# Semidefinite and Second Order Cone Programming Seminar
# Fall 2012
# Lecture 11

**Instructor:** Farid Alizadeh
**Scribe:** Jingnan Fan

11/26/2012

## 1 Overview.

The first part of this lecture is about the calculation of four important quantities for a weighted undirected graph: size of largest clique, size of largest independent set, minimum coloring number and minimum clique-covering number. We first formulate these problems by integer programming and then give a SDP-relaxation.The second part is the general SDP-relaxation for 0/1 integer programming problems.

## 2 Classic graph problems.

### 2.1 Four quantities: definition and ILP formulation.

#### 2.1.1 Definitions.

For undirected graph $G = (V, E)$ where $V = \{1, .., n\}$ with assigned weights $\boldsymbol{w} = \{w_1, ..., w_n\} \geq 0$, we define four quantities on this graph:

- SIZE OF THE LARGEST CLIQUE $\omega(G, \boldsymbol{w})$: A subset of nodes is called a *clique* or a *complete subgraph* if every two nodes in the subset are connected by an edge. It is known that to find the maximum clique is a NP-hard problem and is hard to approximate (by any constant).
- SIZE OF THE LARGEST INDEPENDENT SET $\alpha(G, \boldsymbol{w})$: A subset of nodes is called an *independent set* or an *stable set* if no two nodes in the subset are connected. We will later see that finding a largest independent set is equivalent to a maximum clique problem, so it is a NP-hard problem as well.

- MINIMUM COLORING NUMBER $\chi(G, \boldsymbol{w})$: In the unweighted case, it is the minimum number of colors required to properly color $G$, i.e, no two connected nodes have a same color. As every nodes in a clique must have different colors, we have $\omega(G) \leq \chi(G)$. In some graphs like a clique or bipartite graph, we can have $\omega(G) = \chi(G)$, but it is not always true. In a pentagon, we have $\omega = 2$ and $\chi = 3$. Remark that a coloring for $G$ is actually a covering for $G$ by independent sets, as a subset colored by a same color must be independent. When the graph is weighted, this last definition remains valid, while "covering" means that each node $i \in V$ is covered at least $w_i$ times.(Independent sets can be used for covering multiple times.) We can also consider the coloring problem for a weighted graph as coloring on a graph obtained by "splitting" each node $i \in V$ to $\lceil w_i \rceil$ nodes. So we also have $\omega(G, \boldsymbol{w}) \leq \chi(G, \boldsymbol{w})$.

- MINIMUM CLIQUE COVERING NUMBER $\rho(G, \boldsymbol{w})$: is the minimum number of node-disjoint cliques that cover every node.

### 2.1.2 Properties.

Recall that the *complement* of $G$, denoted by $\bar{G}$, if the graph composed by the same set of nodes $V$ and the complement edge set $\bar{E} = \{(i,j) | i \neq j, (i,j) \notin E\}$.

**Fact 1** *Every clique in $G$ is an independent set in $\bar{G}$, every independent set in $G$ is a clique in $\bar{G}$. Thus the following equalities hold:*

1. $\omega(G, \boldsymbol{w}) = \alpha(\bar{G}, \boldsymbol{w})$
2. $\alpha(G, \boldsymbol{w}) = \omega(\bar{G}, \boldsymbol{w})$
3. $\chi(G, \boldsymbol{w}) = \rho(\bar{G}, \boldsymbol{w})$
4. $\rho(G, \boldsymbol{w}) = \chi(\bar{G}, \boldsymbol{w})$
5. $\alpha(G, \boldsymbol{w}) \leq \rho(G, \boldsymbol{w})$

### 2.1.3 ILP formulation.

We formulate these problems as 0/1 integer linear programming problem:

- <u>ILP FOR $\omega(G, \boldsymbol{w})$</u>

  Let $\boldsymbol{x}$ be the characteristic vector of clique,then $\omega(G, \boldsymbol{w})$ is the optimal value of

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \boldsymbol{w}^T \boldsymbol{x} \\ \text{subject to} \quad & x_i + x_j \leq 1, \forall (i,j) \notin E \\ & x_i \in \{0,1\}, \forall i \in V. \end{aligned} \tag{1}$$

The polytope

$$\text{CLIQUE}(\mathsf{G}) := \text{conv}\{\mathbf{1}_{\mathsf{K}} | \mathsf{K} \text{ is a clique}\}$$

where $\mathbf{1}_K$ is the characteristic vector of $K$[1], is the convex hull of all the feasible 0/1 solutions, then $\omega(G, \boldsymbol{w})$ is also the optimal value of

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{x} \in \text{CLIQUE}(G). \end{aligned} \tag{2}$$

The most obvious LP-relaxation for (1) is

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{x} \\ \text{subject to} \quad & x_i + x_j \leq 1, \forall (i,j) \notin E \\ & 0 \leq x_i \leq 1, \forall i \in V. \end{aligned} \tag{3}$$

For some graph we can have feasible solutions in LP-relaxation (3) form exactly CLIQUE$(G)$, we have

**Theorem 2** *The polyhedra of feasible solutions of LP-relaxation (noted by LP-CLIQUE$(G)$) is exactly CLIQUE$(G)$ if and only if $G$ is a perfect graph, i.e, $\omega(G, \boldsymbol{w}) = \chi(G, \boldsymbol{w})$ for all $\boldsymbol{w}$.*

**Remark 3** *All bipartite graphs are perfect.*

- <u>ILP FOR $\alpha(G, \boldsymbol{w})$</u>

Let $\boldsymbol{x}$ be the characteristic vector of independent set,then $\alpha(G, \boldsymbol{w})$ is the optimal value of

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{x} \\ \text{subject to} \quad & x_i + x_j \leq 1, \forall (i,j) \in E \\ & x_i \in \{0,1\}, \forall i \in V \end{aligned} \tag{4}$$

The polytope

$$\text{INDP}(G) := \text{conv}\{\mathbf{1}_S | S \text{ is an independent set}\}$$

is the convex hull of all the feasible 0/1 solutions, then $\alpha(G, \boldsymbol{w})$ is also the optimal value of

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{x} \in \text{INDP}(G) \end{aligned} \tag{5}$$

[1] Recall that a subset $K \subseteq \{1, 2, \ldots, n\}$ may be represented by a vector of zeros and ones, where the $i^{\text{th}}$ entry is one iff $i \in K$. This vector is called the *characteristic vector* of $K$.

The standard LP-relaxation for (20) is

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \boldsymbol{w}^T \boldsymbol{x} \\ \text{subject to} \quad & x_i + x_j \leq 1, \forall (i,j) \in E \\ & 0 \leq x_i \leq 1, \forall i \in V. \end{aligned} \tag{6}$$

We note the polyhedra of feasible solutions of LP-relaxation by LP-INDP$(G)$.

- <u>ILP FOR $\chi(G, \boldsymbol{w})$</u>

  Let $\boldsymbol{y} = \{y_S | S \text{ is maximal independent set}\}$, $y_S$ is the number of the maximal independent set $S$ used for covering, then $\chi(G, \boldsymbol{w})$ is the optimal value of

$$\begin{aligned} \underset{\boldsymbol{y}}{\text{minimize}} \quad & \sum_{S:\ S \text{ is a maximal independent set}} y_S \\ \text{subject to} \quad & \sum_{S \ni i:\ S \text{ is a maximal independent set}} y_S \geq w_i, \forall i \in V \\ & y_S \in \mathbb{N}, \forall S \text{ is a maximal independent set.} \end{aligned} \tag{7}$$

  The standard LP-relaxation is

$$\begin{aligned} \underset{\boldsymbol{y}}{\text{minimize}} \quad & \sum_{S:\ S \text{ is a maximal independent set}} y_S \\ \text{subject to} \quad & \sum_{S \ni i:\ S \text{ is a maximal independent set}} y_S \geq w_i, \forall i \in V \\ & y_S \geq 0, \forall S \text{ is a maximal independent set.} \end{aligned} \tag{8}$$

- <u>ILP FOR $\rho(G, \boldsymbol{w})$</u>

  Let $\boldsymbol{y} = \{y_K | K \text{ is maximal clique}\}$, $y_K$ is the number of the maximal clique $K$ used for covering, then $\rho(G, \boldsymbol{w})$ is the optimal value of

$$\begin{aligned} \underset{\boldsymbol{y}}{\text{minimize}} \quad & \sum_{K:\ K \text{ is a maximal clique}} y_K \\ \text{subject to} \quad & \sum_{K \ni i:\ K \text{ is a clique}} y_K \geq w_i, \forall i \in V \\ & y_K \in \mathbb{N}, \forall K \text{ is a maximal clique.} \end{aligned} \tag{9}$$

  The standard LP-relaxation is

$$\begin{aligned} \underset{\boldsymbol{y}}{\text{minimize}} \quad & \sum_{K:\ K \text{ is a maximal clique}} y_K \\ \text{subject to} \quad & \sum_{K \ni i:\ K \text{ is a maximal independent set}} y_K \geq w_i, \forall i \in V \\ & y_K \geq 0, \forall K \text{ is a maximal independent set.} \end{aligned} \tag{10}$$

### 2.1.4 Q-CLIQUE and Q-INDP.

The standard LP-relaxation polyhedron LP-CLIQUE($G$) and LP-INDP($G$) are often large compared to CLIQUE($G$) and INDP($G$).So we look for a better LP-relaxation for maximum clique/independent set problems.
Define

$$\text{Q-CLIQUE}(G) := \{\boldsymbol{x} | \boldsymbol{x} \geq 0, \mathbf{1}_S^T \boldsymbol{x} \leq 1, \forall \text{ maximal independent set } S\}$$

and

$$\text{Q-INDP}(G) := \{\boldsymbol{x} | \boldsymbol{x} \geq 0, \mathbf{1}_K^T \boldsymbol{x} \leq 1, \forall \text{ maximal clique } K\}.$$

We have

$$\text{CLIQUE}(G) \subseteq \text{Q-CLIQUE}(G) \subseteq \text{LP-CLIQUE}(G),$$

and

$$\text{INDP}(G) \subseteq \text{Q-INDP}(G) \subseteq \text{LP-INDP}(G),$$

so if we note

$$\bar{\omega}(G, \boldsymbol{w}) := \begin{array}{ll} \max_{\boldsymbol{x}} & \boldsymbol{w}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in \text{Q-CLIQUE}(G), \end{array} \tag{11}$$

and

$$\bar{\alpha}(G, \boldsymbol{w}) := \begin{array}{ll} \max_{\boldsymbol{x}} & \boldsymbol{w}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in \text{Q-INDP}(G), \end{array} \tag{12}$$

we have

$$\bar{\omega}(G, \boldsymbol{w}) \geq \omega(G, \boldsymbol{w}),$$

and

$$\bar{\alpha}(G, \boldsymbol{w}) \geq \alpha(G, \boldsymbol{w}).$$

On the other hand, we notice that the dual problems of (11) and (12) are in fact the LP-relaxation of the problems (7) and (9). Let's note respectively by $\bar{\chi}(G, \boldsymbol{w})$ $\bar{\rho}(G, \boldsymbol{w})$ the optimal values of the dual problems of (11) and (12), then we have

$$\omega(G, \boldsymbol{w}) \leq \bar{\omega}(G, \boldsymbol{w}) = \bar{\chi}(G, \boldsymbol{w}) \leq \chi(G, \boldsymbol{w}),$$

and

$$\alpha(G, \boldsymbol{w}) \leq \bar{\alpha}(G, \boldsymbol{w}) = \bar{\rho}(G, \boldsymbol{w}) \leq \rho(G, \boldsymbol{w}).$$

## 2.2 SDP relaxation.

Let $A(G)$ be the adjacent matrix of $G$ defined by We introduce the *number of Lovàsz*

$$\begin{aligned}\theta(G) \quad := \quad & \min \quad && \lambda_{[1]}(X + \boldsymbol{w}\boldsymbol{w}^T) \\ & \text{subject to:} && X \in \mathbb{S}_n \\ &&& X_{ij} = 0, \forall (i,j) \in E \\ &&& X_{ii} = 0, \forall i \in V \\ = \quad & \min && z \\ & \text{subject to:} && zI_n - X - \boldsymbol{w}\boldsymbol{w}^T \succeq 0 \\ &&& X \in \mathbb{S}_n \\ &&& X_{ij} = 0, \forall (i,j) \in E \\ &&& X_{ii} = 0, \forall i \in V.\end{aligned} \tag{13}$$

By duality,

$$\begin{aligned}\theta(G) \quad := \quad & \max \quad && (\boldsymbol{w}\boldsymbol{w}^T) \bullet Y \\ & \text{subject to:} && Y \in \mathbb{S}_n \\ &&& Y \succeq 0 \\ &&& \text{trace}(Y) = 1 \\ &&& Y_{ij} = 0, \forall (i,j) \notin E.\end{aligned} \tag{14}$$

The key result of the Lovàsz number is that:

**Theorem 4 (Lovasz Sandwich Theorem)**

$$\omega(G, \boldsymbol{w}) \le \theta(G, \boldsymbol{w}) \le \chi(G, \boldsymbol{w}).$$

**Proof:**

- Let's first prove the theorem for the unweighted case, here $\boldsymbol{w}\boldsymbol{w}^T = J_n$. If $K$ is a maximum clique, let $Y_K = \frac{1}{|K|}\sum_{i\in K, j\in K} \mathbf{E}_{ij}$, then we will have $Y_K$ is a feasible solution of (14) and $J_n \bullet Y_K = |K|$. So $\omega(G) \le \theta(G)$.

- There exists a partition of $V$ into independents set $V = \{S_1, ..., S_{\chi(G)}\}$. For all node $i$, note $S(i)$ the only subset in the partition containing $i$. For any $S \subseteq V$, note $\boldsymbol{e}_S = \sum_{i\in S} \boldsymbol{e}_i$. Then let $X = \chi(G)(I_n - \sum_{i=1}^{\chi(G)} \boldsymbol{e}_{S_i}\boldsymbol{e}_{S_i}^T)$. We show that $(\chi(G), X)$ is a feasible solution to (13). For all node $i$, $X_{ii} = \chi(G)(1 - (\boldsymbol{e}_{S(i)}\boldsymbol{e}_{S(i)}^T)_{ii}) = 0$; for all $(i,j) \notin E$ such that $i \ne j$,

$X_{ij} = 0$. Take any $\mathbf{u} \in \mathbb{R}^n$, we have

$$\begin{aligned}
& \mathbf{u}^T(\chi(G)I_n - X - J_n)\mathbf{u} \\
= \ & \chi(G)\mathbf{u}^T(\sum_{i=1}^{\chi(G)} \boldsymbol{e}_{S_i}\boldsymbol{e}_{S_i}^T)\mathbf{u} - \mathbf{u}^T J_n \mathbf{u} \\
= \ & \chi(G)\sum_{i=1}^{\chi(G)} (\boldsymbol{e}_{S_i}^T\mathbf{u})^2 - (\boldsymbol{e}^T\mathbf{u})^2 \\
\geq \ & 0.
\end{aligned}$$

- For weighted case, the idea is to convert the whole problem to an unweighted problem on "replicated graph".

■

# 3 General SDP-relaxation for 0/1 programming.

## 3.1 The SDP relaxation.

Consider the integer programming

$$\max\{\bar{w}^T\bar{x} | \bar{A}\bar{x} \geq b \text{ and } \bar{x}_i \in \{0,1\}, \forall i\}. \tag{15}$$

One method to solve this ILP problem is solve the LP-relaxation and do the cutting plane, but the LP-relaxation often enlarge too much the feasible polyhedra. Lovàsz and Schrijver proposed the following SDP-relaxation which gives a better approximation. First we homogenize the problem by adding a new variable $x_0$ and imposing $x_0 = 1$, thus the homogenize integer programming problem is

$$\begin{aligned}
\max \quad & \boldsymbol{w}^T\boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} \geq 0 \\
& x_0 = 1 \\
& x_i \in \{0,1\}, \forall i,
\end{aligned} \tag{16}$$

where

$$\boldsymbol{x} = \begin{pmatrix} x_0 \\ \bar{\boldsymbol{x}} \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} 0 \\ \bar{\boldsymbol{w}} \end{pmatrix},$$

and

$$A = [-b \mid \bar{A}].$$

The LP-relaxation if the homogenize problem is

$$\begin{aligned}
\max \quad & \boldsymbol{w}^T\boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} \geq 0 \\
& x_0 = 1 \\
& 0 \leq x_i \leq x_0, \forall i.
\end{aligned} \tag{17}$$

Note $\mathcal{I}(P)$ polyhedron of feasible solutions in the integer problem (16) and note $P$ the polyhedron of feasible solutions in the LP-relaxation (17) without the constraint $x_0 = 1$,

$$P = \left\{ \boldsymbol{x} \,\middle|\, \begin{array}{l} A\boldsymbol{x} \geq 0 \\ 0 \leq x_i \leq x_0, \forall i \end{array} \right\}.$$

For $(J_1, J_2)$ a partition of index set of rows of $A$, let

$$\begin{aligned} P_1 &= \left\{ \boldsymbol{x} \,\middle|\, \begin{array}{l} a_j^T \boldsymbol{x} \geq 0, \forall j \in J_1 \\ 0 \leq x_i \leq x_0, \forall i \end{array} \right\} \\ &= \{\boldsymbol{x} | A_1 \boldsymbol{x} \geq 0\} \end{aligned}$$

and

$$\begin{aligned} P_2 &= \left\{ \boldsymbol{x} \,\middle|\, \begin{array}{l} a_j^T \boldsymbol{x} \geq 0, \forall j \in J_2 \\ 0 \leq x_i \leq x_0, \forall i \end{array} \right\} \\ &= \{\boldsymbol{x} | A_2 \boldsymbol{x} \geq 0\}, \end{aligned}$$

then

$$P = P_1 \bigcap P_2.$$

Define

$$M_+(P_1, P_2) := \{X \in \mathbb{S}^{n+1} | X \succeq 0,\ X\boldsymbol{e}_0 = \mathbf{diag}(X),\ (A_1 \otimes A_2)\mathbf{vect}(X) \geq 0\},$$

and

$$N_+(P_1, P_2) := \{\mathbf{diag}(X) | X \in M_+(P_1, P_2)\}.$$

Obviously the problem

$$\begin{array}{ll} \max & \boldsymbol{w}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in N_+(P_1, P_2) \end{array} \tag{18}$$

is a semi definite programming problem. And the main result of Lovàsz and Schrijver is that

**Theorem 5**

$$\mathcal{I}(P) \subseteq N_+(P_1, P_2) \subseteq P.$$

**Proof:**

- Let $\boldsymbol{x} = \begin{pmatrix} 1 \\ \bar{x} \end{pmatrix}$ be a integer solution. Then $X = \boldsymbol{x}\boldsymbol{x}^T \in M_+(P_1, P_2)$. Obviously $X \succeq 0$ and $X\boldsymbol{e}_0 = \mathbf{diag}(X)$. We have also

$$\begin{aligned} & (A_1 \otimes A_2)\mathbf{vect}(X) \\ = \ & \mathbf{vect}(A_2 X A_1^T) \\ = \ & \mathbf{vect}(A_2 \mathbf{x}\mathbf{x}^T A_1) \\ \geq \ & 0. \end{aligned}$$

- As $\boldsymbol{e}_0$ is a row of $A_2$, we have $A_1\mathbf{diag}(X) = A_1 X \boldsymbol{e}_0 \geq 0$. Thus, $N_+(P_1, P_2) \subseteq P_1$. Similarly $N_+(P_1, P_2) \subseteq P_2$.

∎

**Remark 6** *If a solution $\boldsymbol{x} = X\boldsymbol{e}_0$ found by the SDP-relaxation problem (18) satisfies $x_0 = 1$ and $\boldsymbol{rank}(X) = 1$, then $\boldsymbol{x}$ is 0/1 integral. Indeed, when $\boldsymbol{rank}(X) = 1$ and $X = \boldsymbol{x}\boldsymbol{x}^T$, $X\boldsymbol{e}_0 = \boldsymbol{diag}(X)$ implies that $x_i = x_i^2$ for all $i$, so $x_i \in \{0, 1\}$.*

## 3.2 Application to the maximum independent set problem.

Consider the maximum independent set problem:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{maximize}} & \bar{\boldsymbol{w}}^T \bar{\boldsymbol{x}} \\ \text{subject to} & \bar{x}_i + \bar{x}_j \leq 1, \forall (i,j) \in E \\ & \bar{x}_i \in \{0,1\}, \forall i \end{array} \tag{19}$$

And its LP-relaxation is

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{maximize}} & \boldsymbol{w}^T \boldsymbol{x} \\ \text{subject to} & x_0 - x_i - x_j \geq 0, \forall (i,j) \in E \\ & 0 \leq x_i \leq x_0, \forall i \\ & x_0 = 1. \end{array} \tag{20}$$

Let

$$P_1 = \left\{ \boldsymbol{x} \left| \begin{array}{l} x_0 - x_i - x_j \geq 0, \forall (i,j) \in E \\ 0 \leq x_i \leq x_0, \forall i \end{array} \right. \right\}$$

and

$$P_2 = \left\{ \boldsymbol{x} \left| \; 0 \leq x_i \leq x_0, \forall i \right. \right\}.$$

Then the SDP-relaxation of the maximum independent set problem is

$$\begin{array}{ll} \max & \boldsymbol{w}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in N_+(P_1, P_2). \end{array} \tag{21}$$

and we have

$$\text{INDP}(G) \leq N_+(P_1, P_2) \leq \text{LP-INDP}(G).$$

We also observe that points in $\mathcal{N}_+(P_1, P_2)$ satisfy following properties of INDP$(G)$:

- For any independent set $K$,
$$\mathbf{1}_K \boldsymbol{x} \leq 1.$$
- For every odd cycle $C$ with $2k + 1$ nodes,
$$\mathbf{1}_C \boldsymbol{x} \leq k.$$

Although the SDP-relaxation gives a better approximation, it might not work very well in practice. One reason is that we square the number of variables, and another reason is that we do not have so far an analogy of branch and bound method in SDP.