A REVIEW OF THE S-PROCEDURE

Final Presentation of
Semidefinite and Second Order Cone
Programming Seminar
Rutcor, Fall 2012

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12/10/2012

1 Overview.

The s-procedure is an instrumental tool in control theory and robust optimization analysis. It is also used in semi-definite relaxation of quadratic programming. In this survey, we first define the s-procedure and prove the losslessness theorem. And in the second part, we introduce the SDP-relaxation of quadratic programmming, which is an application of the s-procedure.

2 S-lemma.

The s-procedure is a tool verifying the non-negativity of a quadratic function under quadratic constraints. We begin with a first result dealing the quadratic optimization problem under a single quadratic inequality constraints, proved by Yakubovich (1971):

Theorem 1 (S-lemma) $A, B \in \mathbb{S}_n$, *we assume* $x^T Ax > 0$ *is feasible, then we hsave*

$$\left(x^T Ax \geq 0 \ \Rightarrow x^T Bx \geq 0\right)$$

if and only if

$$\exists \lambda \geq 0, \ B \succeq \lambda A$$

Proof: Sufficiency is clear. Assume $\left(x^T Ax \geq 0 \ \Rightarrow x^T Bx \geq 0\right)$. Let

$$S = \{(x^T Ax, x^T Bx) \| x \in \mathbb{R}^n\},$$

and

$$U = \{(u_1, u_2)|u_1 \geq 0, u_2 < 0\}.$$

S is convex by Dines' theorem (1941), and U is a convex cone, as $S \cap U = \emptyset$, there exists a separating hyperplane $c = (c_1, c_2)$ such that $(c, s) \leq 0,\ \forall s \in S$ and $(c, u) \geq 0,\ \forall u \in U$, because

Lemma 2 *Two nonempty convex subset of $\mathbb{R}^n$ can be properly separated by a hyperplane if and only if their relative interiors are disjoint.*

For any $\alpha > 0$, $(1, -\alpha) \in U$, ie, $c_1 \geq \alpha c_2$, so $c_1 \geq 0$. $\bar{x}$ is such that $\bar{x}^T A \bar{x} > 0$, then

$$c_1 \bar{x}^T A \bar{x} + c_2 \bar{x}^T B \bar{x} \leq 0$$

implies that $c_2 < 0$, as $c_1 c_2 \neq 0$. Therefore,

$$x^T B x \geq -\frac{c_1}{c_2} x^T A x,\ \forall x.$$

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3 S-procedure.

The S-procedure is the generation of S-lemma. Let

$$\begin{array}{rcl} \sigma_k : V & \to & \mathbb{R} \\ \sigma_k(y) & = & y^T Q_k y + 2 s_k^T y + r_k, \end{array}$$

for $k = 0, 1, ..., N$, be quadratic fuctions defined on a linear vector space V and consider the two conditions:

- S_1: For all $y \in V$,

$$(\sigma_k(y) \geq 0, \forall k = 1, ..., N) \Rightarrow \sigma_0(y) \geq 0.$$

- S_2: There exists $\tau_k \geq 0, k = 1, ..., N$ such that

$$\sigma_0(y) - \sum_{k=1}^{N} \tau_k \sigma_k(y) \geq 0,\ \forall y \in V.$$

The S-procedure is the method verifying S_1 using S_2. There are some reasons for which it is generally hard to verify S_1:

1. σ_0 is not a convex function in general.
2. The constraint set

$$\{y|\sigma_k(y) \geq 0, \forall k = 1, ..., N\}$$

   is not convex in general.

However S_2 is easy to verify. In fact,

$$
\begin{aligned}
S_2 \quad & \Leftrightarrow \quad \exists \tau_k \geq 0 \text{ s.t. } \sigma_0(y) - \sum_{k=1}^{N} \tau_k \sigma_k(y) \geq 0, \ \forall y \\
& \Leftrightarrow \quad \exists \tau_k \geq 0 \text{ s.t. } \begin{pmatrix} y \\ 1 \end{pmatrix}^T \begin{pmatrix} Q_0 + \sum_{k=1}^N \tau_k Q_k & s_0 + \sum_{k=1}^N \tau_k s_k \\ s_0^T + \sum_{k=1}^N \tau_k s_k^T & r_0 + \sum_{k=1}^N \tau_k r_k \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix} \geq 0, \ \forall y \\
& \Leftrightarrow \quad \exists \tau_k \geq 0 \text{ s.t. } z^T \begin{pmatrix} Q_0 + \sum_{k=1}^N \tau_k Q_k & s_0 + \sum_{k=1}^N \tau_k s_k \\ s_0^T + \sum_{k=1}^N \tau_k s_k^T & r_0 + \sum_{k=1}^N \tau_k r_k \end{pmatrix} z \geq 0, \ \forall z = \begin{pmatrix} y \\ \delta \end{pmatrix} \text{ s.t. } \delta \neq 0 \\
& \Leftrightarrow \quad \exists \tau_k \geq 0 \text{ s.t. } z^T \begin{pmatrix} Q_0 + \sum_{k=1}^N \tau_k Q_k & s_0 + \sum_{k=1}^N \tau_k s_k \\ s_0^T + \sum_{k=1}^N \tau_k s_k^T & r_0 + \sum_{k=1}^N \tau_k r_k \end{pmatrix} z \geq 0, \ \forall z \\
& \Leftrightarrow \quad \exists \tau_k \geq 0 \text{ s.t. } \begin{pmatrix} Q_0 & s_0 \\ s_0^T & r_0 \end{pmatrix} + \sum_{k=1}^{N} \tau_k \begin{pmatrix} Q_k & s_k \\ s_k^T & r_k \end{pmatrix} \succeq 0.
\end{aligned}
$$

Obviously, $S_2 \Rightarrow S_1$. In general $S_1 \nRightarrow S_2$, however there are some special cases when $S_1 \Leftrightarrow S_2$. As in Theorem 1, we need some regularity on functions σ_k:

Definition 3 *The constraint* $\sigma_k(y) \geq 0, \forall k = 1, ..., N$ *is said to be regular if there exists* $y* \in \nu$ *such that* $\sigma_k(y*) > 0, \forall k = 1, ..., N$.

Then we have the following theorem:

Theorem 4 (Yakubovich) *Let*

$$\mathcal{K} = \{(\sigma_0(y), \sigma_1(y), ..., \sigma_N(y)) | y \in V\},$$

$$\mathcal{N} = \{(n_0, n_1, ..., n_N) | n_0 < 0, n_k > 0, k = 1, ..., N\}.$$

If $(\mathcal{K} \bigcap \mathcal{N} = \emptyset) \Rightarrow (conv(\mathcal{K}) \bigcap \mathcal{N} = \emptyset)$*, then under the regularity on constraint, the S-procedure is lossless, i.e.,* $S_1 \Leftrightarrow S_2$.

Proof: Analogy of the proof of Theorem 1. ■

Remark 5 *If* $\mathcal{K}$ *is convex, the S-procedure is lossless.*

Corollary 6 (S-procedure Losslessness in linear case) *In the linear case, where* $\sigma_k(y) = s_k^T y + r_k$*, under the regularity of constraint, the S-procedure is lossless.*

Corollary 7 (S-procedure Losslessness in the case of one quadratic constraint)

1. *Assume* σ_1 *is regular, then the following conditions*

   - S_1*:* $\forall y \in V, \ (\sigma_1(y) \geq 0) \Rightarrow (\sigma_0(y) \geq 0)$

- S_2: *There exists* $\tau \geq 0$ *such that*

$$\begin{pmatrix} Q_0 & s_0 \\ s_0^T & r_0 \end{pmatrix} - \tau \begin{pmatrix} Q_1 & s_1 \\ s_1^T & r_1 \end{pmatrix} \succeq 0$$

are equivalent.

2. *Assume* $\sigma_1(y) = y^T Q_1 y$ *is regular, then the following conditions*

   - S_1: $\forall y \in V,\ (y^T Q_1 y \geq 0,\ y \neq 0) \Rightarrow (y^T Q_0 y > 0)$
   - S_2: *There exists* $\tau \geq 0$ *such that* $Q_0 - \tau Q_1 \succ 0$

   are equivalent.

Proof:

1. $\mathcal{K}$ is convex.
2. Let

$$\epsilon = \min_{\{y | y^T y = 1,\ y^T Q_1 y \geq 0\}} y^T Q_0 y$$

then $y^T(Q_0 - \epsilon I_{\dim(V)})y \geq 0$ for all y such that $y^T Q_1 y \geq 0$. Thus $Q_0 - \epsilon I_{\dim(V)} - \tau Q_1 \succeq 0$.

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4 SDP Relaxation of Quadratically Constrainted Quadratic Programming.

We apply the S-procedure to a genenral case of quadratic programming problems

$$\begin{aligned} \min \quad & y^T Q_0 y + 2 s_0^T y + r_0 \\ \text{s.t.} \quad & y^T Q_k y + 2 s_k^T y + r_k \geq 0,\ k = 1, .., N. \end{aligned} \tag{1}$$

This class of optimization problem includes also integer valued constraints, since $x_i^2 - x_i = 0$ is a valid constraint. Thus the problem is generally NP-hard unless the quadratic functions are convex. Lower bound on the optimization problem (1) can be obtained using the following semidefinite relaxation

$$\begin{aligned} \min \quad & \mathbf{Tr}(Q_0 Y) + 2 s_0^T y + r_0 \\ \text{s.t.} \quad & \mathbf{Tr}(Q_k Y) + 2 s_k^T y + r_k \geq 0,\ k = 1, .., N, \\ & \begin{pmatrix} Y & y \\ y^T & 1 \end{pmatrix} \succeq 0. \end{aligned} \tag{2}$$

To see that (2) is a lower bound of (1) we notice that $\begin{pmatrix} Y & y \\ y^T & 1 \end{pmatrix} \succeq 0$ is equivalent to $Y \succeq yy^T$ by Schur complement formula. Since $\mathbf{Tr}(Q_k y y^T) = y^T Q_k y$ we see that (2) must have a lower optimal value than (1). And for the case of single constraint, we have

Proposition 8 *Assume that* $y^T Q_1 y + 2s_1^T y + r_1 \geq 0$ *is a regular constraint and that the two optimization problems are strictly feasible, then the following optimization problems have the same objective values*

1. *Quadratic programming*

$$\begin{array}{ll} \min & y^T Q_0 y + 2s_0^T y + r_0 \\ s.t. & y^T Q_1 y + 2s_1^T y + r_1 \geq 0. \end{array} \tag{3}$$

2. *Eigen value for LMI*

$$\begin{array}{ll} \max & \gamma \\ s.t. & \begin{pmatrix} Q_0 - \lambda Q_1 & s_0 - \lambda s_1 \\ s_0^T - \lambda s_1^T & r_0 - \lambda r_1 - \gamma \end{pmatrix} \succeq 0 \end{array} \tag{4}$$

3. *Semidefinite relaxation of (3) adn dual of (4)*

$$\begin{array}{ll} \min & \boldsymbol{Tr}(Q_0 Y) + 2s_0^T y + r_0 \\ s.t. & \boldsymbol{Tr}(Q_1 Y) + 2s_k^T y + r_1 \geq 0 \\ & \begin{pmatrix} Y & y \\ y^T & 1 \end{pmatrix} \succeq 0. \end{array} \tag{5}$$

Proof: The direct consequence of Corollary 7. ∎

References

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[3] Derinkuyu, K., Pinar, M., *On the S-Procedure and Some Variants*, 2004.