Semidefinite and Second Order Cone Programming Seminar
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Lecture 4

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1 Overview

We go over duality examples and define complementary slackness conditions for conic LP optimality.

2 Duality (continued)

Let us apply the table of duality presented in the previous lecture to a few examples.

Example 1. Let us consider the following SDP and it's dual:

$$\begin{array}{llllllll}
\min & \mathbf{c}_1^T\mathbf{x}_1 & +\mathbf{c}_2^T\mathbf{x}_2 & +C\cdot X & & & & \\
\text{s.t.} & \mathbf{a}_1^T\mathbf{x}_1 & +\mathbf{a}_2^T\mathbf{x}_2 & +A_1\cdot X & = & b_1 & \leftrightarrow & y_1 \\
 & \mathbf{d}_1^T\mathbf{x}_1 & & +A_2\cdot X & \geq & b_2 & \leftrightarrow & y_2 \\
 & \mathbf{h}^T\mathbf{x}_1 & +\mathbf{g}^T\mathbf{x}_2 & & \succcurlyeq_{\mathcal{Q}} & k & \leftrightarrow & \mathbf{y}_3 \\
 & & \mathbf{x}_2 & & \preccurlyeq_{\mathcal{Q}} & 0 & & \\
 & & & X & \succcurlyeq & 0 & &
\end{array} \tag{1}$$

$$\begin{array}{llllllll}
\max & b_1 y_1 & +b_2 y_2 & +\mathbf{l}^T\mathbf{y}_3 & & & & \\
\text{s.t.} & \mathbf{a}_1 y_1 & +\mathbf{a}_2 y_2 & +h\mathbf{y}_3 & = & \mathbf{c}_1 & \leftrightarrow & \mathbf{x}_1 \\
 & \mathbf{a}_2 y_1 & & +g\mathbf{y}_3 & \succcurlyeq_{\mathcal{Q}} & \mathbf{c}_2 & \leftrightarrow & \mathbf{x}_2 \\
 & A_1\cdot y_1 & +A_2\cdot y_2 & & \preccurlyeq & C & \leftrightarrow & X \\
 & & y_2 & & \geq & 0 & & \\
 & & & \mathbf{y}_3 & \succcurlyeq_{\mathcal{Q}} & 0 & &
\end{array} \tag{2}$$

We note that $\mathcal{Q} = \mathcal{Q}^*$ for second order cone. □

Example 2. Find the dual of the following SDP:

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & \begin{pmatrix} 0 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & x_1+1 \end{pmatrix} \succcurlyeq 0 \end{array}$$

We first note that the optimal solution of the problem satisfies $x_1 = 0$, as otherwise one of the subdeterminants of above matrix satisfies $x_2 \times 0 - x_1 \times x_1 = -x_1^2 < 0$ and therefore not a positive semidefinite matrix satisfying the constraint. To get the dual of the problem, we must introduce a generalized notation for unit vectors. Let

$$E_{ij} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \tag{3}$$

where 1 appears at the i,jth position. The problem reduces to the following using this notation:

$$\begin{array}{lccccccc} \min & & x_1 & & & & & \\ \text{s.t.} & (E_{11}+E_{21}+E_{33})x_1 & & +E_{22}x_2 & \succcurlyeq & -E_{33} & \leftrightarrow & Y \end{array} \tag{4}$$

Therefore the dual can be stated as follows:

$$\begin{array}{lrcl} \max & -E_{33}\cdot Y & & \\ \text{s.t.} & (E_{12}+E_{21}+E_{33})\cdot Y & = & 1 \\ & E_{22}^T \cdot Y & = & 0 \\ & Y & \succcurlyeq & 0 \end{array} \tag{5}$$

where $Y \in \Re^{3\times 3}$ symmetric matrix. Let

$$Y = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_2 & y_4 & y_5 \\ y_3 & y_5 & y_6 \end{pmatrix} \tag{6}$$

The first constraint in the dual $(E_{12}+E_{21}+E_{33})\cdot Y = 1$ implies $2y_2 + y_6 = 1$, in otherwords $y_2 = \frac{1-y_6}{2}$. Also the second constraint $E_{22}\cdot Y = 0$ implies $y_4 = 0$. Therefore we can equivalently represent the dual as follows:

$$\begin{array}{ll} \max & -y_6 \\ \text{s.t.} & \begin{pmatrix} y_1 & \frac{1-y_6}{2} & y_3 \\ \frac{1-y_6}{2} & 0 & y_5 \\ y_3 & y_5 & y_6 \end{pmatrix} \succcurlyeq 0 \end{array}$$

If we again take a look to the first 2×2 submatrix of dual constraint, we figure out it's determinant is $-\left(\frac{(1-y_6)}{2}\right)^2 \leq 0$ which implies the matrix can be PSD only if $y_6 = 1$. Therefore we observe there exists a duality gap in this problem where strong duality relation doesn't hold. □

Let us remember that the duality gap in Example 2 is because the cone $\bar{A}(\mathcal{K})$ defined for the conic LP

$$\begin{array}{lccc} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \succcurlyeq_{\mathcal{K}} & 0 \end{array} \tag{7}$$

where $\bar{A} = \begin{pmatrix} A \\ \mathbf{c}^T \end{pmatrix}$, is not closed. A well-known condition for closedness is the Slater's Condition:

Lemma 1. *If there exists $\mathbf{x}_S \succ_{\mathcal{K}} 0$ (the so-called Slater's point, if and only if $\mathbf{x}_S \in Int(\mathcal{K})$) where $A\mathbf{x} = \mathbf{b}$, then $\bar{A}(\mathcal{K})$ is closed.*

We omit the proof here. But it can be proved, for example by using a result is Rockafellar's text (Chapter?, page?, Lemma ?).

Example 3. Let us consider the Single Facility Location Problem (SFCP). We want to find the location of a single facility on $\mathbf{x} \in \mathfrak{R}^n$ to minimize euclidean distance to a set of m customers whose locations are given by $\mathbf{a}_i \in \mathfrak{R}^n$ for $i = 1, \ldots, m$. Let us also assume that a priority in terms of distance penalty is assigned for each customer by $w_i \in \mathfrak{R}$ for all $i = 1, \ldots, m$. Therefore we want to solve the following optimization problem:

$$\min \quad \sum_{i=1}^{m} w_i \|\mathbf{x} - \mathbf{a}_i\| \tag{8}$$

which is equivalent to the following problem

$$\begin{array}{lcccc} \min & \sum_{i=1}^{m} w_i z_i & & & \\ \text{s.t.} & z_i & \geq & \|\mathbf{x} - \mathbf{a}_i\| & \forall i = 1, \ldots, m \end{array} \tag{9}$$

As this constraint is equivalent to the second order cone constraint, we can equivalently represent the problem as a conic LP as follows:

$$\begin{array}{lcccc} \min & \sum_{i=1}^{m} w_i z_i & & & \\ \text{s.t.} & \begin{pmatrix} z_i \\ x \end{pmatrix} & \succcurlyeq_{\mathcal{Q}} & \begin{pmatrix} 0 \\ \mathbf{a}_i \end{pmatrix} & \forall i = 1, \ldots, m \end{array} \tag{10}$$

or

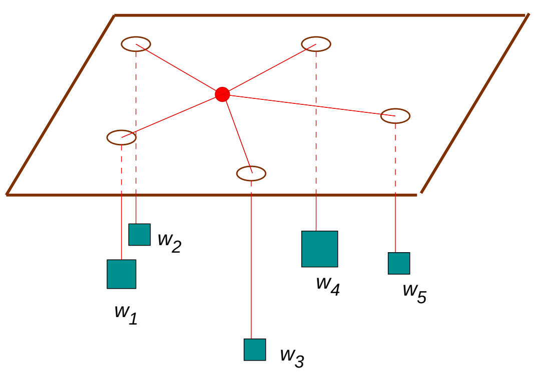
$$\begin{array}{lcccc} \min & \sum_{i=1}^{m} w_i z_i & & & \\ \text{s.t.} & z_i \mathbf{e}_0 + \bar{x} & \succcurlyeq_{\mathcal{Q}} & \begin{pmatrix} 0 \\ \mathbf{a}_i \end{pmatrix} & \forall i = 1, \ldots, m \\ & x_0 & = & 0 & \end{array} \tag{11}$$

Let us associate y_i, $\mathbf{u}_i$ and z as the dual variables of the constraints in this final form respectively. Then the dual of this problem can be given as follows:

$$\begin{array}{llll} \max & \sum_{i=1}^{m} \begin{pmatrix} 0 \\ \mathbf{a}_i \end{pmatrix}^T \begin{pmatrix} y_i \\ \mathbf{u}_i \end{pmatrix} & & \\ \text{s.t.} & \mathbf{e}_0^T \begin{pmatrix} y_i \\ \mathbf{u}_i \end{pmatrix} & = & w_i \quad \forall i = 1, \ldots, m \\ & \sum_{i=1}^{m} \begin{pmatrix} y_i \\ \mathbf{u}_i \end{pmatrix} + z\mathbf{e}_0 & = & 0 \\ & \begin{pmatrix} y_i \\ \mathbf{u}_i \end{pmatrix} & \succcurlyeq_{\mathcal{Q}} & 0 \end{array} \tag{12}$$

It is easy to observe the constraints imply $y_i = w_i$ and $z = -\sum_{i=1}^{m} w_i$. Therefore we can eliminate these redundant variables.

$$\begin{array}{llll} \max & \sum_{i=1}^{m} (\mathbf{a}_i - \mathbf{x})^T \mathbf{u}_i & & \\ \text{s.t.} & \sum_{i=1}^{m} \mathbf{u}_i & = & 0 \\ & \|\mathbf{u}_i\| & \leq & w_i \quad \forall i = 1, \ldots, m \end{array} \tag{13}$$



This dual has a very nice interpretation. Assume we have weights hanging on locations $\mathbf{a}_i \in \mathfrak{R}^n$ from an n dimensional horizontal hyperplane. And we tie a rope to each and every one of these hanging weights and tie them all together on top of this hyperplane. Eventually these all tied together weights are going to come into an equilibrium state where they are stationary. At that state definitely the force applied (as a force in the direction of $\mathbf{u}_i \in \mathfrak{R}^n$ with the strength of $\|\mathbf{u}_i\|$) in each and every one of these weights cannot exceed their own weights otherwise the object should be lifted upwards. Also we note that the forces must come into balance in the stationary state where $\sum_{i=1}^{m} \mathbf{u}_i = 0$. □

Corollary 2. *Let $\mathcal{K} \subseteq \mathfrak{R}^n$ be an arbitrary set. Then the following holds:*

1. *$\mathcal{K}^*$ is a closed convex cone.*

2. $(\mathcal{K}^*)^* = \text{cl}(\text{cone}(\mathcal{K}))$*. Furthermore if* $\mathcal{K}$ *is a closed convex cone, then* $(\mathcal{K}^*)^* = \mathcal{K}$*.*

3. $\mathcal{K}^*$ *is proper if and only if* $\mathcal{K}$ *is proper.*

4. If A *is a bijective linear transformation, then* $(A\mathcal{K})^* = A^{-T}\mathcal{K}^*$*.*

Proof. 1. if $\mathbf{y} \in \mathcal{K}^*$, then $\langle \mathbf{x}, \alpha\mathbf{y}\rangle \geq 0$, for every $\mathbf{x} \in \mathcal{K}$ and every real number $\alpha \geq 0$. So $\mathcal{K}^*$ is a cone. Also if $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{K}^*$, then $\langle \mathbf{y}_1 + \mathbf{y}_2, \mathbf{x}\rangle \geq 0$ for all $\mathbf{x} \in \mathcal{K}$. Thus $\mathcal{K}^*$ is a convex cone. Finally, let $\mathbf{y}_1, \mathbf{y}_2, \ldots$ be any sequence of points in $\mathcal{K}^*$, and $\mathbf{y}$ any limit point of this sequence. Consider for any $\mathbf{x} \in K$, the sequence of values $z_i = \langle \mathbf{x}, \mathbf{y}_i\rangle$. Since the half line $\mathbb{R}_+$ of nonnegative numbers is closed, and $\langle \cdot, \cdot \rangle$ is continuous, it follows that $z = \langle \mathbf{x}, \mathbf{y}\rangle$ is also a limit point of z_i and thus it is nonnegative. Thus, $\mathbf{y} \in \mathcal{K}^*$, and $\mathcal{K}^*$ is a closed and convex cone.

2. If $\mathbf{x} \in \text{cone}(\mathcal{K})$, then $\mathbf{x} = \sum_i \alpha_i \mathbf{x}_i$, for some $\mathbf{x}_i \in \mathcal{K}$ and some $\alpha_i \geq 0$. Thus, for any $\mathbf{y} \in \mathcal{K}^*$ we have $\langle \mathbf{x}, \mathbf{y}\rangle \geq 0$, and thus $\mathbf{x} \in \big(\mathcal{K}^*\big)^*$, implying $\text{cone}\,\mathcal{K} \subseteq \big(\mathcal{K}^*\big)^*$. Since by part 1 $\big(\mathcal{K}^*\big)^*$ is closed, it also contains $\text{cl}\big(\text{cone}(\mathcal{K})\big)$. Conversely, suppose there is some point $\mathbf{y}$ where $\mathbf{y} \notin \text{cl}\,\text{cone}(\mathcal{K})$, but $\mathbf{y} \in \big(\mathcal{K}^*\big)^*$. Then, by the separating hyperplane theorem for closed cones, there is a vector $\mathbf{a}$ such that $\langle \mathbf{a}, \mathbf{z}\rangle \geq 0$ for all $\mathbf{z} \in \text{cl}(\text{cone}\,\mathcal{K})$, and $\langle \mathbf{z}, \mathbf{y}\rangle < 0$. Thus, $\mathbf{a} \in \mathcal{K}^*$. On the other hand $\mathbf{a}$ has negative inner product with a member of $\big(\mathcal{K}^*\big)^*$, namely $\mathbf{y}$, and this is a contradiction. So, if $\mathbf{y} \notin \text{cl}\,\text{cone}(\mathcal{K})$, it cannot be in $\big(\mathcal{K}^*\big)^*$. Thus, $\big(\mathcal{K}^*\big)^* \subseteq \text{cl}\,\text{cone}(\mathcal{K})$, and the assertion is proved.

3. We note that if $\mathcal{K}$ is full dimensional, its dual $\mathcal{K}^*$ cannot contain a line, and so it is pointed. Thus a cone is closed, pointed, convex and full-dimensional if and only if its dual is also closed convex, pointed and full-dimensional.

■

3 Complementary Slackness Conditions for Conic LP

We will work on the following problem:

$$\begin{array}{lccc} \min & \mathbf{c}^T\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \succcurlyeq_{\mathcal{K}} & 0 \end{array} \tag{14}$$

and its dual:

$$\begin{array}{lcccc} \max & \mathbf{b}^T\mathbf{y} & & & \\ \text{s.t.} & A^T\mathbf{y} & +\mathbf{s} & = & \mathbf{b} \\ & & \mathbf{s} & \succcurlyeq_{\mathcal{K}^*} & 0 \end{array} \tag{15}$$

It is easily verified that for any primal-dual feasible solution, we have $\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x} - \mathbf{x}^T A^T \mathbf{y} = \mathbf{c}^T\mathbf{x} - \mathbf{x}^T(\mathbf{c} - \mathbf{s}) =< \mathbf{x}, \mathbf{s} >\geq 0$ as $x \in \mathcal{K}$ and $s \in \mathcal{K}^*$. A well-known corollary to strong duality theorem is that $< \mathbf{x}, \mathbf{s} >= 0$ if and only if $\mathbf{x}$ and $\begin{pmatrix} \mathbf{y} \\ \mathbf{s} \end{pmatrix}$ are primal-dual optimal solutions. Therefore we define the Complementarity set of $\mathcal{K}$ (and $\mathcal{K}^*$) as follows:

$$C(\mathcal{K}, \mathcal{K}^*) = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{s} \end{pmatrix} : \mathbf{x} \succcurlyeq_{\mathcal{K}} 0,\ \mathbf{s} \succcurlyeq_{\mathcal{K}^*} 0,\ < \mathbf{x}, \mathbf{s} >= 0 \right\}. \tag{16}$$

Later on we will show that $C(\mathcal{K}, \mathcal{K}^*)$ is an n dimensional manifold.

Example 4. If $\mathcal{K} = \mathcal{L} = \{x \in \Re^n : x \geq 0\}$, then we get the usual LP complementarity relation:

$$C(\mathcal{L}, \mathcal{L}) = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{s} \end{pmatrix} : \mathbf{x} \geq 0,\ \mathbf{s} \geq 0,\ \mathbf{x}^T\mathbf{s} = 0 \right\}. \tag{17}$$

Note that $\mathcal{L} = \mathcal{L}^*$. □

Example 5. If $\mathcal{K}$ is the PSD cone, i.e. $X \succcurlyeq 0$ and $S \succcurlyeq 0$. Then

$$\begin{aligned} X \cdot S &= \mathrm{tr}(XS) \\ &= \mathrm{tr}(XS^{\frac{1}{2}}S^{\frac{1}{2}}) \\ &= \mathrm{tr}(S^{\frac{1}{2}}XS^{\frac{1}{2}}) = 0 \end{aligned}$$

implies $S^{\frac{1}{2}}XS^{\frac{1}{2}} = 0$. If we apply Spectral Decomposition Theorem again to X we get $S^{\frac{1}{2}}X^{\frac{1}{2}}X^{\frac{1}{2}}S^{\frac{1}{2}} = 0$ or equivalently $\|X^{\frac{1}{2}}S^{\frac{1}{2}}\|^2 = 0$, which is if and only if $X^{\frac{1}{2}}S^{\frac{1}{2}} = 0$. Which implies $XS = 0$ also similarly $SX = 0$, therefore we have

$$\frac{SX + XS}{2} = 0 \tag{18}$$

When at least one of X or S is positive semidefinite (as is the case in SDP) then the converse is also true:

Lemma 3. *If at least one of the symmetric matrices X or S is positive semidefinite, then $\frac{XS+SX}{2} = 0$ implies that $XS = SX = 0$.*

Proof. Let $A = XS$, and assume $X \succcurlyeq 0$. Since $A + A^\top = 0$, A is skew symmetric, and thus its eigenvalues are either zero or purely imaginary. On the other hand since $X \succcurlyeq 0$, $XS = X^{1/2}X^{1/2}S$ has the same eigenvalues as $X^{1/2}SX^{1/2}$ which is symmetric positive semidefinite, and in particular has real eigenvalues. So, XS has only purely imaginary or zero eigenvalues, and only real eigenvalues. This can only happen if all eigenvalues of XS are zero. Since XS is similar to $X^{1/2}SX^{1/2}$, this atrix has only zero eigenvalues, and since it is symmetric we must have $X^{1/2}SX^{1/2} = 0$ which immediately follows that $XS = SX = 0$. ■

(18) is the *Complementary Slackness condition* for the general SDP programming problem. □

Example 6. Let $\mathcal{K} = \mathcal{Q}$, namely, the second order cone. Let us note that $\mathcal{Q} = \mathcal{Q}^*$. Therefore $\mathbf{x} \succcurlyeq_{\mathcal{Q}} 0$ and $\mathbf{s} \succcurlyeq_{\mathcal{Q}} 0$ if and only if $x_0 \geq ||\bar{\mathbf{x}}||$ and $s_0 \geq ||\bar{\mathbf{s}}||$. Complementary Slackness $\mathbf{x}^T\mathbf{s} = 0$ holds if and only if

$$\begin{aligned} x_0 s_0 &= -\sum_{i=1}^{n} x_i s_i \\ &\leq \sum_{i=1}^{n} |x_i s_i| \\ &\leq ||\bar{\mathbf{x}}||||\bar{\mathbf{s}}|| \\ &\leq x_0 s_0 \end{aligned}$$

That implies $x_0 s_0 = ||\bar{\mathbf{x}}||||\bar{\mathbf{s}}||$, therefore $\mathbf{x}$ and $\mathbf{s}$ are parallel. In other words there exists $\alpha \in \Re$ such that $\frac{\mathbf{x}_i}{\mathbf{s}_i} = \alpha$ for all $i = 0, \ldots, n$. Using the fact that $\frac{\mathbf{x}_i^2}{\mathbf{x}_i \mathbf{s}_i} = \alpha$, we can $\alpha = \frac{\sum_{i=1}^{n} \mathbf{x}_i^2}{\sum_{i=1}^{n} \mathbf{x}_i \mathbf{s}_i} = \frac{||\bar{\mathbf{x}}||^2}{-x_0 s_0} = \frac{x_0^2}{-x_0 s_0} = -\frac{x_0}{s_0}$ implies $\frac{\mathbf{x}_i}{\mathbf{s}_i} = -\frac{x_0}{s_0}$, or $x_0 s_i + x_i s_0 = 0$. Together with the $\mathbf{x}^T\mathbf{s} = 0$ these conditions form Complementarity Slackness conditions for Second Order Cone Programming (SOCP). □

Theorem 4. *The set* $C(\mathcal{K}, \mathcal{K}^*) = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{s} \end{pmatrix} : \mathbf{x} \succcurlyeq_{\mathcal{K}} 0,\ \mathbf{s} \succcurlyeq_{\mathcal{K}^*} 0,\ <\mathbf{x}, \mathbf{s}>= 0 \right\}$ *is an* n *dimensional manifold homeomorphic to* $\Re^n$.

Claim 1. *Let* $\mathbf{a} \in \Re^n$ *and* $\mathbf{x} = \Pi_{\mathcal{K}}(\mathbf{a})$ *namely the projection of* $\mathbf{a}$ *onto* $\mathcal{K}$. *Let* $\mathbf{s} = \mathbf{x} - \mathbf{a}$. *Then we have* $<\mathbf{x}, \mathbf{s}>= 0$.

Proof. Let $\mathbf{u} \in \mathcal{K}$ and $\mathbf{u}_\alpha = \alpha\mathbf{u} + (1-\alpha)\mathbf{x}$ for $\alpha \in [0, 1]$. Let $\zeta(\alpha) = ||\mathbf{a} - \mathbf{u}_\alpha||^2$. Then

$$\begin{aligned} \frac{d\zeta(\alpha)}{d\alpha} &= \frac{d}{d\alpha}(\mathbf{a} - \alpha\mathbf{u} - (1-\alpha)\mathbf{x})^T(\mathbf{a} - \alpha\mathbf{u} - (1-\alpha)\mathbf{x}) \\ &= 2(\mathbf{x} - \mathbf{u})^T(\mathbf{a} - \mathbf{x}) \\ &= 2\mathbf{s}^T(\mathbf{u} - \mathbf{x}) \geq 0 \end{aligned}$$

where the last inequality follows from the fact that $\zeta(\alpha)$ is a non-decreasing function. As $\mathbf{u} \in \mathcal{K}$ was chosen arbitrarily, we have $\mathbf{s}^T\mathbf{x} \geq 0$ for $\mathbf{u} = 2\mathbf{x}$ and $\mathbf{s}^T\mathbf{x} \leq 0$ for $\mathbf{u} = \frac{\mathbf{x}}{2}$. Implying $\mathbf{s}^T\mathbf{x} = 0$.

Now we need to show that for every pair $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K}, \mathcal{K}^*)$ there is continuous function sending it to an element of $\mathbb{R}^n$. We will prove this in the next lecture. ■