Semidefinite and Second Order Cone Programming Seminar
Fall 2012
Lecture 9

Instructor: Farid Alizadeh
Scribe: Shunqiao Sun

11/12/2012

1 Overview

This lecture addresses the $\mathbb{K}$-representability, such as $\mathbb{K}$-representable sets, functions as well as operations that preserve the $\mathbb{K}$-representability. Specially, the $\mathbb{K}$-representability of singular value and eigenvalue optimization is studied.

2 $\mathbb{K}$-representable Sets

First, assume that $\mathbb{K}$ is a family of cones and satisfies:

i) $\mathcal{K}_1, \mathcal{K}_2 \in \mathbb{K} \Rightarrow \mathcal{K}_1 \times \mathcal{K}_2 \in \mathbb{K}$

ii) $\mathcal{K} \in \mathbb{K} \Rightarrow \mathcal{K}^* \in \mathbb{K}$

Definition 1 ($\mathbb{K}$-representable) *A set $\mathcal{X}$ is $\mathbb{K}$-representable if there is a cone $\mathcal{K} \in \mathbb{K}$, and linear transformations (matrices) P and Q and vector $\mathbf{b}$ such that*

$$\mathcal{X} = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \exists \mathbf{u} : \mathsf{P}\mathbf{x} + \mathsf{Q}\mathbf{u} \succcurlyeq_{\mathcal{K}} \mathbf{b} \right\}$$

In this case optimization problems

$$\begin{aligned} \min\ & \mathbf{c}^\mathsf{T}\mathbf{y} \\ \text{s.t. } & \mathbf{y} \in \mathcal{X} \end{aligned}$$

can be formulated as some $\mathcal{K}$-LP:

$$\begin{aligned} \min\ & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t. } & \mathsf{P}\mathbf{x} + \mathsf{Q}\mathbf{u} \succcurlyeq_{\mathcal{K}} \mathbf{b} \end{aligned}$$

Here are some typical, but also fundamental examples:

Example 1 LP-representable sets: When $\mathbb{K}$ consists of all nonnegative orthants of dimensions $1, 2, 3, \ldots$, that is $\mathcal{L}_n = \{\mathbf{x} \mid x_i \geq\}$, we say that the $\mathcal{X}$ is *LP-representable.*

It is easy to show that:

Lemma 2 *$\mathcal{X}$ is LP-representable if and only if it is a polyhedron.*

Example 2 Second order (SO)-representable sets If $\mathbb{K}$ is the family of second order cones $\mathcal{Q}_{n+1}$ for $n = 0, 1, \ldots$, then a $\mathbb{K}$-representable set $\mathcal{X}$ is called *SO-representable.*

We will shortly see some prominent SO-representable sets shortly.

Example 3 semidefinite (SD)-representable sets If $\mathbb{K}$ is the family of real, symmetric, positive semidefinite matrices, then a $\mathbb{K}$-representable set $\mathcal{X}$ is called *SD-representable.*

We can replace "real-symmetric" with "complex, Hermitian" in the definition above, but it turns out SD-representability with respect to complex Hermitian matrices can be formulated with SD-representability by real, symmetric matrices. We will see some concrete examples of SD-representable sets below.

2.1 Operations that preserve $\mathbb{K}$-representability

1) **Direct sums (Cartesian products):** If $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n$ are $\mathbb{K}$-representable, then
$\mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n =$

$$\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \succcurlyeq_{K_1 \times \cdots \times \mathcal{K}_n} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\}$$

is $\mathbb{K}$-representable.

2) **Finite intersections:** If $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n$ are $\mathbb{K}$-representable, then $\mathcal{X}_1 \cap \mathcal{X}_2 \cap \cdots \mathcal{X}_n$ is $\mathbb{K}$-representable.

$$\cap \mathcal{X}_i = \left\{ \mathbf{x} \,\middle|\, \exists \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} : \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} \mathbf{x} + \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \succcurlyeq_{K_1 \times \cdots \times \mathcal{K}_n} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\}$$

3) **Affine transformations:** Suppose $\mathcal{A}(\mathbf{x}) = A\mathbf{x} + \mathbf{a}$, and let $\mathcal{X} = \{\mathbf{x} \mid \exists \mathbf{u}, P\mathbf{x} + Q\mathbf{u} \succcurlyeq_{\mathcal{K}} \mathbf{b}\}$ for some matrices P and Q (that is $\mathcal{X}$ is $\mathbb{K}$-representable). To show that $\mathcal{A}(\mathcal{X})$ is also $\mathbb{K}$-representable, let $A \in \mathbb{R}^{m \times n}$. Without loss of generality we may assume that A has full column rank if $m \geqslant n$ and full row rank if $m < n$. We consider two cases:

(a) $m \geqslant n$: In this case, we have

$$Ax = y \Rightarrow A^\top Ax = A^\top y \Rightarrow x = \left(A^\top A\right)^{-1} A^\top y$$

Note that $A^\top A$ is a $n \times n$ *invertible* matrix. Thus, defining $A^+ = \left(A^\top A\right)^{-1} A^\top$, we see that

$$\begin{aligned} \mathcal{A}(\mathcal{X}) &= \{y = Ax + a \mid \exists u : Px + Qu \succcurlyeq_{\mathcal{K}} b\} \\ &= \{y \mid \exists u : PA^+ y + Qu \succcurlyeq_{\mathcal{K}} b + PA^+ a\} \end{aligned}$$

Thus, by setting $P \leftarrow PA^+$ and $b \leftarrow b + PA^+ a$, we have shown that $\mathcal{A}(\mathcal{X})$ is $\mathbb{K}$-representable.

(b) $m < n$: In this case the columns are linearly dependent and A has a nontrivial (right) kernel: $\ker(A) = \{x | Ax = 0\}$. Let $A = U_m[\Sigma, 0]V_n$ be the singular value decomposition of A, that is Σ is $m \times m$ diagonal matrix with positive entries in its diagonal, and U_m and V_n are, respectively, $m \times m$ and $n \times n$ orthogonal matrices. Then, if $Ax = y$, we get

$$\begin{aligned} U_m[\Sigma, 0]V_n x = &\quad y \\ [\Sigma, 0]V_n x = &\quad U_m^\top y \\ \Sigma V_{n,m} x_1 = &\quad U_m^\top y \quad \text{where } V_{n,m} \quad \text{is the first } m \text{ columns of } V_n \\ &\quad \text{and where } x = [x_1; x_2] \quad \text{with } x_1 \text{ the first } m \text{ entries of } x \\ V_{n,m} x_1 = &\quad \Sigma^{-1} U_m^\top y \\ x_1 = &\quad V_{n,m}^\top \Sigma^{-1} U_m^\top y \end{aligned}$$

Thus, setting $P_1 = V_{n,m}^\top \Sigma^{-1} U_m^\top$, we have

$$\begin{aligned} \mathcal{A}(\mathcal{X}) = &\quad \{y = Ax + a \mid \exists u : Px + Qu \succcurlyeq_{\mathcal{K}} b\} \\ = &\quad \{y \mid \exists u, x_2 : PP_1 y + Px_2 + Qu \succcurlyeq_{\mathcal{K}} b + PP_1 a\} \end{aligned}$$

4) **Inverse affine transformations:** Let $\mathcal{X}$ be $\mathbb{K}$-representable, and define $\mathcal{Y} = \{y \mid Ay + a \in \mathcal{X}\}$. Then

$$\mathcal{Y} = \{y \mid \exists u : PAy + Qu \succcurlyeq_{\mathcal{K}} b - Pa\}$$

Thus, $\mathcal{Y}$ is $\mathbb{K}$-representable.

5) **Minkowski sum:** For $\mathbb{K}$-representable sets $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$, the Minkowski sum

$$\mathcal{X}_1 + \mathcal{X}_2 + \cdots + \mathcal{X}_n = \{x_1 + x_2 + \cdots + x_n \mid x_i \in \mathcal{X}_i, \text{ for } i = 1, \ldots, n\}$$

is also $\mathbb{K}$-representable. An easy proof is to note that the Minkowski sum is the affine transformation of the direct sum of $\mathcal{X}_i$: $(x_1; x_2; \ldots; x_n) \to x_1 + \ldots + x_n$.

6) **Polyhedral sets are $\mathbb{K}$-representable:** Consider the set $P = \{x \mid a_i^\top x + b_i \geqslant 0, i = 1, \dots, m\}$. If $v \in \mathcal{K}$, where $\mathcal{K} \in \mathbb{K}$, then

$$P = \left\{x \mid \left((a_1^\top x + b_1)v; \dots; (a_m^\top x + b_m)v\right) \in \mathcal{K} \times \mathcal{K} \times \cdots \times \mathcal{K}\right\}$$

which shows P is $\mathbb{K}$-representable. This fact is significant in that $\mathbb{K}$-representable sets can be constructed by combinations of "component-wise" inequalities and $\succcurlyeq_{\mathcal{K}}$ inequalities, for $\mathcal{K} \in \mathbb{K}$.

7) Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two classes of cones. Furthermore, suppose that every cone in $\mathbb{K}_1$ is $\mathbb{K}_2$-representable. Then every $\mathbb{K}_1$-representable set is also $\mathbb{K}_2$-representable. First, notice that in the case of cones, if $\mathcal{K}_1$ is $\mathbb{K}_2$-representable, then there are matrices P and Q, and cone $\mathcal{K}_2 \in \mathbb{K}_2$ such that

$$\mathcal{K} = \{y \mid \exists v, P_1 y + Q_1 v \succcurlyeq_{\mathcal{K}_2} 0\}$$

Now let X be $\mathbb{K}_1$-representable. Then

$$\begin{aligned} X &= \{x \mid \exists u : Px + Qu \succcurlyeq_{\mathcal{K}_1} b\} \\ &= \{x \mid \exists u, v : P_1(Px + Qu - b) + Q_1 v \succcurlyeq_{\mathcal{K}_2} 0\} \\ &= \{x \mid \exists u, v : P_1 P x + P_1 Q u + Q_1 v \succcurlyeq_{\mathcal{K}_2} P_1 b\} \end{aligned}$$

There are many other operations which preserve $\mathbb{K}$-representability. See Nemirovski's slides of 1997 talk at RUTCOR.

3 $\mathbb{K}$-representable Functions

Definition 3 *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is* $\mathbb{K}$*-representable if its epigraph* $\text{epi}(f) = \{(x, x_0) : f(x) \leq x_0\}$ *is* $\mathbb{K}$*-representable set.*

$$f(x) \leq x_0 \Leftrightarrow \exists u : Px + x_0 \underset{\sim}{P} + Qu \succcurlyeq_{\mathcal{K}} b \Leftrightarrow \left[P | \underset{\sim}{P}\right] \begin{bmatrix} x \\ x_0 \end{bmatrix} + Qu \succcurlyeq_{\mathcal{K}} b.$$

$\mathbb{K}$representable functions are convex and their minimization can be expressed as a $\mathbb{K}$-LP problem: since

$$\min_x f(x) = \begin{array}{l} \min \ x_0 \\ \text{s.t. } f(x) \leqslant x_0 \end{array}$$

And the feasible set epi(f) is $\mathbb{K}$-representable.

Note that if $f(\cdot)$ is piece-wise linear then it is necessarily an LP-representable function, because a convex function is piece-wise linear if and only if its epigraph is polyhedral.

scribe: Shunqiao Sun
Lecture 9 **Date:** 11/12/2012

3.1 Operations Preserving $\mathbb{K}$-representability of Functions

1) Level sets: $\{x \,|f(x) \leqslant a\}$, $\exists u : Px + Qu \geqslant b - a\underset{\sim}{P}$

2) If $f_1, f_{2,...,}f_n$ are $\mathbb{K}$-representable, then $(\alpha_1 f_1 + ... + \alpha_n f_n)(x)$ is also $\mathbb{K}$-representable for $\alpha_i \geqslant 0$. To see this first notice that for a positive real number α and a $\mathbb{K}$-representable function $f(x)$, the function $\alpha f(x)$ is also $\mathbb{K}$-representable. This is because epi$\big(\alpha f(x)\big) = \{(x; x_0) \mid f(x) \leq x_0/\alpha\}$ and this is an affine transformation of the epi(f). Thus we may concentrate on showing that the sum $f_1 + f_2 + \cdots + f_m$ is $\mathbb{K}$-representable. It suffices to show that $f_1 + f_2$ is $\mathbb{K}$ representable.

$$\begin{aligned}\text{epi}(f_1+f_2) = \quad & \big\{(x, x_0) \mid f_1(x) + f_2(x) \leqslant x_0\big\} \\ = \quad & \big\{(x, x_0) \mid \exists y_0 : f_1(x) \leqslant y_0, f_2(x) \leqslant x_0 - y_0\big\} \\ = \quad & \big\{(x, x_0) \mid \exists u_1, u_2, y_0 : P_1 x + y_0 p_1 + Q_1 u_1 \succcurlyeq_{\mathcal{K}_1} 0, \\ & P_2 x + x_0 p_2 - y_0 p_2 + Q_2 u_2 \succcurlyeq_{\mathcal{K}_2} 0\big\}\end{aligned}$$

which shows that epi$(f_1 + f_2)$ is $\mathbb{K}$-representable.

3) If $f_i(x)$ are $\mathbb{K}$-representable for $i = 1, ..., n$, then $f(x) = \max\limits_i\{f_1(x), ..., f_n(x)\}$ is $\mathbb{K}$-representable. This is because epi $f = \bigcap\limits_i$ epi f_i.

4) Functional composition: $f_i : \mathbb{R}^n \to \mathbb{R}$ is $\mathbb{K}$-representable for $i = 1, ..., K$ and $g : \mathbb{R}^K \to \mathbb{R}$ is $\mathbb{K}$-representable. In addition let $g(\cdot)$ be *monotone*, that is if $x_1 \leqslant x_2$ then $g(x_1) \leqslant g(x_2)$. Then $h\big(x\big) = g\big(f_1(x), f_2(x), ..., f_K(x)\big)$ is $\mathbb{K}$-representable.
$f_i(x) \leqslant t \Leftrightarrow \exists u_i : P_i x + t\underset{\sim}{P_i} + Q_i u_i \succcurlyeq_{\mathcal{K}} b_i$
$h(x) \leqslant t \Leftrightarrow \exists \tau = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_K \end{pmatrix}, u_1, ..., u_K, v$ such that
$P_i x + \tau_i \underset{\sim}{P_i} + Q_i u_i \succcurlyeq_{\mathcal{K}_i} b_i \;\; (f_i(x) \leqslant \tau)$,
$P\tau + t\underset{\sim}{P} + Qv \succcurlyeq_{\mathcal{K}} b \;\; (g(\lambda) \leqslant t)$.

5) Suppose $f(x, y)$ is $\mathbb{K}$-representable. Assume for each value of x in the domain $g(x) = \min\limits_y f(x, y)$ is well-defined, that is for each value of x the minimum is attained at a finite value of x. Then if $f(\cdot)$ is $\mathbb{K}$-representable, so is $g(\cdot)$. Let

$$f(x, y) \leqslant t \Leftrightarrow \exists u : P_1 x + P_2 y + t\underset{\sim}{P} + Qy \succcurlyeq_{\mathcal{K}} b$$

Then:

$$g(x) \leqslant t \Leftrightarrow \exists y, u : Px + t\underset{\sim}{P} + [Ry + Qu] \succcurlyeq_{\mathcal{K}} b.$$

Again, there are many more operations that preserve $\mathbb{K}$-representability; see Nemirovski's slides.

scribe:Shunqiao Sun
Lecture 9 **Date:** 11/12/2012

3.2 Second Order -representable Functions

Let $\mathbb{Q}$ be the class of all second order cones $\mathcal{Q}_{n+1} = \left\{\mathcal{Q}_i \in \mathbb{R}^{n_i+1} : n_i = 0, 1, ...\right\}$. The following functions and sets are $\mathbb{Q}$-representable (SO-representable).

1) $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$ is second order representable: $\|\boldsymbol{x}\| \leqslant t \Leftrightarrow \begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$.

2) $\left\{(s, t, \boldsymbol{x}) \middle| st \geqslant \|\boldsymbol{x}\|^2\right\}$, i.e., $\frac{(s+t)^2}{2} - \frac{(s-t)^2}{2} \geqslant \|\boldsymbol{x}\|^2$. Thus, $\begin{pmatrix} \frac{s+t}{2} \\ \frac{s-t}{2} \\ \boldsymbol{x} \end{pmatrix} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$.

   Examples:

   i) $f(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$, $\|x\|^2 \leqslant t \Leftrightarrow \begin{pmatrix} \frac{t+1}{2} \\ \frac{t-1}{2} \\ \boldsymbol{x} \end{pmatrix} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$.

   ii) $f(s, \boldsymbol{x}) = \frac{\|\boldsymbol{x}\|^2}{s}, s > 0$, then $\frac{\|\boldsymbol{x}\|^2}{s} \leqslant t \Leftrightarrow \|\boldsymbol{x}\|^2 \leqslant ts \Leftrightarrow \begin{pmatrix} \frac{s+t}{2} \\ \frac{s-t}{2} \\ \boldsymbol{x} \end{pmatrix} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$.

3) $f(x) = -\prod_{i=1}^{n} x_i^{\pi_i}$, where π_i are rational numbers. $f(x)$ is second order-representable. Examples:

   i) $\underbrace{x_1 x_2}_{z_1^2} \underbrace{x_3 x_4}_{z_2^2} \geqslant t^4 \Leftrightarrow x_1 x_2 \geqslant z_1^2, x_3 x_4 \geqslant z_2^2, z_1 z_2 \geqslant t^2$.

   ii) $x_1^{-\frac{5}{6}} x_2^{-\frac{1}{3}} x_3^{-\frac{1}{2}} \leqslant t \Leftrightarrow 1^6 = 1 \leqslant x_1^5 x_2^2 x_3^3 t^6$.

   The general idea is that by proper algebraic manipulation we can reduce such inequalities into an inequality of the form $x_1 x_2 \ldots x_{2^k} \geqslant 1$, where x_i are not necessarily distinct, and some of them may equal 1. From here, by repeated partitioning into two parts, we can replace this inequality with k inequalities of the form $z_1 z_2 \geq 1$, and construct a second order representation.

4) convex quadratic functions: $f(\boldsymbol{x}) = \boldsymbol{x}^\top A \boldsymbol{x} + \boldsymbol{a}^\top \boldsymbol{x} + b$, where $A \succcurlyeq 0$ that is, there is matrix B such that $A = B^\top B$.

$$f(\boldsymbol{x}) \leq t \Leftrightarrow \|B\boldsymbol{x}\| \leq t - \boldsymbol{a}^\top \boldsymbol{x} - b \Leftrightarrow \begin{pmatrix} \frac{t - \boldsymbol{a}^\top \boldsymbol{x} - b + 1}{2} \\ \frac{t - \boldsymbol{a}^\top \boldsymbol{x} - b - 1}{2} \\ B\boldsymbol{x} \end{pmatrix} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$$

For more semidefinite representable functions and sets see Nemirovski's slides, Alizadeh and Goldfarb paper on second order cone programming, and Lubo et al on applications of SOCP.

3.3 Semidefinite -representable Sets and Functions

Let $\mathbb{P}$ be the class of $n \times n$ positive semidefinite matrices: $\mathbb{P} = \{\mathcal{P}_{n\times n} : n = 1, ...\}$. Then $\mathbb{P}$-representable functions are called SD-representable. First, let us introduce the following notation: For a finites set of real numbers $\{b_1, \ldots, b_k$ define, $b_{[k]}$ to be the k^{th} largest element of B, and $b_{|k|}$ the k^{th} *absolute-value-wise* largest element of B. For instance, for a symmetric matrix X, $\lambda_{[k]}(X)$ is the k^{th} largest eigenvalue of X, and $\lambda_{|k|}(X)$ is its k^{th} absolute-value-wise largest eigenvalue.

We first note that every second order cone $\mathcal{Q}_i$ is itself SD-representable. $\begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix} Q \geq 0 \Leftrightarrow \begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ x_1 & x_0 & & \\ \vdots & & \ddots & \\ x_n & & & x_0 \end{pmatrix} \succcurlyeq 0 \Leftrightarrow x_0 I + \begin{pmatrix} 0 & x_1 & \cdots & x_n \\ x_1 & 0 & & \\ \vdots & & \ddots & \\ x_n & & & 0 \end{pmatrix} \succcurlyeq 0.$

Define $\bar{\bar{x}} = \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$, then $x_0 I + \bar{\bar{x}} \boldsymbol{e}_0^T + \boldsymbol{e}_0 \bar{\bar{x}}^T \succcurlyeq 0 \Leftrightarrow x_0 \geqslant \|\bar{x}\|$.

This immediately implies:

Corollary 4 *All second order-representable sets are also semidefinite-representable. All second order-representable functions are also semidefinite-representable.*

The following functions and sets are $\mathbb{P}$-representable

1) The largest eigenvalue of matrix X, $\lambda_{[1]}(\boldsymbol{x})$ is semidefinite representable: $\lambda_{[1]}(X) \leqslant t \Leftrightarrow tI - X \succcurlyeq 0$.

2) The sum of K largest eigenvalues of matrix X, i.e., $\left(\lambda_{[1]} + \lambda_{[2]} + ... + \lambda_{[K]}\right)(X)$ is semidefinite representable:
$\left(\lambda_{[1]} + \lambda_{[2]} + ... + \lambda_{[K]}\right)(A) \leqslant t \Leftrightarrow tI + X \succcurlyeq A, X \succcurlyeq A$.
Minimize the sum of k largest eigenvalues of matrix X can be written as

$$\begin{aligned} &\min\ t \\ &\text{s.t. } \ t \geqslant \left(\lambda_{[1]} + \lambda_{[2]} + ... + \lambda_{[k]}\right)(A) \end{aligned}$$

Claim: The above minimization problem can be formulated as

$$\begin{aligned} \max\ & A \bullet Y \\ \text{s.t. } & \operatorname{Tr}(Y) = k \\ & I \succcurlyeq Y \succcurlyeq 0 \end{aligned}$$

The dual problem is

$$\begin{aligned} \min\ & kz + \operatorname{Tr}(X) \\ \text{s.t. } & zI + X \succcurlyeq A \\ & X \succcurlyeq 0 \end{aligned}$$

Let the eigenvalue decomposition of A be $A = \lambda_{[1]}\mathbf{q}_1\mathbf{q}_1^\top + \cdots + \lambda_{[n]}\mathbf{q}_n\mathbf{q}_n^\top$. Define $Y = \sum_{i=1}^{k} q_i q_i^\top$. First note that Y is feasible for the maximization problem above, and the value of the objective function can be computed as:

$$
\begin{aligned}
A \bullet Y &= \Big(\sum_{i=1}^{k} \mathbf{q}\mathbf{q}^\top\Big)\bullet\Big(\sum_{i=1}^{k} \lambda_{[i]}\mathbf{q}_i\mathbf{q}_i^\top\Big)\\
&= \lambda_{[1]}(\mathbf{q}_1\mathbf{q}_1^\top)\bullet(\mathbf{q}_1\mathbf{q}_1^\top) + \lambda_{[2]}(\mathbf{q}_2\mathbf{q}_2^\top)\bullet(\mathbf{q}_2\mathbf{q}_2^\top) + \cdots + \lambda_{[k]}(\mathbf{q}_k\mathbf{q}_k^\top)\bullet(\mathbf{q}_k\mathbf{q}_k^\top)\\
&\quad \text{(noting } (\mathbf{q}_i\mathbf{q}_i^\top)\bullet(\mathbf{q}_i\mathbf{q}_i^\top) = 1 \quad \text{and} \quad (\mathbf{q}_i\mathbf{q}_i^\top)\bullet(\mathbf{q}_j\mathbf{q}_j^\top) = 0 \quad \text{for } i \neq j)\\
&= \lambda_{[1]} + \cdots + \lambda_{[k]}.
\end{aligned}
$$

Furthermore, let z be a number chosen so that $\lambda_{[k+1]} \leqslant z \leqslant \lambda_{[k]}$, and $X = \sum_{i=1}^{k} x_i\mathbf{q}_i\mathbf{q}_i^\top$, where $x_i = \lambda_{[i]} - z$, for $i = 1, \ldots, k$. Then again, this choice of X and z is feasible for the minimization problem, and, by construction, the value of the objective function is again $\sum_{i=1}^{l} \lambda_{[i]}(A)$. We conclude that the (Y, X, z) constructed above is primal and dual optimal. As a corollary we have shown that the function $\big(\lambda_{[1]} + \cdots + \lambda_{[k]}\big)(\cdot)$ is an SD-representable function.

3) The weighted sum of eigenvalues is $\big(w_1\lambda_{[1]} + ... + w_k\lambda_{[K]}\big)(X)$. In general, this function is not convex. However, if $w_1 \geqslant w_2 \geqslant ... \geqslant w_k$, then $\big(w_1\lambda_{[1]} + \cdots + \lambda_{[k]}\big)(\cdot)$ is convex, and indeed SD-representable. This can be proved by noting

$$
\begin{aligned}
w_1\lambda_{[1]} + w_2\lambda_{[2]} + \cdots + w_k\lambda_{[k]} = &(w_1 - w_2)\lambda_{[1]}\\
&+ (w_2 - w_3)\big(\lambda_{[1]} + \lambda_{[2]}\big)\\
&+ \cdots\\
&+ (w_{k-1} - w_k)\big(\lambda_{[1]} + \lambda_{[2]} + \cdots + \lambda_{[k-1]}\big)\\
&+ w_k\big(\lambda_{[1]} + \lambda_{[2]} + \cdots + \lambda_{[k]}\big)
\end{aligned}
$$

which expresses the weighted sum, as a nonnegative linear combination of SD-representable functions, and thus, itself is SD-representable.

4) The results above can all be generalized under an umbrella. Let $g(x_1, ..., x_n) : \mathbb{R}^n \to \mathbb{R}$ be symmetric if $g(x_1, ..., x_n) = g(x_{\pi_1}, ..., x_{\pi_n})$, where π is any permutation. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a symmetric semidefinite-representable function, then $h(X) = g\big(\lambda_{[1]}(X), ..., \lambda_{[n]}(X)\big)$ is also semidefinite-representable. To see this let

$$g(\mathbf{x}) \leqslant t \Leftrightarrow \exists \mathbf{u} : P\mathbf{x} + t\underset{\sim}{P} + Q\mathbf{u} \geqslant \mathbf{b}$$

Then, the relation $h(X) \leqslant t$ can be expressed as follows:

$$\begin{aligned} & h(X) \leqslant t \Leftrightarrow \exists \mathbf{u}, x_1, ..., x_n : P(X) + t\mathbf{p} + Q\mathbf{u} \geqslant \mathbf{b}, \\ & x_1 \geqslant x_2 \geqslant ... \geqslant x_n \\ & x_1 + ... + x_k \geqslant \left(\lambda_{[1]} + ... + \lambda_{[k]}\right)(X), k = 1, ..., n \\ & x_1 + ... + x_n = \operatorname{Tr}(X). \end{aligned} \quad (1)$$

The following functions can immediately be seen to be SD-representable as a result of this fact:

(a) **The function** $\operatorname{Det}(X)$ **: defined over symmetric matrices:** We saw that the function $f(\boldsymbol{x}) = x_1 x_2 \dots x_n$ is SO-representable, and thus SD-representable. Therefore, $f\big(\boldsymbol{\lambda}(X)\big) = \operatorname{Det} X$ is SD-representable.

(b) **The function** $\big(\lambda_{|1|} + \lambda_{|2|} + \cdots + \lambda_{|k|}\big)(\cdot)$**:** We see that for a set of n real numbers $\{x_1, \dots, x_n\}$ the function $x_{|1|} + \cdots + x_{|k|}$, that is sum of k absolute-value-wise largest elements, is a *symmetric LP representable*. In fact, the sum of k absolute-value-wise largest elements in $\{x_1, x_2, \dots, x_n\}$ equals the sum of k largest elements in $\{-x_1, x_1, \dots, -x_n, x_n\}$. Since LP-representable functions are SD-representable, it follows that $\big(\lambda_{|1}| + \dots + \lambda_{|k|}\big)(\cdot)$ is SD-representable.

(c) **The sum of** k **largest singular values of an arbitrary** $m \times n$ **matrix** X**:** This follows from the fact that the singular values of X are absolute values of eigenvalues of $Y = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$ (see the Appendix). Thus,

$$\sigma_{[1]}(X) + \cdots + \sigma_{[k]}(X) = \lambda_{|1|}(Y) + \cdots + \lambda_{|k|}(Y)$$

and therefore, SD-representable.

Appendix: Singular Value Decomposition (SVD)

Singular value decomposition of matrices $X \in \mathbb{R}^{m \times n}, m \leqslant n$ is $X = U\Sigma V$, where $U \in \mathbb{R}^{m \times m}$, and $UU^T = I$, and $V \in \mathbb{R}^{n \times n}$, with $VV^T = I$.

If $m \leqslant n$, then $\Sigma = [\Sigma_1, 0]$ where Σ_1 is a $m \times m$ diagonal matrix with positive numbers on its diagonal. These m positive numbers are the *singular values* of X.

If $m > n$, then $\Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}$, and again Σ_1 is a diagonal matrix with positive entries in the diagonal.

As in the case of eigenvalue decomposition, the singular value decomposition can alternatively written as $X = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^\top + \cdots + \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^\top$, where the σ_i are the singular values, and $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are, respectively, are columns of U and V.

It turns out that the singular values σ_i are the square roots of eigenvalues of $XX^\top$ if $m \leqslant n$, and $X^\top X$ if $m > n$:

$$X = U\Sigma V \Rightarrow XX^\top = U\Sigma VV^\top \Sigma^\top U^\top = U\Sigma_1^2 U^\top.$$

Thus, the eigenvalues of $XX^\top$, and of $X^\top X$ are σ_i^2 (plus possibly a number of zeros), and $\mathbf{v}_i$ are eigenvectors of $X^\top X$, and $\mathbf{u}_i$ are eigenvectors of $XX^\top$.

Another useful characterization is the following lemma:

Lemma 5 *Let X be an $m \times n$ matrix, and let $m \leqslant n$, with linearly independent rows. Then the eigenvalues of the matrix*

$$A = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$$

are the singular values of X and their negatives, plus zero with multiplicity of $n - m$: $\{\lambda_i(A) :| i = 1, \dots, m\} = \{\pm\sigma_i(X) \mid i = 1, \dots, m\} \cup \{0\}$.

This can be seen by noting that eigenvectors of A and A^2 are the same, with the corresponding eigenvalues of A^2, squares of eigenvalues of A. But

$$A^2 = \begin{pmatrix} XX^\top & 0 \\ 0 & X^\top X \end{pmatrix}$$

Recalling that eigenvalues of $XX^\top$ and of $X^\top X$ are squares of singular values of X (plus some zeros), the lemma follows.