Proof of Theorem 10.

(i) $\Rightarrow$ (ii). Every strongly miserable game is miserable and hence tame, by Theorem 9. It remains to show that G has no $(0,0)$- and $(1,1)$-positions. Indeed, assume that x is such a position. Then, properties $(\mathfrak{a})$ and $(\mathfrak{c})$ of Definition 5 fail for x, which is contradiction.

(ii) $\Rightarrow$ (i). Let x be a non-swap position of G. Since G is pet, x is a (k,k)-position for some $k \geq 2$. By Lemma 1 and its misère version, there are moves from x to a $(0,i)$-position and to a $(j,0)$-position. Since G is pet, $i = j = 1$. Thus, $(\mathfrak{c})$ holds for x.

(ii) $\Rightarrow$ (iii). This implication is straightforward.

(iii) $\Rightarrow$ (ii). Assume that G has no $(0,0)$-position and prove by induction on $d(x)$ that every position x is either a swap position or a (k,k)-position for some $k \geq 2$. Standardly, the claim can be verified for the case $d(x) \leq 1$.

Suppose that position x is a counterexample with the smallest value of $d(x)$. The following case analysis results in a contradiction:

(a) Case 1: x is a $(1,1)$-position. Then x is movable to a $(0,e)$-position x_1 with $e \neq 1$. Since $d(x_1) < d(x)$, our choice of x implies $e = 1$, which is impossible.

(b) Case 2: x is a $(0,a)$-position (the case where x is a $(0,a)$-position is treated similarly) with $a \geq 2$. Then x is movable to some $(e,1)$-position x_2 with $e \neq 0$. Since $d(x_2) < d(x)$, our choice of x implies $e = 0$, which is impossible.

(c) Case 3: x is a (b,c)-position with $1 \leq b < c$. Then, there must be three followers x_3, x_4, x_5 of x such that

- x_3 is a $(0,i)$-position for some $i \geq 1$,
- x_4 is a $(j,0)$-position for some $j \geq 1$, and
- x_5 is a (k,b)-position for some $k \geq 0$

By the choice of x, we have $j = 1$, and hence, $b \geq 2$. Furthermore, since $b \geq 2$ and $d(x_5) < d(x)$, we have $k = b$ or equivalently $\mathcal{G}(x_5) = \mathcal{G}(x)$, which is impossible.

(iii) $\Leftrightarrow$ (iv). We already proved that (iii) $\Rightarrow$ (ii). Furthermore, (ii) $\Rightarrow$ (iv) results immediately from the definition of pet games. Thus, (iii) $\Rightarrow$ (iv) holds.

(ii) $\Rightarrow$ (v)(resp.(vi)). Assume that G is pet. Let x be a position with $\mathcal{G}(x) = 0$ (resp. $\mathcal{G}^-(x) = 0$). Since G is pet, x must be a $(0,1)$-position (resp. $(1,0)$-position). Since x is not a terminal position, it is movable to a $(l,0)$-position (resp. to a $(0,l)$-position) for some l. Since G is pet, we have $l = 1$, as required.

(v) $\Rightarrow$ (ii). Assume that (v) holds for a game G that is not pet. Then, G contains a position x that is neither swap nor a (k,k)-position for any $k \geq 2$. Due to symmetry, we can assume that x is either

(1) a $(0,0)$-position, or

(2) a $(1,1)$-position, or

(3) an (m,n)-position with $0 \leq m < n$ and $n \geq 2$.

As usual, let us choose such an x with the smallest $d(x)$. Then,

($\star$) every position x' with $d(x') < d(x)$ is a swap or a (k,k)-position for some $k \geq 2$

In case (1) (resp. (2)), x is movable to a position x' with $\mathcal{G}(x') = 1$, by (v) (resp. $\mathcal{G}(x') = 0$, by the SG Theorem). Then, x' is a $(1,0)$- (resp. a $(0,1)$)-position, by ($\star$) and the assumption $d(x') < d(x)$. Hence, $\mathcal{G}^-(x') = 0 = \mathcal{G}^-(x)$ (resp. $\mathcal{G}^-(x') = 1 = \mathcal{G}^-(x)$), resulting in a contradiction.

Since $n \geq 2$, in case (3) there are moves $x \to x'$ and $x \to x''$ such that $\mathcal{G}^-(x') = 0$ and $\mathcal{G}^-(x'') = 1$, by Lemma 1 and its misère version. Since $d(x') < d(x)$ and $d(x'') < d(x)$, by ($\star$) we conclude that x' and x'' are a $(1,0)$- and $(0,1)$-position, respectively. Hence, $m \geq 2$. Since $\mathcal{G}^-(x) = n > m$, there exists a move $x \to x'''$ such that $\mathcal{G}^-(x''') = m$, that is, x''' is a (r,m)-position for some r. Since $d(x''') < d(x)$ and $m \geq 2$, by ($\star$) we have $r = m$. Thus, that $\mathcal{G}(x) = m = \mathcal{G}^-(x''')$, resulting in a contradiction.

(vi) $\Rightarrow$ (ii). This case is similar to the case $(v) \Rightarrow (ii)$. $\square$

Proposition 11. *Strongly miserable games are returnable.*

Proof. Suppose that G is a strongly miserable game and x is its $(0,1)$- (resp. $(1,0)$)-position. If $d(x) \leq 1$, we are done. Assume that $d(x) \geq 2$. Then each follower x' of x is an (i,j)-position with $i > 0$ (resp. $j > 0$), by Lemma 1 (resp. by its misère version). Hence, x' is movable to a $(0,k)$- (resp. $(k,0)$)-position. Then, $k = 1$, since G is strongly miserable (pet). Thus, G is returnable. $\square$

4 Constructive characterizations of domestic, tame, miserable, and strongly miserable games

4.1 General plan

We could make use of Definitions 3 and 5 to verify whether a game is miserable or strongly miserable, but to do so we have to know its swap positions. It may be even more difficult to verify membership in the other considered classes, because the sets $V_{0,0}$ and/or $V_{1,1}$ become also involved. Since the SG values are defined recursively, it looks difficult to guarantee in advance that a given subset contains all, for example, $(0,1)$-positions; see Definition 5.

To avoid this problem and obtain constructive characterizations, we will modify Definitions 3, 5 and obtain Theorems 12, 13, 14, 15 characterizing strongly miserable (pet), miserable, t-miserable (tame), weakly miserable (domestic) games, respectively. In these theorems sets $V_{0,1}$, $V_{1,0}$, $V_{0,0}$, $V_{1,1}$ of Definition 5 are replaced by some "abstract" sets $V'_{0,1}$,

$V'_{1,0}$, $V'_{0,0}$, $V'_{1,1}$. Requiring (almost) the same properties from the new sets, we characterize all above classes and, moreover, show that the old and new sets are equal, that is, $V'_{i,j} = V_{i,j}$ for all $i, j \in \{0, 1\}$.

We will prove only Theorem 12; the remaining three theorems can be proven in a similar way and we leave them to the reader.

4.2 Strongly miserable games

Let us begin with the strongly miserable (pet) games.

Theorem 12. *A game G is strongly miserable if and only if there exist two disjoint sets $V'_{0,1}$ and $V'_{1,0}$ satisfying the following conditions:*

(i) *both sets are independent, that is, there is no move between two positions of one set;*

(ii) *$V'_{0,1}$ contains all terminal positions, $V_T \subseteq V'_{0,1}$;*

(iii) *$V'_{0,1} \setminus V_T$ is movable to $V'_{1,0}$;*

(iv) *$V'_{1,0}$ is movable to $V'_{0,1}$;*

SM(v) *exactly one of the next two conditions holds for each position x:*

- ($\mathfrak{a}'$) *$x \in V'_{0,1} \cup V'_{1,0}$;*
- ($\mathfrak{c}'$) *x is movable to $V'_{0,1}$ and to $V'_{1,0}$.*

Moreover, if all above conditions hold then $V'_{0,1} = V_{0,1}$ and $V'_{1,0} = V_{1,0}$.

Proof. The "only if" part is straightforward, by setting $V'_{0,1} = V_{0,1}$ and $V'_{1,0} = V_{1,0}$. Let us prove the "if" part. Actually, it is enough to prove that $V'_{0,1} = V_{0,1}$ and $V'_{1,0} = V_{1,0}$. It then follows from condition SM(v) that the game does not have $(0, 0)$-position, and so it is stronly miserable by Theorem 10.

As usual, we proceed by induction on $d(x)$ to show the following claims:

(1) If x is a $(0, 1)$-position, then $x \in V'_{0,1}$.

(2) If x is a $(1, 0)$-position, then $x \in V'_{1,0}$.

(3) If x is not a swap position then x is a (k, k)-position for some $k \geq 2$ and, moreover, ($\mathfrak{c}'$) holds for x.

If $d(x) = 0$ then x is a terminal position and (1) holds, since $V'_{0,1}$ contains V_T. If $d(x) = 1$ then x is a $(1, 0)$-position that is movable to terminal position. Moreover, there are no other moves from x. In particular, it means that there is no move from x that terminates in $V'_{1,0}$. The condition SM(v) implies that $x \in V'_{0,1} \cup V'_{1,0}$. By (i), $x \notin V'_{0,1}$ and so $x \in V'_{1,0}$. Thus (2) holds for x.

The claims $(1) - (3)$ are standardly verified for $d(x) = 0$ and $d(x) = 1$. Let us assume that it holds for every position x with $d(x) \leq n$ for some $n \geq 1$ and prove it for x such that $d(x) = n + 1$.

(1) Let x be a $(0,1)$-position. Then, x is not movable to $V'_{0,1} \cap G_{[x]}$, because each position of this set is a $(0,1)$-position, by the inductive hypothesis on (1), meaning x is not movable to $V'_{0,1}$. From this fact and SM(v) it follows that $x \in V'_{0,1} \cup V'_{1,0}$. We show that $x \notin V'_{1,0}$.

Assume on the contradiction that $x \in V'_{1,0}$. It follows from (iv) that x is movable to a position $y \in V'_{0,1}$. By induction, if $y \in (G_{[x]} \cap V'_{0,1}) \setminus \{x\}$ then y is a $(0,1)$-position. But x is a $(0,1)$-position too and, hence, it cannot be movable to such y. This give a contradiction.

Thus, $x \notin V'_{1,0}$, implying that $x \in V'_{0,1}$ or, equivalently, that (1) holds.

(2) Similarly, assuming that x is a $(1,0)$-position. We can show that $x \in V'_{1,0}$.

(3) Assume that x is not a swap position. We show that (3) holds. First, note that x is neither a $(0,0)$- nor a $(1,1)$-position as well, because $(\mathfrak{a}')$ or $(\mathfrak{c}')$ holds for x.

Let x be a (k,l)-position such that either $k \geq 2$ or $l \geq 2$. Without loss of generality, assume that $k \geq 2$. Then, $(\mathfrak{a}')$ fails for x and, hence, $(\mathfrak{c}')$ holds. It follows that x is movable to a position x' in $V'_{0,1}$ and to a position x'' in $V'_{1,0}$. It remains to show that $l = k$.

Let us consider two sets

$$M = \{\mathcal{G}(y) \mid \ y \text{ is a follower of } x\} \quad \text{ and } \quad M^- = \{\mathcal{G}^-(y) \mid \ y \text{ is a follower of } x\}.$$

We have $\{0,1\} \subseteq M$ and $\{0,1\} \subseteq M^-$, since both x' and x'' are followers of x. Moreover, by the inductive hypothesis, if a follower y of x is not a swap position then y is a (m,m)-position. Therefore, $M = M^-$ and, hence,

$$k = \mathcal{G}(x) = \text{mex}(M) = \text{mex}(M^-) = \mathcal{G}^-(x) = l.$$

□

4.3 Miserable games

The miserable games can be characterized in a very similar way; only property SM(v) of Theorem 12 is slightly changed.

Theorem 13. *A game G is miserable if and only if there exist two disjoint sets $V'_{0,1}$ and $V'_{1,0}$ satisfying conditions* (i) – (iv) *of Theorem 12 and every position x satisfies at least one of the following three conditions:*

($\mathfrak{a}'$) $x \in V'_{0,1} \cup V'_{1,0}$.

($\mathfrak{b}'$) *x is not movable to $V'_{0,1} \cup V'_{1,0}$.*

($\mathfrak{c}'$) *x is movable to $V'_{0,1}$ and to $V'_{1,0}$.*

Moreover, if all above conditions hold then $V'_{0,1} = V_{0,1}$ and $V'_{1,0} = V_{1,0}$.

4.4 Tame games

Similarly, we characterize the tame games as follows.

Theorem 14. *A game is tame if and only if there exist four disjoint sets $V'_{0,1}$, $V'_{1,0}$, $V'_{0,0}$, $V'_{1,1}$ satisfying the following conditions:*

(i) *all four sets are independent;*

(ii) $V'_{0,1}$ *contains the terminal position, ,* $V_T \subseteq V'_{0,1}$*;*

(iii) *if* $x \in V'_{0,1}$ *then* x *is movable to* $V'_{1,0}$ *but not to* $V'_{0,0} \cup V'_{1,1}$*;*

(iv) *if* $x \in V'_{1,0}$ *then* x *is movable to* $V'_{0,1}$ *but not to* $V'_{0,0} \cup V'_{1,1}$*;*

(v) *if* $x \in V'_{0,0}$ *then* x *is not movable to* $V'_{0,1} \cup V'_{1,0}$*;*

(vi) *if* $x \in V'_{1,1}$ *then* x *is movable to* $V'_{0,0}$ *but not to* $V'_{0,1} \cup V'_{1,0}$*;*

(vii) *if* $x \notin V'_{0,1} \cup V'_{1,0} \cup V'_{0,0}$ *then* x *is movable to* $V'_{0,1} \cup V'_{1,0} \cup V'_{0,0}$*.*

T(viii) *Every position* x *satisfies at least one of the following three conditions:*

($\mathfrak{a}_\circ{}'$) $x \in V'_{0,1} \cup V'_{1,0} \cup V'_{0,0} \cup V'_{1,1}$*;*

($\mathfrak{c}'$) x *is movable to* $V'_{0,1}$ *and to* $V'_{1,0}$*;*

($\mathfrak{e}'$) x *is movable to* $V'_{0,0}$ *and to* $V'_{1,1}$*;*

Moreover, $V'_{0,1} = V_{0,1}$*,* $V'_{1,0} = V_{0,1}$*,* $V'_{0,0} = V_{0,0}$*, and* $V'_{1,1} = V_{1,1}$ *whenever all above conditions hold.* □

One may ask, whether conditions (i) - (vii) of Theorem 14 themselves result in equalities $V'_{0,1} = V_{0,1}$ and $V'_{1,0} = V_{1,0}$ too. The answer is negative. The game in Figure 9 provides a counterexample with setting $V'_{0,1} = A$, $V'_{1,0} = B$, and $C \in V_{0,1} \setminus V'_{0,1} \neq \emptyset$.

$$A \longleftarrow B \longleftarrow C$$

Figure 9: $V_{0,1} \neq V'_{0,1}$, although conditions (i) - (viii) of Theorem 14 hold.

4.5 Domestic games

Finally, a similar characterization holds for the domestic games.

Theorem 15. *A game is domestic if and only if there exist three disjoint sets* $V'_{0,1}$*,* $V'_{1,0}$*, and* $V'_{0,0}$ *such that the following conditions hold:*

(i) *all three sets are independent;*

(ii) $V'_{0,1}$ *contains all terminal positions;*

(iii) *if* $x \in V'_{0,1}$ *is non-terminal,* x *is movable to* $V'_{1,0}$ *but not to* $V'_{0,0}$*;*

(iv) *If* $x \in V'_{1,0}$*,* x *is movable to* $V'_{0,1}$ *but not to* $V'_{0,0}$*;*

(v) *If* $x \in V'_{0,0}$*,* x *is not movable to* $V'_{0,1} \cup V'_{1,0}$*;*

(vi) *If* $x \notin V'_{0,1} \cup V'_{1,0} \cup V'_{0,0}$*,* x *is movable to* $V'_{0,1} \cup V'_{1,0} \cup V'_{0,0}$*;*

D(vii) *every position* x *satisfies at least one of conditions*

($\mathfrak{a}'$) $x \in V'_{0,1} \cup V'_{1,0}$*;*
($\mathfrak{b}'$) x *is not movable to* $V'_{0,1} \cup V'_{1,0}$*;*
($\mathfrak{c}'$) x *is movable to* $V'_{0,1}$ *and to* $V'_{1,0}$*;*
($\mathfrak{c}_0{}'$) x *is movable to* $V'_{0,1}$ *and to* $V'_{0,0}$*;*
($\mathfrak{c}_1{}'$) x *is movable to* $V'_{1,0}$ *and to* $V'_{0,0}$*.*

Moreover, in all above conditions hold then $V'_{0,1} = V_{0,1}$*,* $V'_{1,0} = V_{0,1}$*, and* $V_{0,0} = V_{0,0}$*.* □

5 Sums of games

We say that a class of games *respects summation* if the sum of any games from this class belongs to it too. In this section, we show the classes of tame, miserable, miserable and forced, miserable and returnable games respect summation. For the tame games, this property was claimed by Conway in [14] and proven in [36]; we suggest a simpler proof.

In contrast, the classes of domestic and of pet (strongly miserable) games do not respect summation. Already the classic n-pile NIM is a counterexample for the second case. Indeed, one-pile NIM is pet but the n-pile NIM, which is the sum of n one-pile NIM games) is not, whenever $n > 1$; see Subsection 6.1 for more details.

The sum of domestic games may be not domestic; Figure 10 gives an example.

5.1 The sum of tame games is tame

Recall that a *swap* position is either a $(0,1)$-position or a $(1,0)$-position. We will call two swap positions *opposite* if one of them is a $(0,1)$-position while the other is a $(1,0)$-position, and we will call them *parallel* otherwise.

Theorem 16. *If games* G_1 *and* G_2 *are tame then their sum* $G_1 + G_2$ *is tame too. Moreover,* $x = (x_1, x_2)$ *is a swap position of* $G_1 + G_2$ *if and only if* x_i *is a swap position of* G_i *for* $i = 1, 2$*. Furthermore,* x *is a* $(1,0)$*-position of* G *if and only if either* x_1 *is a* $(1,0)$*-position in* G_1 *and* x_2 *is a* $(0,1)$*-position in* G_2 *or vise versa.*

The first claim was stated (without a proof) in 1976 by Conway; see Con76 page 178. A proof based on the genus theory appeared in [36]. Here we give an alternative proof based on our characterization of the tame games by Theorem 14.

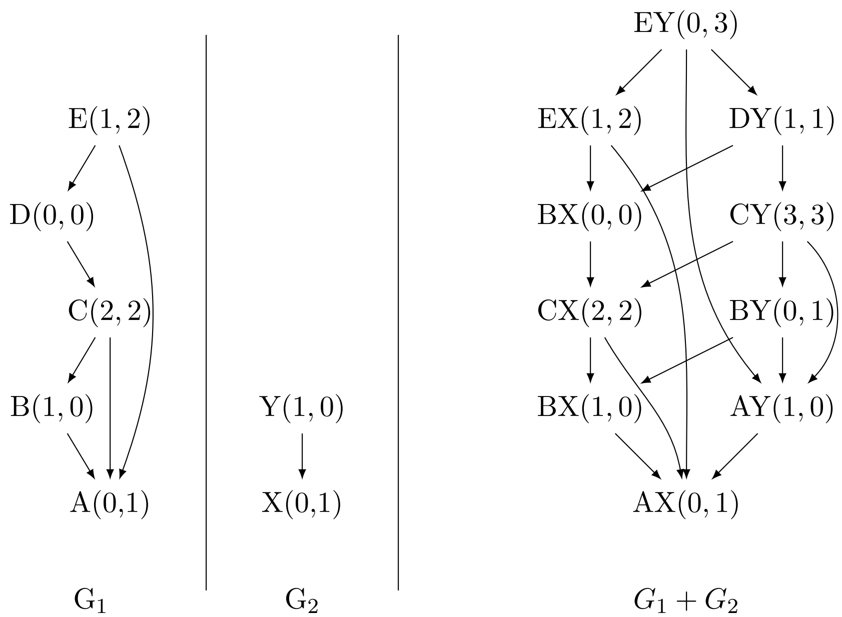

Figure 10: Games G_1 and G_2 are domestic but their sum $G_1 + G_2$ is not. Notation $P(i, j)$ means that P is an (i, j)-position in a summand, while $PQ(i, j)$ means that the sum PQ of P and Q is an (i, j)-position.

Proof. For non-negative integers i, j, k and l, denote by $[(i, j), (k, l)]$ the set of positions $x = (x_1, x_2)$ in the sum $G_1 + G_2$ such that x_1 is an (i, j)-position in G_1 and x_2 is a (k, l)-position in G_2. Let us set

$$\begin{aligned}
V'_{0,1} &= \{[(0,1),(0,1)], [(1,0),(1,0)]\},\\
V'_{1,0} &= \{[(0,1),(1,0)], [(1,0),(0,1)]\},\\
V'_{0,0} &= \{[(0,1),(0,0)], [(0,0),(0,1)], [(n,n),(n,n)] \mid n \in \mathbb{Z}_{\geq 0}\},\\
V'_{1,1} &= \{[(0,0),(1,0)], [(1,0),(0,0)], [(0,1),(1,1)], [(1,1),(0,1)],\\
&\qquad [(n,n),(n+1,n+1)], [(n+1,n+1),(n,n)] \mid n = 2k, k \in \mathbb{Z}_{\geq 0}\}
\end{aligned}$$

Recall that $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers.

It can be verified that the above four sets satisfy conditions (i) - (vii) of Theorem 14. We now prove by induction on $d(x)$ that every position x of the sum $G = G_1 + G_2$ satisfies (at least) one of the conditions $(\mathfrak{a}_\circ')$, $(\mathfrak{c}')$, $(\mathfrak{e}')$ of Theorem 14 and so the sum is tame.

Note that in this proof, when we recall conditions $(\mathfrak{a}_\circ)$, $(\mathfrak{c})$, and $(\mathfrak{e})$ (resp. $(\mathfrak{a}'_\circ)$, $(\mathfrak{c}')$, and $(\mathfrak{e}')$), we refer them in Definition 5 (resp. Theorem 14).

By definition, $x = (x_1, x_2)$ is a terminal position of the sum $G = G_1 + G_2$ if and only if each x_i is a terminal position of the summand G_i, for $i = 1, 2$. Hence, $(\mathfrak{a}_\circ')$ holds for (x_1, x_2). If $d(x_1, x_2) = 1$ then either $d(x_1) = 0$ (x_1 is terminal) and $d(x_2) = 1$ or vise versa and so $(x_1, x_2) \in V'_{1,0}$, meaning $(\mathfrak{a}_\circ')$ holds for (x_1, x_2).

We assume that at least one of the conditions ($\mathfrak{a}_\circ'$), ($\mathfrak{c}'$), ($\mathfrak{e}'$) holds for every position (x_1, x_2) in G such that $d(x_1, x_2) \leq n$ for some $n \geq 1$ and will show that at least one of these conditions holds for each position (x_1, x_2) in G such that $d(x_1, x_2) = n + 1$.

Suppose that ($\mathfrak{a}_\circ'$) fails for $x = (x_1, x_2)$. Then there exists a move from x to a position $x' \in V'_{0,1} \cup V'_{1,0} \cup V'_{0,0}$, by (vii) of Theorem 14. Assume such move $x_1 \to x'_1$ is made in G_1.

(1) Case $x' = (x'_1, x_2) \in V'_{0,1}$. In this case x'_1 and x_2 are two parallel swap positions. Since x_1 is movable to the swap position x'_1, condition ($\mathfrak{a}_\circ$) fails for x_1 and, hence, ($\mathfrak{c}$) or ($\mathfrak{e}$) holds for x, since G_1 is tame.

  (a) If ($\mathfrak{c}$) holds for x_1 then x_1 is movable to a position x''_1 such that x'_1 and x''_1 are two opposite swap positions, then x''_1 and x_2 are two opposite swap positions and, hence, $x'' = (x''_1, x_2) \in V'_{1,0}$, by definition. Recall that x can also be moved to $x' \in V'_{0,1}$. Then ($\mathfrak{c}'$) holds for x.

  (b) If ($\mathfrak{e}$) holds for x_1 then x_1 is movable to some $(0, 0)$-position x'''_1 and to some $(1, 1)$-position x''''_1. It is not difficult to verify that one of these two positions belongs to $V'_{0,0}$, while the other to $V'_{1,1}$ and, hence, ($\mathfrak{e}'$) holds for x.

(2) Case $x' = (x'_1, x_2) \in V'_{1,0}$ is similar to the case (1): just swapping "opposite" and "parallel", as well as "0,1" and "1,0".

(3) Case $x' = (x'_1, x_2) \in V'_{0,0}$. Consider the following three options for x':

  (a) If $(x'_1, x_2) \in [(0, 0), (0, 1)]$ then either x_1 is a (1,1)-position or ($\mathfrak{a}_\circ$) fails for x_1. Yet, the former case cannot occur as otherwise, $x = (x_1, x_2) \in V'_{1,1}$, giving a contradiction. In the latter case, either ($\mathfrak{c}$) or ($\mathfrak{e}$) holds for x_1, since G_1 is tame. It is easily seen that if ($\mathfrak{c}$) (resp. ($\mathfrak{e}$)) holds for x_1 then ($\mathfrak{c}'$) (resp. ($\mathfrak{e}'$)) holds for x.

  (b) Case $x' = (x'_1, x_2) \in [(0, 1), (0, 0)]$ is similar to the case $(x'_1, x_2) \in [(0, 0), (0, 1)]$ treated in (a).

  (c) If both x'_1 and x_2 are (n, n)-positions, we consider two possibilities for n: n is odd and n is even. By checking carefully possible cases for n, one can verify that ($\mathfrak{a}_\circ'$), or ($\mathfrak{c}'$), or ($\mathfrak{e}'$) holds for x. We leave the checking task to the reader.

By induction, we conclude that each position satisfies ($\mathfrak{a}_\circ'$), or ($\mathfrak{c}'$), or ($\mathfrak{e}'$) and, by Theorem 14, sum $G_1 + G_2$ is tame. Moreover, $V_{0,1} = V'_{0,1}$ and $V_{1,0} = V'_{1,0}$, implying that $x = (x_1, x_2)$ is a swap position of the sum $G_1 + G_2$ if and only if x_i is a swap position of the summand G_i for $i = 1, 2$. □

The following obvious generalization results from Theorem 16 and Theorem 2.

Corollary 17. *If games $G_1, \ldots, G_n$ are tame then their sum $G = G_1 + \ldots + G_n$ is tame too. Moreover, a position $x = (x^1, \ldots, x^n)$ of G is a swap position of G if and only if x^i is a swap position of G_i for $i \in \{1, \ldots, n\}$. Furthermore, x is a $(1, 0)$-position if and only if the number of $(1, 0)$-positions in the set $\{x^1, \ldots, x^n\}$ is odd.* □

5.2 The sum of miserable, returnable, and forced games

Theorem 18. *If games G_1 and G_2 are miserable then their sum $G_1 + G_2$ is miserable too. Moreover, $x = (x_1, x_2)$ is a swap position of $G_1 + G_2$ if and only if each x_i is a swap position of G_i for $i = 1, 2$. Furthermore, x is a $(1, 0)$-position of G if and only if either x_1 is a $(1, 0)$-position in G_1 and x_2 is a $(0, 1)$-position in G_2 or vise versa.*

Proof. We proceed by induction on $d(x)$ and prove that every position x in $G = G_1 + G_2$ satisfies condition $(\mathfrak{a})$, or $(\mathfrak{b})$, or $(\mathfrak{c})$ of Definition 5. Note that G_1 and G_2 are tame, by Theorem 9 and, hence, G is tame, by Theorem 16.

Let $x = (x_1, x_2)$ be a position of G. Clearly, $(\mathfrak{a})$ holds when $d(x) = 0$, since in this case both x_1 and x_2 are terminal positions.

Assume that $(\mathfrak{a})$, or $(\mathfrak{b})$, or $(\mathfrak{c})$ holds for every position x with $d(x) \leq n$ for some $n \geq 1$.

Then, by induction, x is either a swap position or a (k, k)-position. We prove that every position x with $d(x) = n + 1$ satisfies $(\mathfrak{a})$, or $(\mathfrak{b})$, or $(\mathfrak{c})$. Assume that $(\mathfrak{a})$ and $(\mathfrak{b})$ fail for x and show that then $(\mathfrak{c})$ holds.

Indeed, $x \notin V_{0,1} \cup V_{1,0}$, since $(\mathfrak{a})$ fails for x, and $x \notin V_{0,0} \cup V_{1,1}$ since $(\mathfrak{b})$ fails for x. Therefore, x is a (m, m)-position for some $m \geq 2$ since G is tame. Furthermore, x is movable to a swap position x', because $(\mathfrak{b})$ fails for x.

Assume that x' is a $(0, 1)$-position.

Furthermore, without loss of generality, we can assume that move $x \to x'$ in G is realized by a move $x_1 \to x'_1$ in G_1. Since G is tame and $x' = (x'_1, x_2)$ is a $(0, 1)$-position, both x'_1 and x_2 are swap positions, by Theorem 16. Moreover, $\mathcal{G}(x'_1) \oplus \mathcal{G}(x_2) = 0$ implies that $\mathcal{G}(x'_1) = \mathcal{G}(x_2)$ and that x'_1 and x_2 are parallel.

In case when x', x'_1, and x_2 are $(1, 0)$- rather than $(0, 1)$-positions, similar arguments are applicable.

Since G_1 is miserable, x_1 satisfies $(\mathfrak{a})$, or $(\mathfrak{b})$, or $(\mathfrak{c})$.

Since x_1 is movable to x'_1, which is a $(0, 1)$-position, $(\mathfrak{b})$ fails for x_1. We claim that $(\mathfrak{a})$ fails for x_1. Indeed, otherwise x_1 is a swap position. Note that x_2 is also a swap position and so x is a swap position by Theorem 16. But this contradicts our assumption that x is a (m, m)-position. Therefore $(\mathfrak{a})$ fails and $(\mathfrak{c})$ holds for x_1.

Then, there is also a move from x_1 to a $(1, 0)$-position x''_1. Note that x'_1 and x''_1 are opposite while x'_1 and x_2 are parallel. Hence, x''_1 and x_2 are opposite. By Theorem 16, $x'' = (x''_1, x_2)$ is a swap position. Moreover, it is a $(1, 0)$-position and a follower of x. Thus, $(\mathfrak{c})$ holds for x.

Then, by induction, $(\mathfrak{a})$, or $(\mathfrak{b})$, or $(\mathfrak{c})$ holds for every position. Therefore, G is miserable. □

The following obvious generalization result from Theorem 18 and Theorem 2.

Corollary 19. *If games $G_1, \ldots, G_n$ are miserable then their sum $G = G_1 + \ldots + G_n$ is miserable too. Moreover, a position $x = (x_1, \ldots, x_n)$ of G is a swap position of G if and only if x_i is a swap position of G_i for $i \in \{1, \ldots, n\}$. Furthermore, x is a $(1, 0)$-position if and only if the number of $(1, 0)$-positions in set $\{x_1, \ldots, x_n\}$ is odd.* □

The classes forced or returnable miserable games respect the summation.

Proposition 20. *The sum of miserable games is returnable whenever all summands are.*

Proof. It is sufficient to prove that $G_1 + G_2$ is returnable whenever G_1 and G_2 are miserable and returnable. Let $x = (x_1, x_2)$ be a swap position in G. By Theorem 18, both x_1 and x_2 are swap positions. Assume that x is movable to some x' in $G_1 + G_2$. Without loss of generality, assume that this move is realized by the move $x_1 \to x_1'$ in G_1. Since G_1 is returnable, there exists a move $x_1' \to x_1''$ in G_1 such that x_1 and x_1'' are either both $(0,1)$-positions or both $(1,0)$-positions. Set $x' = (x_1', x_2)$ and $x'' = (x_1'', x_2)$ and consider moves $x \to x'$ and $x' \to x''$ in $G_1 + G_2$. By Theorems 2 and 18, x and x'' are either both $(0,1)$-positions or both $(1,0)$-positions in G. □

Proposition 21. *The sum of miserable games is forced whenever all summands are forced.*

Proof. It is sufficient to prove that $G = G_1 + G_2$ is forced whenever G_1 and G_2 are miserable and forced. Let $x = (x_1, x_2)$ be a swap position in G. By Theorem 18, both x_1 and x_2 are swap positions. If x_1' is an immediate follower of x_1 in G_1 then x_1' and $x' = (x_1', x_2)$ are swap positions, by Theorem 18. Moreover, Theorems 2 and 18 imply that if x is a $(0,1)$-position (resp. $(1,0)$-position) then x' is a $(1,0)$-position (resp. $(0,1)$-position). These arguments are applicable to any immediate follower of x in G. □

6 Applications

In this section, we show that many classical games fall into classes considered above.

6.1 The game of Nim

This game is played with k piles of tokens. By each move a player chooses one pile and removes an arbitrary (positive) number of tokens from it. The complete analysis of Nim is was given by Charles Bouton in [8], who solved both the normal and misère versions.

Let us start with the trivial case $k = 1$. The next statement is obvious.

Lemma 22. *One-pile* Nim *is a strongly miserable game with exactly one* $(0,1)$*-position, which is the terminal position, and exactly one* $(1,0)$*-position, which is the single pile of size* 1*, while the pile of size* n *is an* (n,n)*-position for all* $n \geq 2$. □

Already the two-pile Nim is not strongly miserable. For example, Nim$(2,2)$ is a $(0,0)$-position.

Proposition 23. *The game of* Nim *is miserable and forced.*

Proof. By Lemma 22 and Theorem 18, Nim is miserable. Let us show that it is forced. Let $x = (x_1, \dots, x_k)$ be a swap position. By Theorem 18, each Nim(x_i) is a swap position, implying either $x_i = 0$ or $x_i = 1$ for every i. Obviously, every move from a swap positions ends in another swap position and changes the parity of the number of ones. □

The above arguments also prove that the $(0,1)$- and $(1,0)$-positions alternate. This immediately results in the following characterization of the sets $V_{0,1}$ and $V_{1,0}$.

Proposition 24. $V_{0,1} = \{(\underbrace{1,\ldots,1}_{2k \text{ entries } 1}) \mid k \geq 0\}$ *and* $V_{1,0} = \{ (\underbrace{1,\ldots,1}_{2k+1 \text{ entries } 1}) \mid k \geq 0\}$. □

6.2 Subtraction games

Subtraction game, denoted by $\mathcal{S}(X)$, is played with a pile of tokens and a set X of positive integers, which may be finite or infinite. A move is to choose an element of X and remove this number of tokens from the pile. Various aspects of this game are exposed in [1, 2, 3, 10, 16].

In [16], Ferguson shows that in any subtraction game each non-terminal 0-position is movable to a 1- position. This and Theorem 10 imply the following statement.

Proposition 25. *Subtraction games are strongly miserable.* □

Since the proof by Ferguson [16] is very short and elegant, we copy it here for the reader's convenience.

Proposition 26 ([16])**.** *Every subtraction game satisfies property* (v) of Theorem 10.

Proposition 26 is based on the following lemma.

Lemma 27 ([16])**.** *Set* $k = \min(X)$. *Then* $\mathcal{G}(x) = 0$ *if and only if* $\mathcal{G}(x+k) = 1$.

Proof. Since $k \in X$, $\mathcal{G}(x) = 0$ implies $\mathcal{G}(x+k) \neq 0$ for all x.

For the necessary condition, assume on the contradiction that there exists the smallest x such that $\mathcal{G}(x) = 0$ and $\mathcal{G}(x+k) > 1$. By the definition of SG values, there exists $s \in X$ such that $\mathcal{G}(x+k-s) = 1$. Since $k = \min(X)$, $k - s \leq 0$. Moreover, $x + k - s \geq k$ or $x - s \geq 0$ (otherwise, there is no move from $x + k - s$ while $\mathcal{G}(x+k-s) = 1$). Furthermore, $\mathcal{G}(x) = 0$ implies $\mathcal{G}(x-s) > 0$. Thus there exists $s' \in X$ such that $\mathcal{G}(x-s-s') = 0$ by the definition of SG values.

Let $y = x - s - s'$. Then $y < x$ and $\mathcal{G}(y) = 0$, implying that $\mathcal{G}(y+k) = 1$, by the choice of the smallest x. However, the last equation implies that $\mathcal{G}(y+k+s') \neq 1$ or, equivalently, $\mathcal{G}(x-s+k) \neq 1$, contradicting $\mathcal{G}(x+k-s) = 1$ as above.

Conversely, if $\mathcal{G}(x) = 1$ and $\mathcal{G}(x-k) \neq 0$, there exists $s \in X$ such that $\mathcal{G}(x-k-s) = 0$. By the necessary condition, $\mathcal{G}(x-s) = 1$, which contradicts $\mathcal{G}(x) = 1$. □

Proof of Proposition 26. Given any non-terminal x such that $\mathcal{G}(x) = 0$, one has $\mathcal{G}(x-k) \neq 0$, where k is the smallest element of X. This implies that there is an $s \in X$ such that $\mathcal{G}(x-k-s) = 0$. From Lemma 27, $\mathcal{G}(x-s) = 1$. □

6.3 Game Mark

A game played with a single pile is called a single-pile Nim-like game if two players take turns removing tokens from that pile. After Subsections 6.1 and 6.2, one may ask whether each single-pile Nim-like game is strongly miserable. The answer is negative. Moreover, such a game may be not even domestic. For example, let us consider the following single-pile Nim-like game suggested by Fraenkel [19] and called Mark. By one move a pile of size n should be reduced to either $n-1$ or $\lfloor \frac{n}{2} \rfloor$.

Proposition 28. *Game* Mark *is not domestic.*

Proof. It is not difficult to verify that 8 is a (0,2)-position. □

6.4 Game Euclid

In 1969 Cole and Davie [12] introduced game Euclid. It is played with two piles of tokens. By one move a player has to remove from the greater pile any number of tokens that is an integer multiple of the size of the smaller pile. The game ends when one of the piles is empty. A position of two piles of sizes x and y is denoted by (x, y). It was shown in [12] that (x, y) is a $\mathcal{P}$-position if and only if $x < y < \phi x$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio [12].

In 1997, Grossman [21] proposed a modification of this game in which the entries must stay positive. In particular, move $(x, y) \to (x, 0)$ is not allowed even y is a multiple of x. Thus, the terminal positions of this game are (x, x) for some positive x.

Note that Grossman's variant is not the misère version of Euclid by Cole and Davie. Also note that in the literature, for examples [24, 31, 33, 35], name "Euclid" commonly refers to Grossman's, rather than Cole and Davie's variant.

The SG function of Grossman's variant was solved in [35] and that of the original game Euclid was solved later in [11], where it was shown that these two SG functions are very similar. Some other variants were also studied in [9, 13, 30].

We now analyze the miserability of these two games. The miserability of Grossman's variant was analyzed in [24].

Proposition 29. *Both Cole and Davie's game and Grossman's game of* Euclid *are miserable and forced.*

Proof. We first prove that Cole and Davie's game miserable. Set $V'_{0,1} = \{(0, x), (x, 0) \mid x \in \mathbb{Z}_{>0}\}$ and $V'_{1,0} = \{(x, x) \mid x \in \mathbb{Z}_{>0}\}$ in which $\mathbb{Z}_{>0}$ is the set of positive integers. Note that if $v \in V'_{0,1}$, then v is a terminal and, hence, a $(0, 1)$-position. If $v \in V'_{1,0}$ then v is movable to a terminal position and, moveover, this is the only move available from v; hence, v is a $(1, 0)$-position.

It is easily seen that if $v \notin V'_{0,1} \cup V'_{1,0}$ then either v is not movable to $V'_{0,1} \cup V'_{1,0}$ or v is movable to $V'_{0,1}$ and to $V'_{1,0}$. Then, by Theorem 13, the game is miserable and, moreover, $V_{0,1} = V'_{0,1}$ and $V_{1,0} = V'_{1,0}$. It follows also that this game is forced.

For Grossman's game, we set $V'_{0,1} = \{(x, x) \mid x \in \mathbb{Z}_{>0}\}$ and $V'_{1,0} = \{(x, 2x), (2x, x) \mid x \in \mathbb{Z}_{>0}\}$ and the same arguments work. □

6.5 Game WYTHOFF

The WYTHOFF game [40] is a modification of the two-pile NIM in which a player by one move is allowed to remove either

(i) an arbitrary number of tokens from one pile, or

(ii) the same number of tokens from both.

Two piles of sizes x and y define a position (x, y). By symmetry, (x, y) and (y, x) are equivalent; we will assume that $x \leq y$ unless the converse is explicitly said.

Let $(x_n, y_n)_{n\geq 0}$, where $x_i < x_j$ if $i < j$, be the sequence of $\mathcal{P}$-positions of the game. Wythoff [40] proved that (x_n, y_n) is a $\mathcal{P}$-position if and only if $x_n = \lfloor \phi n \rfloor$ and $y_n = \lfloor \phi^2 n \rfloor$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. Note that $\lfloor \phi^2 n \rfloor = \lfloor \phi n \rfloor + n$.

The game WYTHOFF and numerous modifications of it were studied intensively in the literature: [4, 7, 15, 17, 18, 20, 27, 29, 39]. However, no explicit formula is known for the SG function of this game. In [20], Fraenkel analyzed the misère version of WYTHOFF and characterized its $\mathcal{P}$-positions. Interestingly, the $\mathcal{P}$-positions of the normal and misère versions ofWYTHOFF differ only by six positions: $\{(0, 0), (1, 2), (2, 1)\} \in V_P^N \setminus V_P^M$, while $\{(0, 1), (1, 0), (2, 2)\} \in V_P^M \setminus V_P^N$. Here V_P^N (resp. V_P^M) is the set $\mathcal{P}$-positions in the normal (resp. misère) version. One can check this claim by comparing [15, Proposition 2] and [20, Theorem 2.1]. Using these results, one can verify directly that the gameWYTHOFF is miserable. Here, we provide an alternative proof using Theorem 13.

Proposition 30. *Game* WYTHOFF *is miserable.*

Proof. Let us set $V'_{0,1} = \{(0, 0), (1, 2), (2, 1)\}$ and $V'_{1,0} = \{(0, 1), (1, 0), (2, 2)\}$. One can easily verify the containments $V'_{0,1} \subseteq V_{0,1}$ and $V'_{1,0} \subseteq V_{1,0}$. Let (x, y) be a position that does not belong to $V'_{0,1} \cup V'_{1,0}$. It is easily seen that either (x, y) is not movable to $V'_{0,1} \cup V'_{1,0}$ or (x, y) is movable to both $V'_{0,1}$ and $V'_{1,0}$. Thus, by Theorem 13, the game WYTHOFF is miserable and, moreover, $V_{0,1} = V'_{0,1}$ and $V_{1,0} = V'_{1,0}$. □

Note that $(3, 5)$ is a $(0, 0)$-position and, thus, WYTHOFF is not strongly miserable.

Proposition 31. WYTHOFF *is returnable but not forced.*

Proof. There is a move from (2, 2), which is a $(1, 0)$-position, to (1, 1), which is a (2, 2)-position; hence, the game is not forced. It is easily seen that the game is returnable. □

6.6 Game WYT(a)

In [17] Fraenkel, for any positive integer a, introduced the following generalization WYT(a) of the game WYTHOFF. This game is also played with two piles of tokens and by one move a player is allowed

(i) to remove an arbitrary number of tokens from one pile, or

(ii) to remove k tokens from one pile and l tokens from the other pile such that $|k-l| < a$.

The game WYT(a) was studied by Fraenkel [17, 20]. Note that WYT(1) is WYTHOFF and, hence, it is miserable.

Proposition 32. *Game* WYT(a) *is strongly miserable whenever* $a \geq 2$.

We first recall results on $\mathcal{P}$-positions of the normal and misère versions.

Proposition 33 ([17])**.** *For* $a \geq 2$*, the sequence* $(x_n, y_n)_{n \geq 0}$ *of* $\mathcal{P}$*-positions of* WYT(a) *satisfies the following conditions:*

(i) $(x_0, y_0) = (0, 0)$;

(ii) *for* $n \geq 1$, $x_n = \text{mex}\{x_i, y_i \mid 0 \leq i < n\}$ *and* $y_n = x_n + an$.

Proposition 34 ([20])**.** *For* $a \geq 2$*, the sequence* $(x'_n, y'_n)_{n \geq 0}$ *of* $\mathcal{P}$*-positions of misère* WYT(a) *satisfies the following conditions:*

(i) $(x'_0, y'_0) = (0, 1)$;

(ii) *for* $n \geq 1$, $x'_n = \text{mex}\{x'_i, y'_i \mid 0 \leq i < n\}$ *and* $y'_n = x'_n + an + 1$.

Corollary 35. *For* $a \geq 2$*, two sets of* $\mathcal{P}$*-positions of* WYT(a) *and its misère version are disjoint.*

Proof. Let (x_m, y_m) be a $\mathcal{P}$-position of WYT(a) and let (x'_n, y'_n) be a $\mathcal{P}$-position of misère WYT(a). If these two positions are coincident then $x_m = x'_n$ and $x_m + am = x'_n + an + 1$. One can then simplify to obtain the equation $a(m - n) = 1$, giving a contradiction as 1 cannot be multiple of a. □

Proof of Proposition 32. It follows immediately from Corollary 35 and Theorem 10 (iii). □

6.7 Game WYT(a, b)

Game WYT(a, b) was introduced in [27], for any two non-negative integers a and b, as follows. Like WYTHOFF, it is played with two piles of tokens. By one move a player is allowed to delete $x \geq 0$ tokens from one pile and $y \geq 0$ tokens from the other such that $x + y > 0$ and $(\min(x, y) < b$ or $|x - y| < a)$. Note that WYT(0, 1) is the two-pile NIM, WYT(1, 1) is WYTHOFF, and WYT(a, 1) is WYT(a).

The following recursive solution of the normal and misère versions of the game was given in [27]

Given an integer $b \geq 1$ and a finite set S of m non-negative integers $s_1, \ldots, s_m$ such that $s_1 < \cdots < s_m$, let us set $s_0 = -b$ and $s_{m+1} = +\infty$. Then, there exists the smallest index $i \in \{0, 1, \ldots, m\}$ such that $s_{i+1} - s_i > b$. Let us define a function mex_b of S as follows:

$$\text{mex}_\text{b}(S) = s_i + b$$

It is easily seen that $\text{mex}_\text{b}(\emptyset) = 0$ and that $\text{mex}_\text{b}(\text{S})$ equals $\text{mex}(S)$ when $b = 1$, that is, $\text{mex}_1 = \text{mex}$.

The $\mathcal{P}$-positions of the normal and its misère versions of game WYT(a, b) are characterized in [27] as follows.

Proposition 36 ([27]). *The sequence $(x_n, y_n)_{n\geq 0}$ of the $\mathcal{P}$-positions of the normal version of game* Wyt(a, b) *satisfies the following recursion:*

$$x_n = \mathrm{mex_b}\{x_i, y_i \mid 0 \leq i < n\}, \quad y_n = x_n + an.$$

□

Proposition 37 ([27]). *The sequence $(x'_n, y'_n)_{n\geq 0}$ of the $\mathcal{P}$-positions of misère version of game* Wyt(a, b) *satisfies the following recursion:*

(i) *if $a = 1$, then $(x'_0, y'_0) = (b+1, b+1)$ and $x'_n = \mathrm{mex_b}\{x'_i, y'_i \mid 0 \leq i < n\}$, $\quad y'_n = x'_n + an$;*

(ii) *if $a \geq 2$, then $x'_n = \mathrm{mex_b}\{x'_i, y'_i \mid 0 \leq i < n\}$, $\quad y'_n = x'_n + an + 1$.* □

Proposition 38 ([27]). *Game* Wyt(a, b) *is strongly miserable whenever $a \geq 2$.* □

Proof. We only need to show that the normal and misère versions of Wyt(a, b) do not share $\mathcal{P}$-positions, or in other words, that there is no $(0, 0)$-position. Then, game Wyt(a, b) is strongly miserable, by Theorem 10.

Let (x_n, y_n) and and (x'_m, y'_m) be $\mathcal{P}$-positions of the normal and misère versions of Wyt(a, b), respectively. Suppose these two positions coincide, $x'_m = x_m$ and $y'_m = y_n$. By Propositions 36 and 37 for case $a \geq 2$, one obtains equality $a(n - m) = 1$, which is a contradiction since 1 cannot be a multiple of a. □

The case $a \leq 1$ was studied in [27, 25]. Combining these results with Proposition 38 we obtain the following criterion.

Proposition 39. *Game* Wyt(a, b) *is miserable and returnable if ($a = 1$ and $b \geq 1$) or ($b = 1$ and $a \leq 1$). Otherwise, the game is strongly miserable.* □

6.8 Moore's Nim$_{n,\leq k}$ and its variants

6.8.1 Moore's Nim$_{n,\leq k}$

The following game was introduced in 1910 by Moore [34]. Let k and n be two positive integers such that $k \leq n$. By one move a player has to reduce (strictly) at least 1 and at most k from given n piles of $(x_1, \ldots, x_n)$ tokens. Moore denoted this game by Nim$_k$, but we will use notation Nim$_{n,\leq k}$ to include n.

Obviously, Nim$_{n,\leq k}$ turns into the standard Nim when $k = 1$. Moore generalized Bouton's solution of Nim as follows.

First, present $(x_1, \ldots, x_n)$ as binary numbers:

$x_i = x_{i_0} + x_{i_1}2 + x_{i_2}2^2 + \cdots + x_{i_m}2^m$. $i \in [n] = \{1, \ldots, n\}$.

Then, take their bitwise sum modulo $k+1$ for each index j: $y_j = (\sum_{i=1}^{n} x_{i_j}) \bmod (k + 1)$ and denote by $M(x)$ the obtained number $y_0y_1 \ldots y_m$, in base $k + 1$). Moore [34] shows that $(x_1, \ldots, x_n)$ is a $\mathcal{P}$-position if and only if $M(x) = 0$. Moreover, 1-positions can be characterized similarly.

Theorem 40 ([32])**.** *For any $k \in \{1, \ldots, n\}$ and $m \in \{0, 1\}$, we have: $\mathcal{G}(x) = m$ if and only if $M(x) = m$.* □

This result was obtained in 1980 by Jenkyns and Mayberry; see also [6] for a slightly simpler proof.

We will show that game of $\text{Nim}_{n,\leq k}$ is miserable. For $k = 1$, it is known.

Proposition 41. *The game of $\text{Nim}_{n,\leq k}$ is miserable for $2 \leq k < n$. Moreover, let $x = (x_1, \ldots, x_n)$ be a position in $\text{Nim}_{n,\leq k}$ and l be the number of non-empty piles in x. Then*

(a) x is a $(0,1)$-position if and only if $x_i \leq 1$ for all i and $l \equiv 0 \bmod (k+1)$;

(b) x is a $(1,0)$-position if and only if $x_i \leq 1$ for all i and $l \equiv 1 \bmod (k+1)$.

Proof. Set

$$V'_{0,1} = \{(x_1, \ldots, x_n) \mid \ x_i \leq 1 \text{ for all } i \text{ and } l \equiv 0 \bmod (k+1)\}$$

and

$$V'_{1,0} = \{(x_1, \ldots, x_n) \mid \ x_i \leq 1 \text{ for all } i \text{ and } l \equiv 1 \bmod (k+1)\}.$$

We verify the conditions (i) - M(v) of Theorem 13. Condition (i) holds since there is no move between two arbitrary positions in each set since such a move must reduce $k+1$ piles. Condition (ii) holds sine $V'_{0,1}$ contains the terminal position $(0, 0, \ldots, 0)$. Condition (iii) holds since from every non-terminal position in $V'_{0,1}$, the move removing exactly k tokens terminates in $V'_{1,0}$. Condition (iv) holds since from every position in $V'_{1,0}$, the move removing exactly one token terminates in $V'_{0,1}$. It remains to verify condition M(v).

Let x be a position not in the set $V'_{0,1} \cup V'_{1,0}$. If there is no move from x that terminates in $V'_{0,1} \cup V'_{1,0}$ then the condition M(v) holds and we are done. Assume that this is not the case. Then there exists one move M_1 from x that terminates in either $V'_{0,1}$ or $V'_{1,0}$. We need to prove that x is movable to both $V'_{0,1}$ and $V'_{1,0}$

Note that a move from x reduces at most k piles $x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}$ for a permutation π, meaning

$$(M_1): \quad (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \to (x_{\pi(1)} - y_1, x_{\pi(2)} - y_2, \ldots, x_{\pi(k)} - y_k)$$

with at least some $y_j \geq 1$.

(1) If the move (M_1) terminates in $V'_{0,1}$, then it leaves $m(k+1)$ entries of size 1.

(a) If $x_{\pi(i)} - y_i = 1$ for all i, then the corresponding move

$$(M_2): \quad (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \to (x_{\pi(1)} - y_1 - 1, x_{\pi(2)} - y_2 - 1, \ldots, x_{\pi(k)} - y_k - 1)$$

terminates in $V'_{1,0}$, leaving $(m-1)(k+1)+1$ entries of size 1.

(b) If $x_{\pi(i)} - y_i = 0$ for some i, then either there exist i_0 such that $y_{i_0} \geq 2$ or there exist i_0 and j_0 such that $y_{i_0} \geq 1$ and $y_{j_0} \geq 1$. In fact, if otherwise, $x \in V'_{1,0}$, giving a contradiction. In either of cases, we can choose $(y'_1, y'_2, \ldots, y'_k)$ such that $0 \leq y'_i \leq y_i$ and $y'_1 + y'_2 + \cdots + y'_k = y_1 + y_2 + \cdots + y_k - 1$. Then the corresponding move

$$(M_3) \quad (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \to (x_{\pi(1)} - y'_1, x_{\pi(2)} - y'_2, \ldots, x_{\pi(k)} - y'_k)$$

terminates in $V'_{1,0}$, leaving $m(k+1)+1$ entries of size 1.

(2) If the move (M_1) terminates in $V'_{1,0}$, then it leaves $m(k+1)+1$ entries of size 1.

(a) If $x_{\pi(i_0)} > y_{i_0}$ for some i_0, then we define $y'_i = y_i$ for all i, except for $y'_{i_0} = x_{\pi(i_0)}$. The move

$$(M_4) \quad (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \to (x_{\pi(1)} - y'_1, x_{\pi(2)} - y'_2, \ldots, x_{\pi(k)} - y'_k)$$

terminates in $V'_{0,1}$, leaves $m(k+1)$ entries of size 1. Here (M_4) imitates (M_1) before removing the whole pile $x_{\pi(i_0)}$.

(b) If $x_{\pi(i)} = y_i$ for all i, we consider two cases.

(i). If $y_{i_0} = 0$ for some i_0, we can choose some pile $x_{j_0} \notin \{x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}\}$ of size 1 which is not touched in the move (M_1). Then the move

$$(M_5): \quad (x_{j_0}, x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \to (0, x_{\pi(1)} - y_1, x_{\pi(2)} - y_2, \ldots, x_{\pi(k)} - y_k)$$

terminates in $V'_{0,1}$. Note that (M_5) imitates (M_1) before removing the pile x_{j_0}, resulting in $m(k+1)$ entries of size 1.

(ii). If $x_{\pi(i)} = y_i > 0$ for all i, then there exists i_0 such that $x_{i_0} \geq 2$. Otherwise, $x \in V'_{0,1}$. Now, we have $y_{i_0} - 1 = x_{i_0} - 1 \geq 1$. Then the move

$$(M_6): \quad (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \to (x_{\pi(1)} - (y_1 - 1), x_{\pi(2)} - (y_2 - 1), \ldots, x_{\pi(k)} - (y_k - 1))$$

terminates in $V'_{0,1}$, leaving $(m+1)(k+1)$ entries of size 1.

$\square$

6.8.2 An extension of Nim$_{n,\leq k}$

We extend Nim$_{n,\leq k}$ to a game called Extended Nim$_{n,\leq k}$ that has an extra pile with x_0 tokens. By one move, it is allowed to reduce x_0 and at most k of the remaining n piles. Note that at least one pile must be reduced strictly; reducing x_0 is not compulsory and reducing only x_0 is legal. When $k = n-1$, the game Extended Nim$_{n,\leq n-1}$ is called Extended Complementary Nim, or Exco-Nim, for short, [5]. For $n \geq 3$, the Sprague-Grundy function of Exco-Nim was characterized in [5] as follows.

Theorem 42 ([5]). *Given an* EXCO-NIM *game with $n \geq 3$, let us define*

$$u(x) = \sum_{i=0}^{n} x_i, \quad m(x) = \min_{i=1}^{n} x_i, \quad y = u(x) - n\, m(x), \quad z = \frac{1}{2}(y^2 + y + 2).$$

The Sprague-Grundy function of this game is given by the following explicit formula:

$$\mathcal{G}(x) \;=\; \begin{cases} u(x), & \text{if } m(x) < z; \\ (z-1) + \big((m-z) \mod (y+1)\big), & \text{otherwise.} \end{cases}$$

In case $x_0 = 0$, this result was obtained in 1980 by Jenkyns and Mayberry [32].
Somewhat surprisingly, case $n = 2$ Is much more difficult; see [5] for partial results.

Furthermore, the following criterion of miserability holds.

Proposition 43. *Let $n \geq 3$ and $1 \leq k < n$. Game* EXTENDED $\text{NIM}_{n,\leq k}$ *is miserable. Moreover, $x = (x_0, x_1, \ldots, x_n)$ is a $(0,1)$- (resp. $(1,0)$)-position if and only if $x_0 = 0$ and $(x_1, \ldots, x_n)$ is a $(0,1)$- (resp. $(1,0)$)-position of* $\text{NIM}_{n,\leq k}$. □

The proof is essentially similar to that of Proposition 41 and we leave it to the reader.

6.8.3 EXACT k-NIM

Let us consider a modification of $\text{NIM}_{n,\leq k}$ in which by one move a player must (strictly) reduce exactly k piles. We denote this game by $\text{NIM}_{n,=k}$. A closed formula for its SG function was obtained in [6] for the case $n \leq 2k$.

For $0 < k < n$ and $x = (x_1, \ldots, x_n)$, define the tetris function $T_{n,k}(x)$ as follows. Set $x^0 = x$, subtract 1 from the k largest coordinates of x^0 getting the next n-vector x^1, do the same for x^1 getting x^2, etc., as long as x^i remains non-negative. Let x^t be the last such n-vector, which contains more than $n - k$ zero coordinates. Then, $T_{n,k}(x) = t$.

It was proven in [6] that $\mathcal{G}(x) = T_{n,k}(x)$ when $n < 2k$.

Case $n = 2k$ is more difficult. In this case EXACT k-NIM appears to be surprisingly close to Complementary NIM.

Theorem 44 ([6]). *For each position $x = (x_1, \ldots, x_n)$ of the game* $\text{NIM}_{n,=k}$ *with $n = 2k$, set*

$$m(x) = \min_{i=1}^{n} x_i, \quad y(x) = T_{n,k}(x_1 - m(x), \ldots, x_n - m(x)), \quad z = \frac{1}{2}(y^2 + y + 2).$$

Then, for $n = 2k \geq 4$ the SG function of $\text{NIM}_{n,=k}$ *is given by the formula*

$$\mathcal{G}(x) \;=\; \begin{cases} T_{n,k}(x), & \text{if } m(x) < z(x); \\ z(x) - 1 + \Big(\big(m(x) - z(x)\big) \mod \big(y(x)+1\big)\Big), & \text{otherwise.} \end{cases}$$

Thus, case $n < 2k$ is relatively simple, $n = 2k$ is more difficult, while $n > 2k$ looks intractable.

Let us show that the game is miserable when $n \leq 2k$. We start with the case $n = 2k$.

Proposition 45. *Game* $\text{NIM}_{2k,=k}$ *is miserable. Moreover,* $x = (x_1, \ldots, x_n)$ *is a*

(i) $(0,1)$*-position if and only if* $x_1 = \cdots = x_{k+1} \le 1$*;*

(ii) $(1,0)$*-position if and only if Tetris function* $T_{2k,k}(x) = 1$*.*

Proof. We leave to the reader to check that two sets $V'_{0,1} = \{x = (x_1, \ldots, x_n) \mid x_1 = \cdots = x_{k+1} \le 1\}$ and $V'_{1,0} = \{x \mid d(x) = 1\}$ satisfy conditions in Theorem 13; hence the game is miserable with $V_{0,1} = V'_{0,1}$ and $V_{1,0} = V'_{1,0}$. □

Recall that $d(x)$ is the largest number of moves from x to the terminal position.

Proposition 46. *Game* $\text{NIM}_{n,=k}$ *with* $n < 2k$ *is strongly miserable.*

Proof. Note that if $x = (x_1, \ldots, x_n)$ is a $\mathcal{P}$-position then x is terminal. Indeed, every non-terminal position is movable to the terminal position by eliminating the piles $x_1, \ldots, x_k$ and, thus, leaving at most $n - k = k - 1$ nonempty piles. By definition, a positions with at most $k - 1$ nonempty piles is terminal.

In other words, $\mathcal{G}(x) = 0$ if and only if x is the terminal position, which is also a $(0,1)$-position. In particular, there are no $(0,0)$-position and, by Theorem 10, the game is strongly miserable. □

Many games $\text{NIM}_{n,=k}$ with $k < n/2$ are not even domestic. For example, our computations show that $(1,2,3,3,3)$ is a $(0,2)$-position of $\text{NIM}_{5,=2}$.

6.8.4 SLOW k-NIM

Let us now consider a modification of $\text{NIM}_{n,\le k}$ in which a move consists of choosing at least one and at most k from n piles and removing *exactly one* token from each of them. The obtained game is denoted by $\text{NIM}^1_{n,\le k}$; it was analyzed in [28]. Let us recall results concerning its SG function for $n = 2, 3$.

Given a nonnegative integer n-vector $x = (x_1, \ldots, x_n)$ with $x_1 \le \cdots \le x_n$, its *parity vector* $p(x)$ is defined as the n-vector $p(x) = (p(x_1), \ldots, p(x_n))$ whose coordinates take values $\{e, o\}$ such that $p(x_i) = e$ if x_i is even and $p(x_i) = o$ if x_i is odd.

Proposition 47 ([28])**.** *The SG values for* $n = k = 2$ *and* $n = 3, k = 2$ *are uniquely defined by* $p(x)$ *as follows:*

(i) *For* $n = k = 2$*,*

$$\mathcal{G}(x) = \begin{cases} 0, & \textit{if } p(x) = (e,e); \\ 1, & \textit{if } p(x) = (e,o); \\ 2, & \textit{if } p(x) = (o,o); \\ 3, & \textit{if } p(x) = (o,e). \end{cases}$$

(ii) *For* $n = 3, k = 2$,

$$\mathcal{G}(x) = \begin{cases} 0, & \textit{if } p(x) \in \{(e,e,e),(o,o,o)\}; \\ 1, & \textit{if } p(x) \in \{(e,e,o),(o,o,e)\}; \\ 2, & \textit{if } p(x) \in \{(e,o,o),(o,e,e)\}; \\ 3, & \textit{if } p(x) \in \{(e,o,e),(o,e,o)\}. \end{cases}$$

Relations between the normal and misère versions are summarized by the following statement.

Proposition 48. *For* $k \geq n-1$, *the game of* SLOW k-NIM *is miserable. Moreover,*

(i) *if* $k = n$, $V_{0,1} = \{(0,0,\dots,0,2j) \mid j \in \mathbb{Z}_{\geq}\}$ *and* $V_{1,0} = \{(0,0,\dots,0,2j+1) \mid j \in \mathbb{Z}_{\geq}\}$;

(ii) *if* $k = n-1$, $V_{0,1} = \{(i,i,\dots,i,i+2j) \mid i,j \in \mathbb{Z}_{\geq}\}$ *and* $V_{1,0} = \{(i,i,\dots,i,i+2j+1) \mid i,j \in \mathbb{Z}_{\geq}\}$.

Proof. For $k = n$ and For $k = n-1$ let us respectively set

$V'_{0,1} = \{(0,0,\dots,0,2j) \mid j \in \mathbb{Z}_{\geq}\}$ and $V'_{1,0} = \{(0,0,\dots,0,2j+1) \mid j \in \mathbb{Z}_{\geq}\}$.

$V'_{0,1} = \{(i,i,\dots,i,i+2j) \mid i,j \in \mathbb{Z}_{\geq}\}$ and $V'_{1,0} = \{(i,i,\dots,i,i+2j+1) \mid i,j \in \mathbb{Z}_{\geq}\}$.

We leave to the reader to verify that these two sets $V'_{0,1}$ and $V'_{1,0}$ satisfy all conditions of Theorem 13 and, hence, the game is miserable with $V_{0,1} = V'_{0,1}$ and $V_{1,0} = V'_{1,0}$. □

Our computations show that game $\text{NIM}^1_{4,\leq 2}$ is not domestic; for example, $(1,1,2,3)$ is a $(4,0)$-positions. Thus, case $k = n-2$ differs a lot from the case $k = n-1$ corresponding to the Complementary NIM.

6.9 Heap overlapping NIM

The following generalization of the game of NIM is introduced in [5] and called HO-NIM, where HO stands for "Heap Overlapping". A position of this game involves a ground set V and a family of its subsets $\mathcal{H} = \{H_1,\dots,H_n\}$. A move from such a position consists of choosing an arbitrary non-empty subset S of a set H_i and deleting $S \cap H_j$ from each set H_j, thus, getting a new position $\{H_1 \setminus S,\dots,H_n \setminus S\}$. Note that HO-NIM $(\mathcal{H})$ becomes classic NIM if the subsets H_i are pairwise disjoint.

The following equivalent representation of HO-NIM $\{H_1,\dots,H_n\}$ will be more convenient. Set $[n] = \{1,\dots,n\}$ and let E be the set of all nonempty subsets of $[n]$. The non-negative integer weight function $x(e) = \#\{\bigcap_{i \in e} H_i \setminus \bigcup_{j \notin e} H_j\}$ on E denotes the cardinality of the set of elements belonging to all sets H_i with $i \in e$ and to no other. Obviously, the following equality holds:

$$|V| = |\bigcup_{1 \leq i \leq n} H_i| = \sum_{e \in E} x(e).$$

Then, HO-NIM $\{H_1, \ldots, H_n\}$ can be represented as $\mathcal{F} = (x(e) \mid e \in E)$, where a move is to choose some $i \in [n]$ and reduce arbitrarily (but at least one strictly) the weights $x(e)$ such that $i \in e$.

The classic NIM $(H_1, \ldots, H_n)$ is HO-NIM $\mathcal{F} = \{x(1), \ldots, x(n)\}$, where $x(i)$ is the size of the pile H_i. Let us give one more example. Let $V = \{a, b, c, d, e, f\}$ be a ground set and let $H_1 = \{a, b, c\}$, $H_2 = \{a, d\}$, $H_3 = \{c, e\}$, and $H_4 = \{b, c, f\}$. Then, HO-NIM (H_1, H_2, H_3, H_4) can be represented as $\mathcal{F} = \{x(2), x(3), x(4), x(1, 2), x(1, 4), x(1, 3, 4)\}$, where $x(2) = |\{d\}|$, $x(3) = |\{e\}|$, $x(4) = |\{f\}|$, $x(1, 2) = |\{a\}|$, $x(1, 4) = |\{b\}|$, $x(1, 3, 4) = |\{c\}|$ and $x(e) = 0$ for all other subsets $e \subseteq [n]$. Standardly, by $|S|$ we denote the cardinality of the set S.

More results on HO-NIM games are discussed in [5]. We will show that this class of games contains examples of domestic, miserable, and strongly miserable games.

Let us start with hypergraph $\mathcal{F}\binom{n}{n-1} = \{x([n]); x([n] \setminus \{i\}) \mid i \in [n]\}$. For example, $\mathcal{F}\binom{2}{1} = \{x(1, 2); x(1), x(2)\}$ and $\mathcal{F}\binom{3}{2} = \{x(1, 2, 3); x(1, 2), x(1, 3), x(2, 3)\}$. We will add a semicolon (;) to separate $x([n])$ from other $x(e)$. When $x([n]) = 0$, HO-NIM turns into NIMB$(n, n-1)$.

Remark 49. For each $n \geq 3$, the SG function for $\mathcal{F}\binom{n}{n-1}$ is evaluated by Theorem 42.

For each position x in HO-NIM, denote by $[x]$ the set of x and all positions that are symmetric to x. For example, in the game $\mathcal{F}\binom{3}{2}$, the two positions $(1; 1, 1, 0)$ and $(1; 0, 1, 1)$ are symmetric to $(1; 1, 0, 1)$. Positions in $[x]$ are viewed as identical to x.

For the following results we only sketch briefly the proofs leaving the rest to the reader.

Proposition 50. *Game* HO-NIM $\mathcal{F}\binom{n}{n-1}$ *is miserable and returnable if* $n \geq 2$ *and it is forced when* $n = 2$.

Proof. Set $x^0 = (0; 0, \ldots, 0)$, $x^e = (0; 1, \ldots, 1)$, $x' = (1; 0, \ldots, 0)$, $x^1 = (0; 1, 0, \ldots, 0)$, $x^2 = (0; 0, 1, 0, \ldots, 0)$, $\ldots$, $x^n = (0; 0, \ldots, 0, 1)$, $V'_{0,1} = \{x^0, x^e\}$, and $V'_{1,0} = \{x', x^i \mid i \in [n]\}$.

It is easily seen that $V'_{0,1} \subseteq V_{0,1}$ and $V'_{1,0} \subseteq V_{1,0}$. Let $x \notin V'_{0,1} \cup V'_{1,0}$. One can verify that x is movable to $V'_{0,1}$ if and only if x is movable to $V'_{1,0}$. By Theorem 13 the game is miserable and $V'_{0,1} = V_{0,1}$ and $V'_{1,0} = V_{1,0}$.

Note that the game is not strongly miserable; for example, position $(0; 2, 2)$ in the game $\mathcal{F}\binom{2}{1} = \{x(1, 2); x(1), x(2)\}$, which becomes NIM$(2, 2)$, is a $(0, 0)$-position.

Note also that position x^0 is terminal. From x' and each x^i, where $i \in [n]$, there is only one move, which ends in the terminal position. If $n = 2$ then there are exactly two moves from x^e, to x^1 and to x^2, which are $(1, 0)$-positions. Hence, the game is forced.

If $n > 2$, we show that it is returnable. Note that x^0 is the terminal position. From each position x reachable from x^e there is a move to x^0 by removing all tokens. Also note that from every position $x \in V_{1,0}$, there is exactly one move only, removing the unique token, and that move leads to the terminal position. Thus, the game is returnable.

Thus, HO-NIM $\mathcal{F}\binom{n}{n-1}$ is returnable for any $n \geq 2$ and forced for $n = 2$. $\square$

For $n \geq 3$, we consider HO-NIM game whose hypergraph is corresponding to cycle

$$\mathcal{F}(C_n) = \{x(1, 2, \ldots, n); x(1, 2), x(2, 3), x(3, 4), \ldots, x(n-1, n), x(n, 1)\}.$$

For examples, we have

$$\mathcal{F}(C_3) = \mathcal{F}\binom{3}{2} = \{x(1,2,3); x(1,2), x(1,3), x(2,3)\},$$
$$\mathcal{F}(C_4) = \{x(1,2,3,4); x(1,2), x(2,3), x(3,4), x(4,1)\}.$$

By Proposition 50, $\mathcal{F}(C_3)$ is miserable. Let us show that $\mathcal{F}(C_4)$ is still miserable, while $\mathcal{F}(C_5)$ is not tame and $\mathcal{F}(C_6)$ is not domestic.

Proposition 51. Ho-Nim $\mathcal{F}(C_4)$ *is miserable and forced.*

Proof. Set $V'_{0,1} = \{(0;0,0,0,0), [(0;0,1,0,1)]\}$ and $V'_{1,0} = \{[(0;0,0,0,1)]; (1;0,0,0,0)\}$. By Theorem 13, the game is miserable; moreover, $V'_{0,1} = V_{0,1}$ and $V'_{1,0} = V_{1,0}$. Furthermore, every move from a position in $V_{1,0}$ ends in $(0;0,0,0,0)$, which is the (unique) terminal, while every move from a position of $V_{0,1}$ terminates in $V_{1,0}$. Hence, the game is forced. Note that the $(0,0)$-positions of this game are $\{(0;a,b,a,b) \mid a,b \in \mathbb{Z}_{\geq 0}\}$. □

Proposition 52. Ho-Nim $\mathcal{F}(C_5)$ *is domestic but not tame.*

Proof. Our computations show that $x = (0;0,1,1,1,2)$ is a $(5,1)$-position, hence, the game is not tame. However, it is domestic. Indeed, the $(0,0)$-positions form the set

$$\begin{aligned} V_{0,0} &= \{[0;c,a+c,b+c,c,a+b+c] \mid a,b,c \in \mathbb{Z}_{\geq 0}\} \\ &= \{[(0;0,a,b,0,a+b) + (0;c,c,c,c,c)] \mid a,b,c \in \mathbb{Z}_{\geq 0}\}. \end{aligned} \tag{2}$$

Let us set $V'_{0,0} = V_{0,0}$, and

$$V'_{0,1} = \{(0;0,0,0,0,0), (0;1,1,1,1,1), [(0;0,0,1,0,1)]\} = \{x^0, x^5, [x^2]\},$$

$$V'_{1,0} = \{(1;0,0,0,0,0), [(0;0,0,0,0,1)], [(0;0,1,1,1,1)]\} = \{y^1, [x^1], [x^4]\}.$$

By Theorem 15, the game is domestic. □

Remark 53. Ho-Nim $\mathcal{F}(C_6)$ is not domestic. Position $(0;1,1,1,1,1,1)$ is a $(0,2)$-position.

For $n \geq 3$, we consider Ho-Nim

$$\mathcal{F}(K_n) = \{x([n]); x(1,2), x(1,3), \ldots, x(1,n), x(2,3), \ldots, x(2,n), \ldots, x(n-1,n)\}.$$

For example,

$$\mathcal{F}(K_4) = \{x(1,2,3,4); x(1,2), x(1,3), x(1,4), x(2,3), x(2,4), x(3,4)\}.$$

Note that $\mathcal{F}(K_3)$ is $\mathcal{F}(C_3)$, which is miserable.

Remark 54.

(i) Ho-Nim $\mathcal{F}(K_4)$ is not domestic. The position $(0;0,1,1,2,2,1)$ is a $(2,0)$-position.

(ii) HO-NIM $\mathcal{F}(K_5)$ is not domestic. The position $(0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ is a $(0, 2)$-position.

For $n \geq 2$, let us consider path

$$\mathcal{F}(P_n) = \{x([n]); x(1), x(1, 2), x(2, 3) \dots, x(n-2, n-1), x(n-1, n), x(n)\}.$$

Note that $\mathcal{F}(P_2) = \{(1, 2); (1), (2)\} = \mathcal{F}\binom{3}{2}$, which is miserable and forced.

Proposition 55. *The game $\mathcal{F}(P_3) = \{x(1, 2, 3); x(1), x(1, 2), x(2, 3), x(3)\}$ is domestic.*

Proof. One can check that $x = (x(1, 2, 3); x(1), x(1, 2), x(2, 3), x(3))$ is a $(0, 0)$-position if and only if either

$$x(1) \leq x(3), x(1, 2, 3) = x(2, 3) = 0, x(1) + x(1, 2) = x(3)$$

or

$$x(1) \geq x(3), x(1, 2, 3) = x(1, 2) = 0, x(2) + x(2, 3) = x(1).$$

Set $V'_{0,0} = V_{0,0}$ and

$$V'_{0,1} = \{(0; 0, 0, 0, 0), [(0; 0, 1, 0, 1)], (0; 1, 0, 0, 1)\} = \{x^0, [x^2], x'^2\}$$

and

$$V'_{1,0} = \{(1; 0, 0, 0, 0), [(0; 0, 0, 0, 1)], [(0; 0, 0, 1, 0)], (0; 1, 1, 1, 1)\} = \{y^1, [x^1], [x'^1], x^4\}.$$

It then follows from Theorem 15 that the game is domestic. Note that it is not tame, since $(0; 1, 1, 1, 2)$ is a $(5, 1)$-position. □

We have given several HO-NIM games including miserable ones, domestic but not tame ones, and not domestic ones. One may question if there exists a HO-NIM game that is strongly miserable. We next answer this question.

Proposition 56. *Let $F = ([n], E)$ be a hypergraph on a ground set V such that $\bigcap_{e \in E} e \neq \emptyset$. Then the* HO-NIM *game $\mathcal{F} = \{x(e) \mid e \in E\}$ is strongly miserable.*

Proof. Without lost of generality, we can assume that $1 \in e$ for all $e \in E$. Let $E = \{e_1, \dots, e_n\}$ for some $n \geq 1$. By one move a player reduces at least one and at most k piles (weights) of $\{x(e_1), \dots, x(e_n)\}$ for some $k \in [n]$. This is because $H_1 = V$. Thus, the game is equivalent with NIMB(n, n), which is exactly one plie NIM whose size is $x(e_1) + \dots + x(e_n)$. Recall that this game is strongly miserable (Lemma 22). □

We conclude the discussion on HO-NIM games with the next two four pile examples. Computations show that both are domestic, yet, we have no rigorous proof.

Conjecture 57. *Game $\mathcal{F} = \{x(1, 2, 3, 4); x(1, 3), x(1, 4), x(2, 3), x(2, 4); x(1, 2, 3)\}$ is domestic.*

Note that this game is not tame, since $(0; 1, 1, 1, 1, 1)$ is a $(1, 5)$-position.

Conjecture 58. *Game $\mathcal{F} = \{x(1, 2, 3, 4); x(1, 4), x(2, 4), x(3, 4); x(1, 2, 3)\}$ is domestic.*

Note that this game is not tame, since $(0; 1, 2, 2; 2)$ is a (7,1)-position.

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