Operations Research, Midterm Exam
March 8, 2007

Professor A. Prekopa

Name:

---

1. Consider the Markov chain with transition probability matrix:

$$P = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(a) (3 points) Draw the graph of the Markov chain.

(b) (3 points) Find the classes.

(c) (4 points) Find the periods.

2. Consider the Markov chain defined recursively by:

$$X_{n+1} = \begin{cases} X_n + \xi_{n+1}, & \text{if } X_n + \xi_{n+1} \le 2 \\ X_n + \xi_{n+1} - 3, & \text{if } X_n + \xi_{n+1} \ge 3, \end{cases}$$

where ξ_1, ξ_2, ... are i.i.d.,

$$P(\xi_i = 0) = p_0, P(\xi_i = 1) = p_1, P(\xi_i = 2) = p_2,$$

$p_0 > 0, p_1 > 0, p_2 > 0$, $p_0 + p_1 + p_2 = 1$, and $X_0 = 0$.

(a) (2 points) What are the states of the Markov chain?

(b) (6 points) Determine the one-step transition probability matrix P.

(c) (6 points) Show that $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist, $j = 0, 1, 2$ and calculate the limiting probabilities π_0, π_1, π_2.

3. Consider the Markov chain defined recursively by:

$$X_{n+1} = \begin{cases} X_n + \xi_{n+1}, & \text{if } X_n + \xi_{n+1} \leq 3 \\ X_n + \xi_{n+1} - 3, & \text{if } X_n + \xi_{n+1} > 3, \end{cases}$$

where ξ_1, ξ_2, $\ldots$ are i.i.d.,

$$P(\xi_i = 0) = p_0, P(\xi_i = 1) = p_1, P(\xi_i = 2) = p_2,$$

$p_0 > 0, p_1 > 0, p_2 > 0$, $p_0 + p_1 + p_2 = 1$, and $X_0 = 0$.

(a) (2 points) What are the states of the Markov chain?

(b) (6 points) Determine the one-step transition probability matrix P.

(c) (6 points) Show that $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist, $j = 0, 1, 2$ and calculate the limiting probabilities π_0, π_1, π_2.

(d) (8 points) Determine which states are transient and which states are recurrent.

4. Consider the Markov chain with transition probability matrix:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix},$$

and initial probabilities $p_0 = p_1 = \frac{1}{2}$.

(a) (3 points) Compute the two-step transition probability matrix.

(b) (6 points) Find $P(X_1 = 1, X_3 = 0)$.

(c) (5 points) Find $P(X_3 = 0)$.

(d) (6 points) Find $P(X_1 = 1, X_2 = 0 | X_3 = 0)$.

(e) (3 points) Show that $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist, $j = 0, 1$, and calculate the limiting probabilities π_0, π_1.

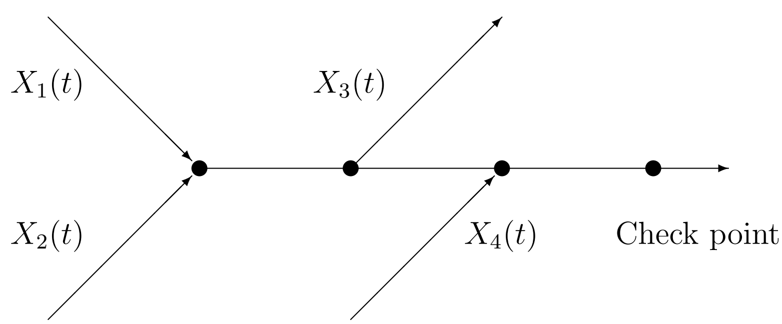
5. Consider the Markov chain with transition probability matrix:

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

(a) (10 points) Determine the probabilities of absorption into state 1: u_0, u_3.

(b) (10 points) Determine the expected time to absorption into state 1: ν_0, ν_3.

6. Consider the road system with Poisson traffic flows depicted below:



where the rates of $X_1(t)$, $X_2(t)$, $X_3(t)$, $X_4(t)$ are λ_1, λ_2, λ_3, λ_4, respectively. Assume that $\lambda_1 = 30$ cars/min, $\lambda_2 = 10$ cars/min, $\lambda_4 = 6$ cars/min, and each car joins traffic flow $X_3(t)$ independently and with probability $\frac{1}{8}$

a) (5 points) Find the parameter of the resulting traffic flow, i.e., the one that passes the check point.

b) (5 points) Find the probability that 4 cars pass the check point during a time interval of 5 seconds.

7. (15 points) Calls arrive to a telephone center according to a Poisson process with rate 10 calls/min. Call durations are independent, exponentially distributed random variables, where each has expectation 80 seconds. We observe the call arrivals in the time interval $(0, t)$, where $t = 2$ hours. Find the probability that at the end of the second hour exactly 5 call conversations are going on.

8. Claims are filed to an insurance company according to a homogeneous Poisson process with rate 40/year. Claims are i.i.d. exponential random variables with expectation \$8,000. Let X be the total claim in 18 months.

a) (5 points) Find $E(X)$.

b) (5 points) Find $Var(X)$.

Remark: The total points are 124. If you get x points, your grade will be 100% ($x \geq 100$), or x% ($x < 100$).