Chapter 6

The complete graph in two and three dimensions

The complete graph on n vertices, denote K_n, is the simple graph which has all possible edges. In this chapter we provide drawings and lower bounds for K_n, both in two and three dimensions. In two dimensions, we create drawings in each of the three models. In three dimensions, we create drawings in the unlimited-growth model.

From these drawings, we can obtain a drawing of any simple graph by deleting edges. These drawings may not be useful from a practical point of view, but they provide insight into the type of bounds we may expect.

The complete graph also serves for providing lower bounds. If we can show that any drawing of K_n must have a certain number of bends, then no algorithm can perform better than these bounds. We present here lower bounds for the number of bends in two and three dimensions, and lower bounds for the volume in three dimensions. Our drawings are asymptotically optimal in the grid-size and the number of bends. In two dimensions, they are even optimal within a small additive term in the number of bends.

6.1 Drawings in two dimensions

6.1.1 Kandinsky-model

We now show how to draw K_n in the Kandinsky-model. This construction is joint work with Michael Kaufmann.

Theorem 6.1 *If n is divisible by 8, then K_n can be embedded in the Kandinsky-model in a grid of width and height $\frac{m}{2} + \frac{3}{8}n$ with $m - \frac{n}{2}$ bends.*

Proof: We divide the vertices into four groups of equal size. Enumerate the vertices in each group as $\{v_1^{(i)}, \ldots, v_{n/4}^{(i)}\}$. We place the vertices "as a diamond," i.e., on a coarse $\frac{n}{2} \times \frac{n}{2}$-grid centered at $(0, 0)$, such that vertex $v_k^{(i)}$ is placed in the ith quadrant and such that the absolute value of the coordinates is $(k, \frac{n}{4} + 1 - k)$.

For each $1 \leq k \leq \frac{n}{4}$, we define the *$k$-square* to be the four vertices $v_k^{(i)}$, $i = 1, \ldots 4$. These vertices induce a K_4 with 6 edges. Four of these edges are drawn straight, the other two edges are drawn with two bends each. We use two columns for these bends if $k \leq \frac{n}{8}$, and two rows otherwise.

For any $1 \leq i \leq \frac{n}{4}$, any $j > i$, and $k = 1, \ldots, 4$, we define the *kth i,j-cross* as the group of vertices $\{v_i^{(k)}, v_j^{(k)}, v_i^{(k-1)}, v_j^{(k+1)}\}$ (all additions are modulo 4). An i,j-cross induces a K_4, and four of its edges were not in a k-square. We draw these four edges with one bend each, using two rows and two columns. This is possible since by $j > i$ the ith vertical and the jth horizontal coarse grid-line cross inside the diamond. See also Figure 6.1.

The number of crosses is $4 \sum_{i=1}^{n/4} i(\frac{n}{4} - i - 1) = \frac{n^2}{8} - \frac{n}{2}$, each accounts for 4 edges, 4 bends, and uses two rows and columns. There are $\frac{n}{4}$ squares, each accounts for 6 edges and 4 bends. Half of the squares use 4 columns and 5 rows, the other half uses 5 columns and 4 rows.

Each edge is in a square or in a cross. Thus, the total number of bends is $4(\frac{n^2}{8} - \frac{n}{2}) + 4\frac{n}{4} = m - \frac{n}{2}$. The width and height each is $2(\frac{n^2}{8} - \frac{n}{2}) + 4\frac{n}{8} + 5\frac{n}{8} = \frac{n^2}{4} + \frac{n}{8}$. Since the complete graph has $m = \binom{n}{2} = \frac{n^2}{2} - \frac{n}{2}$ edges, we can write the number of bends as $\frac{m}{2} + \frac{3}{8}n$. $\square$

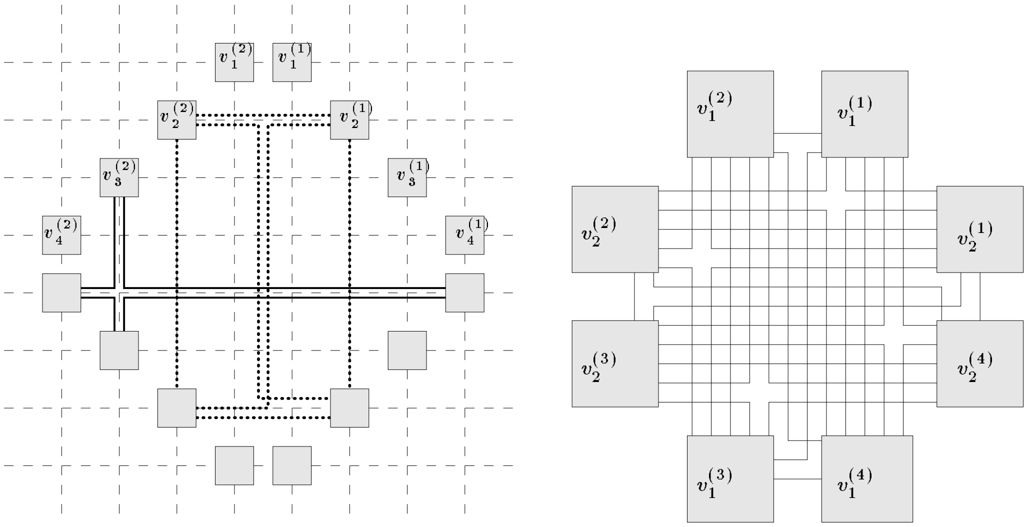

Figure 6.1: The construction in the Kandinsky-model. We show the 2-square (dotted), and the third 3,4-cross (solid). On the right side is the completed drawing of K_8.

6.1.2 TSS-model

The number of bends and the area can be improved in the TSS-model by introducing more grid-lines that are shared by two vertices. Assume that n is divisible by four, and place the vertices as before as a diamond in a coarse grid. Between any grid-line of the coarse-grid, insert a new grid-line. We let the vertices in the right half overlap the row below, and the vertices in the left half overlap the row above. We let the vertices in the top half overlap the column to the right, and the vertices in the bottom half overlap the column to the left. See also Figure 6.2.

In this construction, for each vertex v we have a vertical and horizontal line of visibility to infinity, i.e., there exists a line crossing v and no other vertex. This observation is crucial for creating a drawing in the TSS-model as opposed to the unlimited-growth model. Along these lines of visibility we can add a new grid-line and thus increase the size of the drawing, and then route a new edge.

We split the edges into four different types. As before, we have four i,j-crosses for any choice of $1 \leq i < j \leq \frac{n}{4}$. However, an $i,i+1$-cross is now called a T. We route the crosses that are not T's exactly as before. They cover 4 edges, use two rows and columns, and create 4 bends. Assume that we have a T with the vertices $\{v_i^{(k)}, v_{i+1}^{(k)}, v_i^{(k+1)}, v_{i+1}^{(k-1)}\}$. If $k = 1, 3$ then there is a (newly added) row containing both

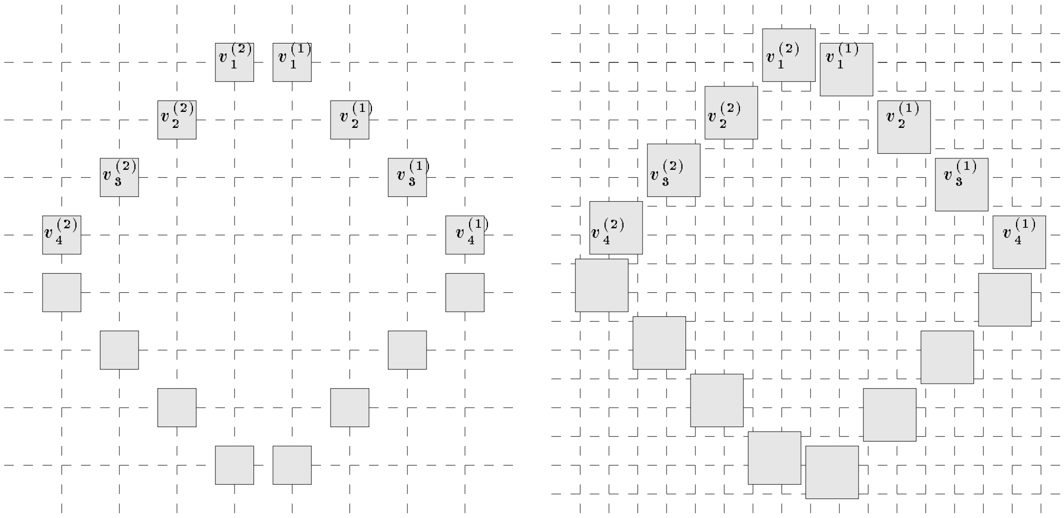

Figure 6.2: We start with the placement as a diamond, but then double the grid and let each node overlap one more row and one more column.

$v_{i+1}^{(k+1)}$ and $v_i^{(k)}$. If $k = 2, 4$ then there is a (newly added) column containing both $v_i^{(k+1)}$ and $v_{i+1}^{(k)}$. Thus, we can draw the T with 3 bends, using two rows and two columns. Here, if $k = 1, 3$, we should make sure that the two columns are placed at the line of sight from $v_i^{(k-1)}$ (otherwise, the TSS-model might be violated for $v_i^{(k)}$). We proceed similarly for the rows if $k = 2, 4$.

The squares also fall into two classes. An i-square is called *extreme* if $i \in \{1, \frac{n}{4}\}$. Consider a non-extreme i-square As before, four of the edges can be drawn without a bend. To draw one of the diagonal edges, say from $v_i^{(1)}$ to $v_i^{(3)}$, we add a column at the vertical line of visibility of $v_i^{(1)}$ and a row at the horizontal line of visibility of $v_i^{(3)}$. Thus, the drawing of the i-square has 6 edges, 2 bends, and needs 4 rows and columns. The extreme i-squares are drawn the same way, except that one of the diagonals can be drawn as a straight line. Thus, the extreme squares contribute only 1 bend, and need 3 rows and 4 columns ($i = 1$) respectively 4 rows and 3 columns ($i = \frac{n}{4}$).

There are $n - 4$ T's and 2 extreme squares. Consequently, the number of bends is

$$3(n-4) + 4(\frac{n^2}{8} - \frac{n}{2} - (n-4)) + 2 + 2(\frac{n}{4} - 2) = \frac{n^2}{2} - \frac{5}{2}n + 2 = m - 2n + 2.$$

The number of grid-lines in each direction is

$$2(\frac{n^2}{8} - \frac{n}{2}) + 4(\frac{n}{4} - 1) + 3 = \frac{n^2}{4} - 1 = \frac{m}{2} + \frac{n}{4} - 1.$$

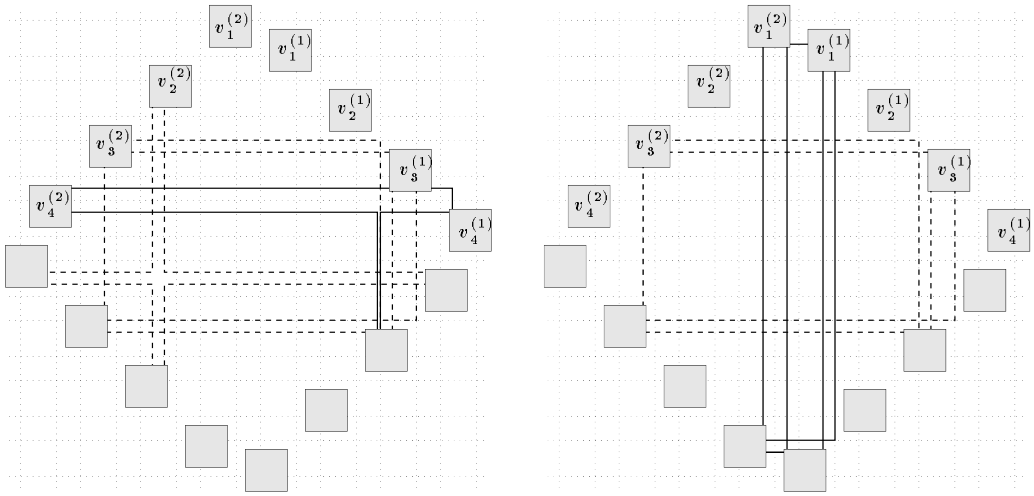

Figure 6.3: We stretch the drawing a little, to show the lines of visibility (dotted) more clearly. In the left picture, we show a T and a cross. In the right picture, we show a square and an extreme square.

Theorem 6.2 *If n is divisible by 4, then K_n can be embedded in the TSS-model in a grid of width and height $\frac{m}{2} + \frac{n}{4} - 1$ with $m - 2n + 2$ bends.*

6.1.3 Unlimited-growth model

For the unlimited-growth model, we can decrease the number of bends and the grid-size even further. Here, we use again the lines of visibility. Assume that n is divisible by 4, and create a drawing of K_{n-4} in the TSS-model as described above. For each node in this K_{n-4}, add a row and a column along the lines of visibility. Then, add four stretched boxes on the four sides, and connect them with each other, using three more rows and columns.

The number of bends in the resulting drawing is

$$\binom{n-4}{2} - 2(n-4) + 2 = \frac{n^2-n}{2} - 4n + 10 - 2n + 10 = m - 6n + 20$$

The number of grid-lines in each direction is

$$\frac{1}{2}\binom{n-4}{2} + \frac{n-4}{4} - 1 + (n-4) + 3 = \frac{m}{2} - 2n + 5 + \frac{5}{4}n - 2 = \frac{m}{2} - \frac{3}{4}n + 3.$$

Theorem 6.3 *If n is divisible by 4, then K_n can be embedded in the unlimited-growth model in a grid of width and height $\frac{m}{2} - \frac{3}{4}n + 3$ with $m - 6n + 20$ bends.*

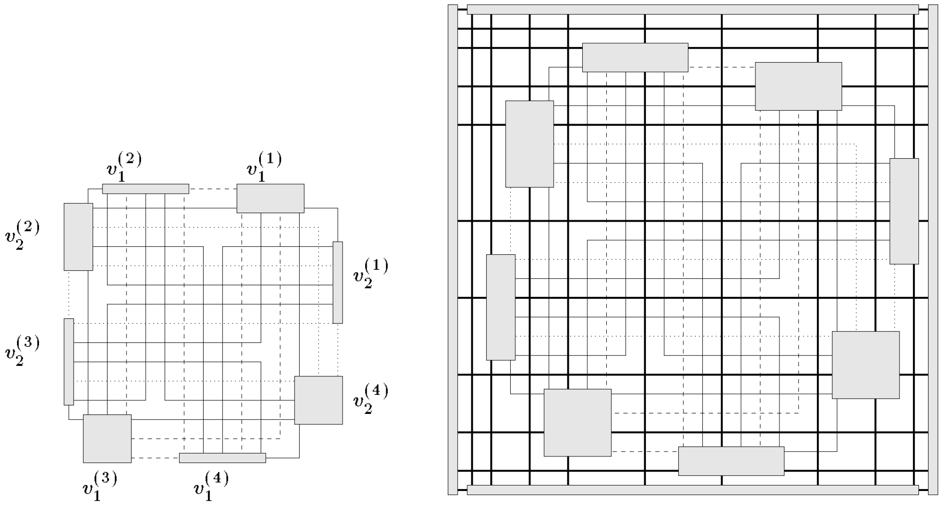

Figure 6.4: K_8 in the TSS-model and K_{12} in the unlimited-growth model.

In particular, the drawing in the unlimited-growth model is optimal in the number of bends (see Lemma 2.3.2).

6.2 Lower bounds in two dimensions

In Lemma 2.3.2 we presented a lower bound on the number of bends of $m - 6n + 20$. In the Kandinsky-model, we can prove a significantly better bound for the complete graph.

Theorem 6.4 *Any drawing of K_n in the Kandinsky-model has at least $m - n$ bends.*

Proof: Assume that we have a drawing Γ of K_n. Let A be the number of vertical slots that contain vertices. Let B be the number of horizontal slots that contain vertices. The number of vertices in the ith vertical slot is denoted by a_i, while the number of vertices in the jth vertical slot is denoted by b_i, hence $\sum_{i=1}^{A} a_i = \sum_{i=1}^{B} b_i = n$.

The edges split into three groups. We define E_a as the set of the edges with both endpoints in the same vertical slot, E_b as the set of edges with both endpoints are in the same horizontal slot, and $E_c = E - E_a - E_b$. Since nodes are placed on a coarse grid, we have $E_a \cap E_b = \emptyset$.

Since we have the complete graph, $|E_a| = \sum_{i=1}^{A} \binom{a_i}{2}$. Of these edges, $\sum_{i=1}^{A}(a_i - 1)$ can be drawn without bends. On the other hand, the remaining edges in E_a must have at least two bends by the bend-or-end property. Therefore, the number of bends in E_a is at least $2\sum_{i=1}^{A}[\binom{a_i}{2} - (a_i - 1)]$. Similarly, the number of bends in E_b is at least $2\sum_{i=1}^{B}[\binom{b_i}{2} - (b_i - 1)]$.

Each edge in E_c has at least one bend. So the total number of bends is at least

$$\begin{aligned}
& m - \sum_{i=1}^{A}\binom{a_i}{2} - \sum_{i=1}^{B}\binom{b_i}{2} + 2\sum_{i=1}^{A}\left[\binom{a_i}{2} - (a_i - 1)\right] + 2\sum_{i=1}^{B}\left[\binom{b_i}{2} - (b_i - 1)\right] \\
= \; & m + \sum_{i=1}^{A}\binom{a_i}{2} + \sum_{i=1}^{B}\binom{b_i}{2} - 4n + 2A + 2B \\
= \; & m + \sum_{i=1}^{A} a_i^2/2 + \sum_{i=1}^{B} b_i^2/2 - 5n + 2A + 2B.
\end{aligned}$$

Given values of A, B, this expression is minimal if the a_i's have the same value, and the b_i's have the same value; so $a_i = \frac{n}{A}$ and $b_i = \frac{n}{B}$. The number of bends then is $m + n^2/2A + n^2/2B - 5n + 2A + 2B$. Minimizing for A and B, we arrive at $A, B = \frac{n}{2}$, therefore the number of bends is at least $m + 2n - 5n + 2n = m - n$. $\square$

This result shows that our construction of K_n with $m - \frac{n}{2}$ bends in the Kandinsky-model was not far from optimality.

As a corollary, we can obtain a lower bound for any simple graph with many edges. Assume that G is a simple graph, and let Γ be a drawing of G in the Kandinsky-model with b bends. One can show that in any drawing in the Kandinsky-model, we can add an edge between any two nodes using at most 3 bends. Thus, if G has n nodes and m edges, then we can add all edges in $K_n - G$ with at most 3 bends each to Γ. Thus, we obtain a drawing of K_n in the Kandinsky-model with at most $b + 3(\binom{n}{2} - m)$ bends. By the above result, we must have

$$b \geq \binom{n}{2} - n - 3(\binom{n}{2} - m) = 3m - n^2.$$

Corollary 6.5 *For any simple graph G, any drawing in the Kandinsky-model has at least $3m - n^2$ bends.*

6.3 Drawings in three dimensions

In this section, we give three constructions for K_n in three-dimensional space. Each gives a drawing in the unlimited-growth model, but they use a different number of bends per edge. The results in this section are joint work with Tom Shermer, Sue Whitesides, and Stephen Wismath.

We follow the three-phase method, but with a different port assignment technique. We place the nodes as points in a two-dimensional xy-plane. Any node placement is feasible, as long as nodes are on disjoint points. Then we place the bends of edges, also in the same xy-plane. Edges may (for now) overlap and cross each other. However, as opposed to the construction in two dimensions, edges are not allowed to intersect or overlap nodes other than their endpoints. This can always be achieved if we allow three bends per edge and route it along newly introduced off-rows and off-columns of nodes. Depending on the node placement, routings with less bends may be possible.

If we have such a node placement and edge routing, then we can utilize the third dimension for port assignment. We introduce some number of z-planes, and assign edges to z-planes such that within each z-plane there is neither a crossing nor overlap. We create drawings in the unlimited-growth model, therefore we may stretch each node to a $1 \times 1 \times Z$-box, such that each edge attaches at the boxes of its endpoints.

For any node placement and any edge routing we can find a feasible port assignment, since in the worst case we can assign one z-plane to each edge. Two examples: If we place the nodes in general position, then we can route each edge with one bend, and therefore get an $n \times n \times m$-grid and one bend per edge. If we arrange the nodes in a $\sqrt{n} \times \sqrt{n}$-grid and double the grid to get off-grid-lines, then we can route each edge with three bends, so we get a $2\sqrt{n} \times 2\sqrt{n} \times m$-grid with three bends per edge.

To achieve a smaller grid, we want to find a node placement and edge routing such that in average more than a constant number of edges is placed in each z-plane. We give three constructions here, with between one and three bends per edge. The more bends we allow, the smaller the volume. The first two constructions have volume $\mathcal{O}(n^3)$, the third has volume $\mathcal{O}(n^{5/2})$.

6.3.1 A small drawing with one bend per edge

In this section, we embed K_n with at most one bend per edge. First, we embed it in an $n \times n \times n$-grid. We take two such drawings of a $K_{\frac{n}{2}}$ and merge them together to a drawing of the K_n in an $\frac{n}{2} \times n \times \frac{n}{2}$-grid. We assume that n is divisible by 4.

Drawing in an $n \times n \times n$-grid

Enumerate the vertices as $v_1, \ldots, v_n$. Place vertex v_i at (i, i), then the nodes are in general position. Route each edge with one bend above the diagonal, so if $e = (v_i, v_j)$ with $i < j$, we place the bend at (i, j). Therefore, no node or part of an edge is placed at a point (x, y) with $y < x$.

The crucial observation for port assignment is that K_n can be split into n outer-planar graphs which have the same order of vertices on the outer-face (in other words, the *page number* of K_n is at most n). To prove this, define edge sets E_i^a, E_i^b, $i = 1, \ldots, \frac{n}{2}$ as $E_i^a = \{(v_{i-l+1}, v_{i+l}), l = 1, \ldots, \frac{n}{2}\}$ and $E_i^b = \{(v_{i-l}, v_{i+l}), l = 1, \ldots, \frac{n}{2} - 1\}$ (all additions are modulo n).

Lemma 6.3.1 *Each edge belongs to exactly one of the sets E_i^a, E_i^b, $i = 1, \ldots, \frac{n}{2}$.*

Proof: Let $1 \leq j < k \leq n$ be given. If $k - j$ is even, then define $i = \frac{k+j}{2}$ (modulo $\frac{n}{2}$), and $l = \frac{k-j}{2}$ (modulo $\frac{n}{2}$), which is greater than 0 since $j \neq k$. Then E_i^b contains the edge $(v_{i-l}, v_{i+l}) = (v_j, v_k)$. If $k - j$ is odd, define $i = \frac{k+j-1}{2}$ and $l = \frac{k-j+1}{2}$ (both modulo $\frac{n}{2}$), then E_i^a contains edge $(v_{i-l+1}, v_{i+l}) = (v_j, v_k)$. $\square$

One can immediately verify that there is no crossing between any two edges in E_i^a, and no crossing between any two edges in E_i^b. See also Figures 6.5-6.6. So we can assign n z-planes and route the edges without crossings or overlap.

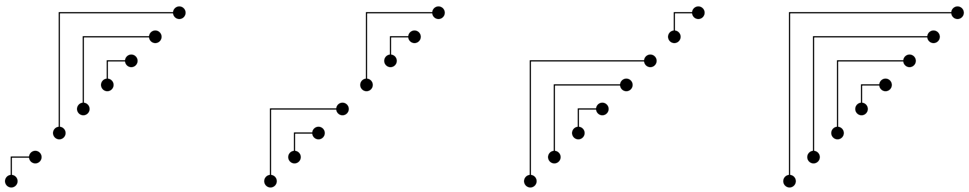

Figure 6.5: The sets $E_a^1, \ldots, E_a^4$ for K_8.

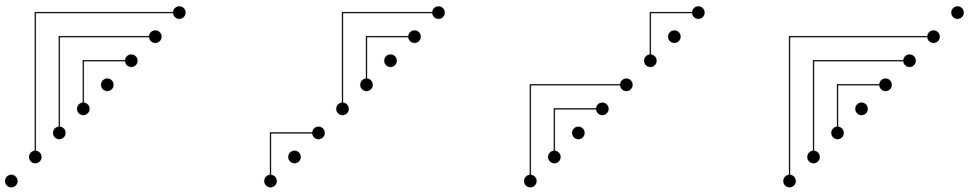

Figure 6.6: The sets $E_b^1, \ldots, E_b^4$ for K_8.

Lemma 6.3.2 *There exists an embedding of K_n in an $n \times n \times n$-grid with one bend per edge such that the points $\{(x, y, z) : y < x\}$ are unused.*

Drawing in an $\frac{n}{2} \times n \times \frac{n}{2}$-grid

Assume that n is divisible by four. We take two $K_{n/2}$'s, denote them by K^1 and K^2. We draw K^1 and K^2 exactly as described above, thus each drawing has grid-size $\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}$. We flip the drawing of K^2 along the xz-plane of coordinate 0, so now all elements of the drawing of K^2 have a negative y-coordinate. Then we rotate the drawing of K^2 such that node v_j of K^2 now overlaps the points $(x, -j, j)$, where $1 \leq x \leq \frac{n}{2}$.

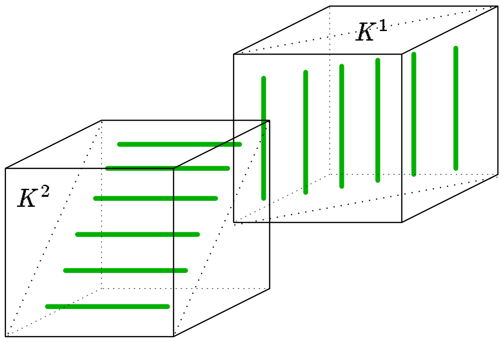

Figure 6.7: Take two $K_{\frac{n}{2}}$'s, draw them as above, and flip and rotate one of them. We show only the boxes of the nodes, which have size $1 \times 1 \times \frac{n}{2}$.

Each node v_i in K^1 can see each node v_j in K^2 along the line $(i, i, j) - (i, -j, j)$. Therefore, these edges can be routed as straight lines. This gives a drawing of K_n. The y-plane of coordinate 0 is unused, so we can delete it. We get a drawing with $X = Z = \frac{n}{2}$ and $Y = n$. There are $n^2/4$ edges drawn without a bend, and all other edges have one bend, so the number of bends is $n^2/4 - n/2$.

Theorem 6.6 *For n divisible by 4, K_n can be drawn in a $\frac{n}{2} \times n \times \frac{n}{2}$-grid with at most one bend per edge. The total number of bends is $n^2/4 - n/2$.*

6.3.2 A smaller drawing with two bends per edge

In this section, we embed K_n with at most two bends per edge. First, we embed it in an $n \times \frac{n}{2} \times n$-grid. We take two such drawings of a $K_{\frac{n}{2}}$ and merge them together to a drawing of side-length $\frac{n}{2}$. We assume that n is divisible by 4.

Drawing in an $n \times \frac{n}{2} \times n$-grid

Enumerate the vertices as $\{v_1, \ldots, v_n\}$ and place v_i at $(i, 1)$. If $e = (v_i, v_j)$ with $i < j$, then let $y = \lceil \frac{j-i}{2} \rceil$. Route edge e along the points $(i, 1), (i, y), (j, y), (j, 1)$, so with two bends if $y > 0$ and without a bend otherwise.

Define the edge sets E_i^a and E_i^b as above. One immediately verifies that with this edge routing again there is no crossing between two edges in the same set. So n z-planes suffice. Since the maximal y-coordinate was $\lceil \frac{n-1}{2} \rceil$, we get an $n \times \frac{n}{2} \times n$-grid. The edges (v_i, v_{i+1}), $i = 1, \ldots, n-1$ are drawn straight, all other edges have two bends, so the number of bends is $n^2 - 3n + 2$.

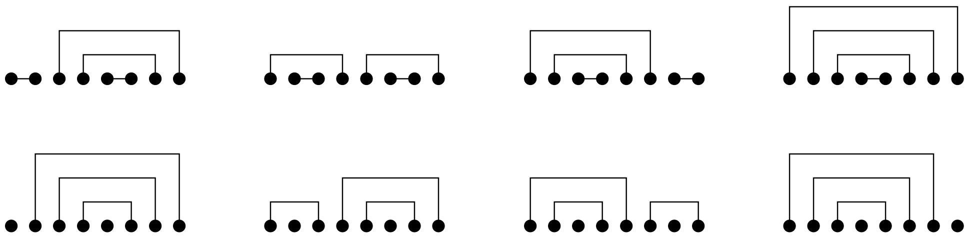

Figure 6.8: The edge sets of K_8 drawn with two bends per edge.

Lemma 6.3.3 *There exists an embedding of K_n in an $n \times \frac{n}{2} \times n$-grid with $n^2 - 3n + 2$ bends and at most two bends per edge, such that node v_i overlaps the points $(i, 1, z)$, where $1 \leq z \leq n$.*

Drawing in an $\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}$-grid

Assume that n is divisible by four. We take two $K_{n/2}$'s, denote them by K^1 and K^2. We draw K^1 and K^2 exactly as described above, thus each drawing has grid-size $\frac{n}{2} \times \frac{n}{4} \times \frac{n}{2}$. We flip the drawing of K^2 along the xz-plane of coordinate 0, so now all elements of the drawing have a negative y-coordinate. Then we rotate the drawing of K^2 such that node v_j of K^2 now overlaps the points $(x, -1, j)$, where $1 \leq x \leq \frac{n}{2}$.

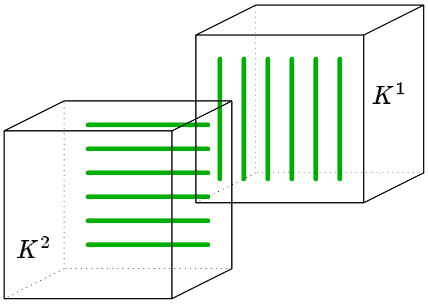

Figure 6.9: Take two $K_{\frac{n}{2}}$'s, draw them as above, and flip and rotate one of them.

Each node v_i in K^1 can see each node v_j in K^2 along the line $(i, 1, j) - (i, -1, j)$. Therefore, these edges can be routed as straight lines. This gives a drawing of K_n. The y-plane of coordinate 0 is unused, so we can delete it. We get a drawing with $X = Y = Z = \frac{n}{2}$. The number of bends is $2(n^2/4 + \frac{3}{2}n + 2) = n^2/2 + 3n + 4$.

Theorem 6.7 *For n divisible by 4, K_n can be drawn in a $\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}$-grid with $n^2/2 + 3n + 4$ bends and at most two bends per edge.*

6.3.3 An optimal drawing with three bends per edge

In this section, we embed K_n with at most three bends per edge and volume $\mathcal{O}(n^{5/2})$. We assume that $n = N^2$ is a square. Enumerate the vertices as pairs of numbers (i, j), $1 \leq i \leq N, 1 \leq j \leq N$, and place node (i, j) at $(2i, 2j)$ during node placement.

Each edge is routed with three bends, using the rows and columns with odd coordinates. Let e connect the vertices (i_1, j_1) and (i_2, j_2). After possible renaming, we may assume that $i_1 \leq i_2$, and if $i_1 = i_2$ then $j_1 < j_2$. We call e an L-edge if $j_1 < j_2$ and a Γ-edge otherwise.

For each L-edge, we route the edge along the points $(2i_1, 2j_1), (2i_1 + 1, 2j_1), (2i_1 + 1, 2j_2 + 1), (2i_2, 2j_2 + 1), (2i_2, 2j_2)$, thus with three bends. Each Γ-edge is routed as $(2i_1, 2j_1), (2i_1 + 1, 2j_1), (2i_1 + 1, 2j_2 - 1), (2i_2, 2j_2 - 1), (2i_2, 2j_2)$.

If we take the routings of the L-edges and rotate them clockwise by 90 degree, then we get the routings of the Γ-edges. Therefore, we only need to find z-planes for the

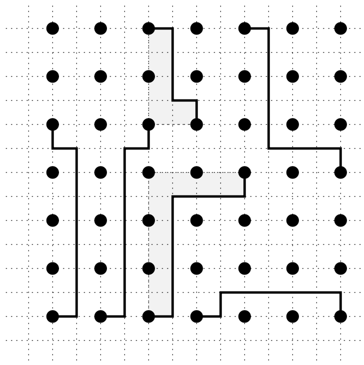

Figure 6.10: We show some L-edges and Γ-edges. From the shading, it should be clear why we chose these names.

L-edges, we can then double their number and also get z-planes for the Γ-edges.

We split the L-edges into $N(N-1)$ groups E_{d_x,d_y}, with $0 \leq d_x \leq N-1$ and $1 \leq d_y \leq N-1$. Each group E_{d_x,d_y} consists of those edges $((i_1,j_1),(i_2,j_2))$ for which $i_2 = i_1 + d_x$ and $j_2 = j_1 + d_y$. These groups cover all L-edges since we have $i_1 \leq i_2$ and $j_1 < j_2$ for any L-edge.

Each group E_{d_x,d_y} is split into at most $d_x + d_y$ sets of edges. For $p = 0, \ldots, d_x + d_y - 1$, let $E^p_{d_x,d_y}$ be the edges in E_{d_x,d_y} for which $j_2 - i_1 = p$ modulo $(d_x + d_y)$. In other words, the corners of the L-edges in $E^p_{d_x,d_y}$ lie on diagonals which intersect the y-axis at the value $2p$ modulo $(2d_x + 2d_y)$. See Figure 6.11. It can be verified that no two edges in $E^p_{d_x,d_y}$ overlap or intersect. Also, $E^p_{d_x,d_y}$ contains edges only if $p \leq 2N - d_x - d_y$.

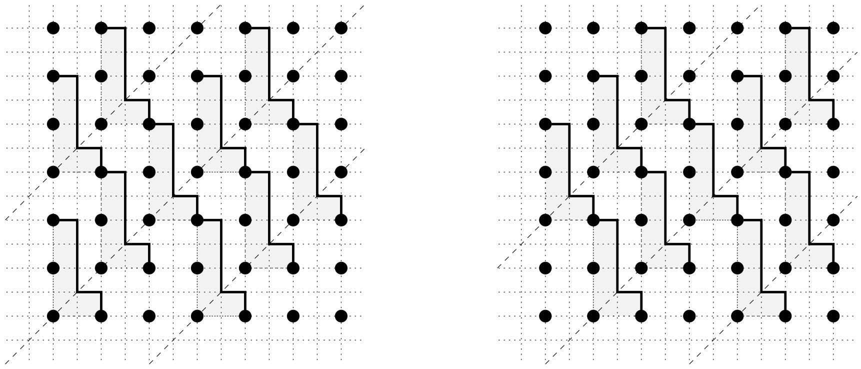

Figure 6.11: The edge sets $E^0_{1,2}$ and $E^2_{1,2}$. No two edges intersect or overlap, since the corners of the L's are placed on diagonals with $2(d_x + d_y)$ vertical distance.

So we assign one z-plane to each set $E^p_{d_x,d_y}$, and obtain a legal drawing of the L-edges. We double the number of z-planes to have space for Γ-edges as well, and then get a drawing of G. The width and depth is $2N = 2\sqrt{n}$. The height is

$$2\sum_{d_x=0}^{N-1}\sum_{d_y=1}^{N-1}\min\{d_x+d_y, 2N-d_x-d_y\}.$$

Some analysis shows that this some is at most

$$2\left[\sum_{k=1}^{N-1}k(2k-1)+(N-1)N\right] = \frac{2(N-1)N(2N-1)}{3} - (N-1)N + 2(N-1)N \leq \frac{4}{3}N^3.$$

Every edge has three bends, with the exception of the $2N(N-1) = 2n - 2\sqrt{n}$ edges that can be drawn without a bend. So the total number of bends is $3(n^2/2 - n/2) - 3(2n - 2\sqrt{n}) = \frac{3}{2}n^2 - \frac{15}{2}n + 6\sqrt{n}$.

Theorem 6.8 *If $n = N^2$ is a square, then K_n can be drawn in a $2N \times 2N \times \frac{4}{3}N^3$-grid (so volume $\frac{16}{3}n^{5/2}$) with $\frac{3}{2}n^2 - \frac{15}{2}n + 6\sqrt{n}$ bends and at most three bends per edge.*

6.4 Lower bounds in three dimensions

Now we prove that the last construction was asymptotically optimal in the volume and the number of bends. The results in this section are joint work with Tom Shermer, Sue Whitesides, and Stephen Wismath.

6.4.1 Lower bounds for the grid-size

In this section, we show that no drawing of K_n can have volume smaller than $\Omega(n^{5/2})$. Let some drawing of K_n with grid-size $X \times Y \times Z$ be given. Our proof splits into three cases. Only in the second case do we get a bound of $\Omega(n^{5/2})$, in the other cases we get a better bound of $\Omega(n^3)$. Let $0 < c_1 < c_2 < 1$ be some constants, we determine suitable values for them later.

A stab crossing many nodes

Assume there exists a z-stab crossing $k \geq c_1 n$ nodes. Let $v_1, \ldots, v_k$ be the crossed nodes in the order while traversing the z-stab. Since nodes are drawn as boxes, there exists a z-plane P_z (not necessarily with integer coordinate) such that $v_1, \ldots, v_{k/2}$ are entirely below P_z, while $v_{k/2+1}, \ldots, v_k$ are entirely above P_z (for simplicity of notation assume that k is even).

Since we have the complete graph, $\frac{1}{4}k^2$ edges connect these two groups, and cross the plane P_z. So P_z contains at least $\frac{1}{4}k^2$ points, hence $XY \geq \frac{1}{4}c_1^2 n^2$. Furthermore, since the z-stab intersects $c_1 n$ nodes, we have $Z \geq c_1 n$ and the volume is at least $\frac{1}{4}c_1^3 n^3$.

A plane crossing many nodes

Assume now that no x-stab, y-stab or z-stab crosses $c_1 n$ nodes, but there exists a z-plane P_z that crosses at least $c_2 n$ nodes. By scanning the z-plane from left to right, i.e., from smaller to larger x-coordinate, we can find an x-plane P_x (not necessarily with integral coordinate) such that at least $\frac{1}{2}(c_2 - c_1)n$ vertices are entirely to the left of P_x, and at least $\frac{1}{2}(c_2 - c_1)n$ vertices are entirely to the right of P_x. If there were no such plane, then one y-stab would contain at least $c_1 n$ nodes, which we excluded.

There are $\frac{1}{4}(c_2 - c_1)^2 n^2$ edges between these vertices, so $YZ \geq \frac{1}{4}(c_2 - c_1)^2 n^2$. By applying exactly the same argument in the y-direction, we get $XZ \geq \frac{1}{4}(c_2 - c_1)^2 n^2$. Finally, we have $XY \geq c_2 n$, since P_z crosses $c_2 n$ nodes. Consequently, the volume is $\sqrt{YZ \cdot XZ \cdot XY} \geq \frac{1}{4}(c_2 - c_1)^2 \sqrt{c_2}\, n^{5/2}$.

No plane crossing many nodes

Assume now that no plane crosses $c_2 n$ nodes. By scanning the x-planes from left to right, we can find an x-plane P_x (not necessarily with integral coordinate) such that at least $\frac{1}{2}(1 - c_2)n$ vertices are entirely left of P_x, and at least $\frac{1}{2}(1 - c_2)n$ vertices are entirely right of P_x. Consequently, P_x contains at least $\frac{1}{4}(1 - c_2)^2 n^2$ points, and $YZ \geq \frac{1}{4}(1 - c_2)^2 n^2$. The exact same argument holds for the other two directions. Therefore, the volume is at least $(\frac{1}{4}(1 - c_2)^2 n^2)^{3/2} = \frac{1}{8}(1 - c_2)^3 n^3$.

Lemma 6.4.1 *For any constants $0 < c_1 < c_2 < 1$, any drawing of K_n has volume at least $\frac{1}{4}\min\{c_1^3 n^3, (c_2 - c_1)^2 \sqrt{c_2}\, n^{5/2}, \frac{1}{2}(1 - c_2)^3 n^3\}$.*

The choice of c_1 and c_2 depends on the type of lower bound that we want to show. If we are interested in asymptotic behavior, then the only relevant term is the middle term. Given any $\epsilon > 0$, we can choose c_1 so close to 0 and c_2 so close to 1 that $(c_2 - c_1)^2 \sqrt{c_2} > 1 - 4\epsilon$. For n sufficiently large, the second term determines the minimum, and therefore the volume is at least $(\frac{1}{4} - \epsilon)n^{5/2}$.

If we want an estimation that holds for small n as well, then we choose the constants accordingly. For example, we could choose $c_1 = 0.34$ and $c_2 = 0.57$, then each term is at least $0.009 n^{5/2}$ for any integer n.

Theorem 6.9 *Any three-dimensional orthogonal drawing of K_n has volume $cn^{5/2}$ for a constant $c > 0.009$. For any $\epsilon > 0$ there exists an N such that for any $n \geq N$ any drawing of K_n has volume $(\frac{1}{4} - \epsilon)n^{5/2}$.*

Our construction gave an $2\sqrt{n} \times 2\sqrt{n} \times 2\sqrt{n}^3$-grid, which is asymptotically optimal.

6.4.2 Lower bounds for the number of bends

We drew K_n with $\mathcal{O}(n^2)$ bends, and we can prove that this is asymptotically optimal.

A graph without uni-directional visibility representation

If the number of bends is small, then there exists a large subgraph without a bend, which therefore has a tri-directional visibility representation. Its edges split into three classes, depending on the direction of visibility. Each class of edges forms a graph that has a uni-directional visibility representation. Our lower bound now depends on the finding a graph without a uni-directional visibility representation.

Lemma 6.4.2 *[FHW96] There exists a constant c, $c \leq 59$, such that K_c has no three-dimensional uni-directional visibility representation.*

A graph that needs a bend

With the aid of c, we now find a (larger) constant c' such that any three-dimensional drawing of $K_{c'}$ must have a bend. The *3-Ramsey number* $R(r, b, g)$ is the smallest number such that if we take a $K_{R(r,b,g)}$ and color its edges arbitrarily with three colors red, blue and green, then the red sub-graph contains a K_r, or the blue subgraph contains a K_b, or the green sub-graph contains a K_g. The theory of Ramsey-numbers is a well-studied field of combinatorics, see for example [GRS80]. We need here only one result: For any number d, the 3-Ramsey-number $R(d, d, d)$ is finite, and in particular $R(d, d, d) \leq 3^d$.

Let $c' = R(c, c, c)$, and consider any drawing of $K_{c'}$. If it has no bend, then color edges red if they are parallel to the x-axis, blue if they are parallel to the y-axis, and green if they are parallel to the z-axis. By the choice of c' as the Ramsey-number, we must have a K_c in one of the subgraphs defined by this coloring. But then we have a three-dimensional uni-directional visibility representation of K_c, a contradiction. So $K_{c'}$ must have a bend in any drawing.

Lemma 6.4.3 *There exists a constant c', such that any three-dimensional drawing of $K_{c'}$ has a bend.*

K_n needs $\Omega(n^2)$ bends

Now let $n > c'$. The graph K_n contains $\binom{n}{c'}$ copies of a $K_{c'}$. Each of these copies must have a bend. Any edge of K_n belongs to exactly $\binom{n}{c'-2}$ of these copies of $K_{c'}$. Consequently, the number of edges with a bend must be at least

$$\frac{\binom{n}{c'}}{\binom{n}{c'-2}} = \frac{n!}{c'!(n-c')!}\frac{(c'-1)!(n-c'+2)!}{n!} = \frac{(n-c'+2)(n-c'+1)}{c'(c'-1)}.$$

Since c' is a constant, the number of bends therefore is at least $\frac{n^2}{c'^2} - \mathcal{O}(n)$.

Theorem 6.10 *There exists a constant c' such that in any drawing of K_n, $n > c'$, there are at least $\frac{n^2}{c'^2} - \mathcal{O}(n)$ bends.*

Unfortunately, this lower bound is of value only for asymptotic behavior, since we have $c \leq 59$, and $c' \leq 3^{59}$, and the number of bends is at least $3^{-118}n^2 \approx 10^{-56}n^2$. It remains open to find a lower bound with a better constant. In particular, we conjecture that a much smaller value of c' suffices.

6.5 Table of the results

We conclude this chapter with two tables containing an overview of the achieved results.

K_n in two dimensions			
	Kandinsky-model	TSS-model	unl.-growth model
Construction			
Area	$(\frac{m}{2} + \frac{3}{8}n)^2$	$(\frac{m}{2} + \frac{n}{4} - 1)^2$	$(\frac{m}{2} - \frac{3}{4}n + 3)^2$
Bends	$m - \frac{n}{2}$	$m - 2n + 2$	$m - 6n + 20$
Lower Bounds			
Area	$\Omega(m^2)$	$\Omega(m^2)$	$\Omega(m^2)$
Bends	$m - n$	$m - 6n + 20$	$m - 6n + 20$

Table 6.1: Upper and lower bounds for the K_n in two dimensions. One can see that the results are quite good for the number of bends, while improvement is needed for the lower bound on the area.

K_n in three dimensions				
	unlimited-growth model			Kandinsky-model/ deg-restr. model
	One bend	Two bends	Three bends	
Construction				
Volume	$\frac{1}{4}n^3$	$\frac{1}{8}n^3$	$\frac{16}{3}n^{5/2}$	$n \times n \times m$
Bends	$\frac{n^2}{4} - \frac{n}{2}$	$\frac{n^2}{2} - 3n + 4$	$\frac{3}{2}n^2 - \frac{15}{2}n + 6\sqrt{n}$	$2m$
Lower Bounds				
Volume	$(\frac{1}{4} - \epsilon)n^{5/3}$	$(\frac{1}{4} - \epsilon)n^{5/3}$	$(\frac{1}{4} - \epsilon)n^{5/3}$	$(\frac{1}{4} - \epsilon)n^{5/3}$
Bends	$\Omega(n^2)$	$\Omega(n^2)$	$\Omega(n^2)$	$\Omega(n^2)$

Table 6.2: Upper and lower bounds for the K_n in three-dimensional drawings. Some open problems remain. We would like to improve the lower bound on the number of bends. Is there a better lower bound on the volume if we restrict the drawing to have only one or two bends per edge? Are there better constructions and lower bounds in the Kandinsky-model or the degree-restricted model?