Applications of Robust Optimization in Signal Processing: Beamforming and Power Control
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1 Overview

In this presentation, we study the applications of robust optimization in signal processing: robust beamforming in array and power control in wireless networks. First, the background of beamforming and the traditional diagonal loading (DL) method are introduced. Then, the robust beamforming designs are proposed by considering the worst-case distortionless and probabilistic constraints of the steering vector, respectively. It is shown that the robust beamforming design can be formulated as a seconder order cone programming (SOCP) problem or a semidefinite programming (SDP) problem, and then can be solved efficiently. In the last, the robust power control problem in cognitive radio networks can be formulated as a SOCP problem by considering the worst-case uncertainties in the wireless channel gains.

2 Background of Beamforming

Beamforming has been widely used in wireless communications and radar systems. In practice, the performance of beamforming may undergo degradation due to the mismatch in the steering vectors. The mismatch could be a result of imperfect array calibration, look direction and signal pointing errors. Array signal processing is very sensitive to these mismatches and robust design is required [1-5].

Consider a narrow-band beamformer:

$$y(k) = \boldsymbol{w}^H \boldsymbol{x}(k),$$

where k is the time index, $\boldsymbol{x}(k) = [x_1(k), ..., x_M(k)]^T \in \mathbb{C}^{M\times 1}$ is the complex output of the array. $\boldsymbol{w} = [w_1, ..., w_M]^T \in \mathbb{C}^{M\times 1}$ is the complex beamforming

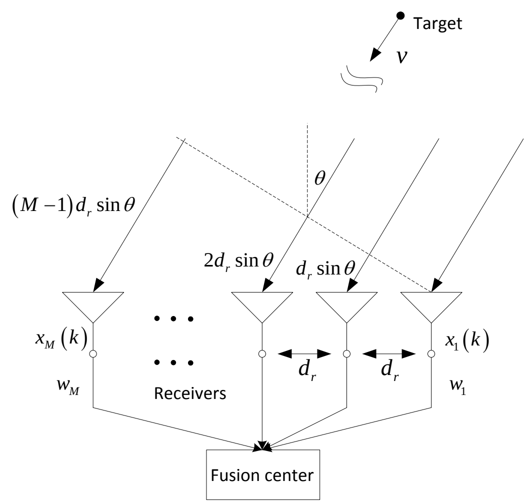

Figure 1: Uniform Linear Array Model.

weights. M is the number of sensors in the array (see Figure 1). The observation vector is

$$\mathbf{x}(k) = s(k)\,\mathbf{a} + \mathbf{i}(k) + \mathbf{n}(k), \tag{1}$$

where $s(k)$ is the signal waveform; $\mathbf{i}(k)$ and $\mathbf{n}(k)$ are the interference and noise, respectively. Here, $\mathbf{a}$ the steering vector and is defined as

$$\mathbf{a} = \left[1, e^{j\frac{2\pi}{\lambda} d_r \sin\theta}, ..., e^{j\frac{2\pi}{\lambda}(M-1) d_r \sin\theta}\right]^T.$$

The signal-to-interference-plus-noise ratio (SINR) is given as

$$\mathrm{SINR} = \frac{\sigma_s^2 \left|\boldsymbol{w}^H \mathbf{a}\right|^2}{\boldsymbol{w}^H \mathbf{R}_{i+n} \boldsymbol{w}}, \tag{2}$$

where $\mathbf{R}_{i+n} = E\left\{(\mathbf{i}(k) + \mathbf{n}(k))(\mathbf{i}(k) + \mathbf{n}(k))^H\right\}$ is the interference-plus-noise covariance matrix, and σ_s^2 is the signal power. The beamforming vector $\boldsymbol{w}$ is chosen to maximize the SINR. The beamforming vector design problem can be formulated as

$$\min_{\boldsymbol{w}} \; \boldsymbol{w}^H \mathbf{R}_{i+n} \boldsymbol{w} \quad \text{s.t.} \quad \boldsymbol{w}^H \mathbf{a} = 1. \tag{3}$$

It is well known that the solution of problem (3) is

$$\boldsymbol{w}_{opt} = \alpha \mathbf{R}_{i+n}^{-1} \mathbf{a}, \tag{4}$$

where $\alpha = \left(\mathbf{a}^H \mathbf{R}_{i+n}^{-1} \mathbf{a}\right)^{-1}$. The solution is referred as the minimum variance distortionless response (MVDR) beamformer.

In practice, it is difficult to know the interference-plus-noise covariance matrix $\mathbf{R}_{i+n}$. So, the sample covariance matrix

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}(n) \mathbf{x}^H(n) \tag{5}$$

is used instead of $\mathbf{R}_{i+n}$. And the sample version beamforming design problem is written as

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^H \hat{\mathbf{R}} \boldsymbol{w} \quad \text{s.t.} \quad \boldsymbol{w}^H \mathbf{a} = 1. \tag{6}$$

The solution of (6) is referred as the sample matrix inversion (SMI)-based minimum variance (MV) beamforming and is written as

$$\boldsymbol{w}_{SMI} = \alpha \hat{\mathbf{R}}^{-1} \mathbf{a}, \tag{7}$$

where $\alpha = \left(\mathbf{a}^H \hat{\mathbf{R}}^{-1} \mathbf{a}\right)^{-1}$. For the signal free case, the optimal SINR is

$$\mathrm{SINR}_{opt} = \sigma_s^2 \mathbf{a}^H \mathbf{R}_{i+n}^{-1} \mathbf{a}. \tag{8}$$

The fact that sample covariance matrix is used instead of the interference-plus noise covariance matrix would dramatically affect the beamforming performance when there is mismatch in the steering vector. The reason is as follows.

In the mismatch case, the actual steering vector $\tilde{\mathbf{a}}$ that characterizes the spatial signature of the signal is

$$\tilde{\mathbf{a}} = \mathbf{a} + \Delta \neq \mathbf{a}, \tag{9}$$

where Δ is unknown complex vector that describes the mismatch of steering vector. Therefore, the SMI beamformer would view the desired signal components in the samples as interference and try to suppress them. This suppression is called "signal cancellation" phenomenon which will degrade the SINR. A little mismatch in the steering vector would lead to severe degradation in the SINR.

One popular method to deal with the mismatch issue is diagonal loading. The basic idea of this method is to replace the sample covariance matrix $\hat{\mathbf{R}}$ with the diagonal loading covariance matrix

$$\hat{\mathbf{R}}_{dl} = \xi \mathbf{I} + \hat{\mathbf{R}} \tag{10}$$

in the SMI expression (7). Here, ξ is a diagonal loading factor and $\mathbf{I}$ is the identity matrix. The diagonal loading method increase the variance of the artificial white noise by the amount of ξ. This modification would force the beamformer to put more effort to minimize the white noise rather than the interference. As we know, when there is mismatch in the steering vector, the desired signal components are suppressed as interference. As the beamformer puts more effort in suppressing the white noise, the signal cancellation is reduced. But if ξ is too large, the beamformer fails to suppress strong interference since most effort is used to suppress the white noise. How to choose a proper diagonal loading factor ξ is not clear.

3 Uncertainty in Steering Vector: SOCP Formulation

3.1 Problem Formulation

In [1], the norm of the steering vector distortion Δ is assumed to be bounded

$$\|\Delta\| \leq \varepsilon, \tag{11}$$

where $\varepsilon > 0$ is some constant. The steering vector belongs to the set

$$\mathcal{A}(\varepsilon) = \{\mathbf{c} \,|\, \mathbf{c} = \mathbf{a} + \boldsymbol{e}, \|\boldsymbol{e}\| \leq \varepsilon\}. \tag{12}$$

In [3], the steering vector is assumed to be lie in an n-dimensional ellipsoid

$$\zeta = \{\mathbf{P}\mu + \mathbf{c} \,|\, \|\mu\| \leq 1\}. \tag{13}$$

The set ζ describes an ellipsoid whose center is $\mathbf{c}$ and whose principal semiaxes are the unit-norm left singular vectors of $\mathbf{P}$ scaled by the corresponding singular values.

For all vectors belonging to $\mathcal{A}(\varepsilon)$, the absolute value of the array response should not be smaller than one

$$\left|\boldsymbol{w}^{\mathrm{H}}\mathbf{c}\right| \geq 1 \quad \text{for all} \quad \mathbf{c} \in \mathcal{A}(\varepsilon). \tag{14}$$

With (14), the robust beamforming can be formulated as

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^{\mathrm{H}}\hat{\mathbf{R}}\boldsymbol{w} \quad \text{s.t.} \quad \left|\boldsymbol{w}^{\mathrm{H}}\mathbf{c}\right| \geq 1 \quad \text{for all} \quad \mathbf{c} \in \mathcal{A}(\varepsilon). \tag{15}$$

For each $\mathbf{c} \in \mathcal{A}(\varepsilon)$, the constraint $\left|\boldsymbol{w}^{\mathrm{H}}\mathbf{c}\right| \geq 1$ is nonconvex. Since there is an infinite number of vectors $\mathbf{c}$ in $\mathcal{A}(\varepsilon)$, there is an infinite number of constraints. Therefore, (15) is a semi-infinite nonconvex quadratic problem. We consider the worst-case constraint in (15) and rewrite the problem as

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^{\mathrm{H}}\hat{\mathbf{R}}\boldsymbol{w} \qquad \text{s.t.} \quad \min_{\mathbf{c}\in\mathcal{A}(\varepsilon)} \left|\boldsymbol{w}^{\mathrm{H}}\mathbf{c}\right| \geq 1. \tag{16}$$

We rewrite the constraint in (16) as

$$\min_{\boldsymbol{e}\in\mathcal{D}(\varepsilon)} \left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a} + \boldsymbol{w}^{\mathrm{H}}\boldsymbol{e}\right| \geq 1, \tag{17}$$

where $\mathcal{D}(\varepsilon) = \{\boldsymbol{e} \,|\, \|\boldsymbol{e}\| \leq \varepsilon\}$. Applying the triangle and Cauchy-Schwarz inequalities, we have

$$\left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a} + \boldsymbol{w}^{\mathrm{H}}\boldsymbol{e}\right| \geq \left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right| - \left|\boldsymbol{w}^{\mathrm{H}}\boldsymbol{e}\right| \geq \left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right| - \varepsilon\|\boldsymbol{w}\|. \tag{18}$$

The equation holds with $\left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right| > \varepsilon\|\boldsymbol{w}\|$ and $\boldsymbol{e} = -\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\varepsilon e^{j\phi}$, where $\phi = angle\left\{\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right\}$. So, we have the worst-case constraint as

$$\min_{\mathbf{c}\in\mathrm{D}(\varepsilon)} \left|\boldsymbol{w}^{\mathrm{H}}\mathbf{c}\right| = \left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right| - \varepsilon\|\boldsymbol{w}\|. \tag{19}$$

The semi-infinite programming (SIP) problem (16) can be formulated as

$$\min_{\boldsymbol{w}} \boldsymbol{w}^{\mathrm{H}}\hat{\mathbf{R}}\boldsymbol{w} \quad \text{s.t. } \left|\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}\right| - \varepsilon \left\|\boldsymbol{w}\right\| \geq 1. \tag{20}$$

In problem (20), the constraint is nonconvex due to the absolute value operation. An important observation is that the cost function in (20) is unchanged when there is an arbitrary phase rotation in $\boldsymbol{w}$. So if $\boldsymbol{w}_0$ is the optimal solution to (20), we can rotate rotate the phase of $\boldsymbol{w}_0$ without affecting the objective function, so that $\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}$ is real. Thus, we can choose $\boldsymbol{w}$ such that

$$\mathrm{Re}\left\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}\right\} \geq 0, \quad \mathrm{Im}\left\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}\right\} = 0. \tag{21}$$

So the robust beamforming problem can be rewritten as

$$\min_{\boldsymbol{w}} \boldsymbol{w}^{\mathrm{H}}\hat{\mathbf{R}}\boldsymbol{w} \quad \text{s.t. } \boldsymbol{w}^{\mathrm{H}}\boldsymbol{a} \geq \varepsilon \left\|\boldsymbol{w}\right\| + 1, \ \mathrm{Im}\left\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}\right\} = 0. \tag{22}$$

3.2 Relationship with SMI Beamformer

It is easy to see the constraint in (16) is equivalent to

$$\min_{\boldsymbol{c}\in A(\varepsilon)} \left|\boldsymbol{w}^{\mathrm{H}}\boldsymbol{c}\right| = 1. \tag{23}$$

So the constraint in (22) is equivalent to equality constraint $\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a} = \varepsilon \left\|\boldsymbol{w}\right\| + 1$. We can rewrite the problem

$$\min_{\boldsymbol{w}} \boldsymbol{w}^{\mathrm{H}}\hat{\mathbf{R}}\boldsymbol{w} \quad \text{s.t. } \left|\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a} - 1\right|^2 = \varepsilon^2 \boldsymbol{w}^{\mathrm{H}}\boldsymbol{w}. \tag{24}$$

The Lagrange function is

$$\mathrm{H}\left(\boldsymbol{w}, \lambda\right) = \boldsymbol{w}^{\mathrm{H}}\hat{\mathbf{R}}\boldsymbol{w} + \lambda\left(\varepsilon^2\boldsymbol{w}^{\mathrm{H}}\boldsymbol{w} - \boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}\boldsymbol{a}^{\mathrm{H}}\boldsymbol{w} + \boldsymbol{w}^{\mathrm{H}}\boldsymbol{a} + \boldsymbol{a}^{\mathrm{H}}\boldsymbol{w} - 1\right), \tag{25}$$

where λ is a Lagrange multipler. Taking the gradient of $\mathrm{H}\left(\boldsymbol{w}, \lambda\right)$ and equating it to zero yields

$$\boldsymbol{w} = -\lambda\left(\hat{\mathbf{R}} + \lambda\varepsilon^2\mathbf{I} - \lambda\boldsymbol{a}\boldsymbol{a}^{\mathrm{H}}\right)^{-1}\boldsymbol{a}. \tag{26}$$

With matrix inversion lemma, we get

$$\boldsymbol{w} = \frac{\lambda}{\lambda\boldsymbol{a}^{\mathrm{H}}\left(\hat{\mathbf{R}} + \lambda\varepsilon^2\mathbf{I}\right)^{-1}\boldsymbol{a} - 1}\left(\hat{\mathbf{R}} + \lambda\varepsilon^2\mathbf{I}\right)^{-1}\boldsymbol{a}. \tag{27}$$

This indicates that the proposed robust beamforming belongs to diagonal loading methods.

3.3 SOCP Problem Implementation

First, we convert the quadratic objective function in (22) to a linear one. Assume the Cholesky factorization of $\hat{\mathbf{R}}$ as $\hat{\mathbf{R}} = \mathbf{U}^{\mathrm{H}}\mathbf{U}$. Then, $\boldsymbol{w}^{\mathrm{H}}\hat{\mathbf{R}}\boldsymbol{w} = \|\mathbf{U}\boldsymbol{w}\|^2$. Therefore, the problem (22) can be converted into the following problem

$$\min_{\tau,\boldsymbol{w}} \tau \quad \text{s.t.} \quad \|\mathbf{U}\boldsymbol{w}\| \leq \tau, \ \ \varepsilon\|\boldsymbol{w}\| \leq \boldsymbol{w}^{\mathrm{H}}\mathbf{a} - 1, \ \ \mathrm{Im}\left\{\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right\} = 0. \tag{28}$$

By introducing

$$\begin{aligned}
\breve{\boldsymbol{w}} &= \left[\mathrm{Re}\{\boldsymbol{w}\}^{\mathrm{T}}, \mathrm{Im}\{\boldsymbol{w}\}^{\mathrm{T}}\right]^{\mathrm{T}} \\
\breve{\mathbf{a}} &= \left[\mathrm{Re}\{\mathbf{a}\}^{\mathrm{T}}, \mathrm{Im}\{\mathbf{a}\}^{\mathrm{T}}\right]^{\mathrm{T}} \\
\bar{\mathbf{a}} &= \left[\mathrm{Im}\{\mathbf{a}\}^{\mathrm{T}}, -\mathrm{Re}\{\mathbf{a}\}^{\mathrm{T}}\right]^{\mathrm{T}} \\
\breve{\mathbf{U}} &= \begin{bmatrix} \mathrm{Re}\{\mathbf{U}\} & -\mathrm{Im}\{\mathbf{U}\} \\ \mathrm{Im}\{\mathbf{U}\} & \mathrm{Re}\{\mathbf{U}\} \end{bmatrix},
\end{aligned}$$

the problem (28) can be written as

$$\min_{\tau,\breve{\boldsymbol{w}}} \tau \quad \text{s.t.} \quad \left\|\breve{\mathbf{U}}\breve{\boldsymbol{w}}\right\| \leq \tau, \varepsilon\left\|\breve{\boldsymbol{w}}\right\| \leq \breve{\boldsymbol{w}}^{\mathrm{T}}\breve{\mathbf{a}} - 1, \ \ \breve{\boldsymbol{w}}^{\mathrm{T}}\bar{\mathbf{a}} = 0. \tag{29}$$

The problem (29) is a SOCP problem which can be efficiently solved.

4 Uncertainties in Steering Vector and Covariance Matrix: SDP Formulation

In this section, we consider the scenario that there are uncertainties in both the steering vector and covariance matrix and the uncertainties are known to belong to a convex set $\mathcal{U}$ of $\mathbb{C}^M \times \mathbb{H}^M_{++}$. Here, $\mathbb{H}^M_{++}$ denotes the set of all Hermitian positive definite matrices of size $M \times M$. The convexity means that, for any two pairs $(\mathbf{a}_1, \Sigma_1)$ and $(\mathbf{a}_2, \Sigma_2)$ in $\mathcal{U}$,

$$\eta(\mathbf{a}_1, \Sigma_1) + (1-\eta)(\mathbf{a}_2, \Sigma_2) \in \mathcal{U}, \ \ \forall\eta \in (0,1). \tag{30}$$

Here, we use Σ to denote the covariance. Given weight vector $\boldsymbol{w}$, the worst-case SINR analysis problem of a finding a steering vector and a covariance matrix that minimize the SINR is

$$\min \ \ \mathsf{SINR}(\boldsymbol{w}, \mathbf{a}, \Sigma) \qquad \text{s.t.} \quad (\mathbf{a}, \Sigma) \in \mathcal{U}, \tag{31}$$

where the variables are $\mathbf{a}$ and Σ. The optimal value of the this problem is the worst-case SINR, denoted as $\mathsf{SINR}_{wc}(\boldsymbol{w})$,

$$\mathsf{SINR}_{wc}(\boldsymbol{w}) = \inf_{(\mathbf{a},\Sigma)\in\mathcal{U}} \mathsf{SINR}(\boldsymbol{w}, \mathbf{a}, \Sigma).$$

The problem (31) is a convex optimization problem as the $\mathsf{SINR}\,(\boldsymbol{w}, \boldsymbol{a}, \Sigma)$ is a convex function of $\boldsymbol{a}$ and Σ for a given weight vector $\boldsymbol{w}$.

The robust beamforming considering the worst-case SINR maximization is formulated as

$$\max \quad \mathsf{SINR}_{wc}\,(\boldsymbol{w}) \qquad \text{s.t.} \quad \boldsymbol{w} \neq 0 \tag{32}$$

with variable $\boldsymbol{w}$. The problem (32) is nonconvex. In [2], a proposition is given as below

Proposition: If $(\boldsymbol{a}^*, \Sigma^*)$ solve the problem

$$\min \sup_{\boldsymbol{w} \neq 0} \mathsf{SINR}\,(\boldsymbol{w}, \boldsymbol{a}, \Sigma) \qquad \text{s.t.} \quad (\boldsymbol{a}, \Sigma) \in \mathcal{U} \tag{33}$$

with variables $\boldsymbol{a} \in \mathbb{C}^M$ and $\Sigma \in \mathbb{H}^M$. Then $(\boldsymbol{w}^*, \boldsymbol{a}^*, \Sigma^*)$ with $\boldsymbol{w}^* = \Sigma^{*-1}\boldsymbol{a}^*$ satisfies the saddle-point property

$$\begin{aligned} \mathsf{SINR}\,(\boldsymbol{w}, \boldsymbol{a}^*, \Sigma^*) \leq \mathsf{SINR}\,(\boldsymbol{w}^*, \boldsymbol{a}^*, \Sigma^*) \leq \mathsf{SINR}\,(\boldsymbol{w}^*, \boldsymbol{a}, \Sigma)\,, \\ \forall \boldsymbol{w} \in \mathbb{C}^M \backslash \{0\}, \forall\,(\boldsymbol{a}, \Sigma) \in \mathcal{U}. \end{aligned} \tag{34}$$

From the saddle-point property, we can show that

$$\begin{aligned} \mathsf{SINR}\,(\boldsymbol{w}^*, \boldsymbol{a}^*, \Sigma^*) &= \inf_{(\boldsymbol{a},\Sigma)\in\mathcal{U}} \mathsf{SINR}\,(\boldsymbol{w}^*, \boldsymbol{a}, \Sigma) = \sup_{\boldsymbol{w}\neq 0} \mathsf{SINR}\,(\boldsymbol{w}, \boldsymbol{a}^*, \Sigma^*) \\ &= \inf_{(\boldsymbol{a},\Sigma)\in\mathcal{U}} \sup_{\boldsymbol{w}\neq 0} \mathsf{SINR}\,(\boldsymbol{w}, \boldsymbol{a}, \Sigma) = \sup_{\boldsymbol{w}\neq 0} \inf_{(\boldsymbol{a},\Sigma)\in\mathcal{U}} \mathsf{SINR}\,(\boldsymbol{w}, \boldsymbol{a}, \Sigma)\,. \end{aligned}$$

We conclude that $\boldsymbol{w}^* = \Sigma^{-1}\boldsymbol{a}$ solves the worst-case SINR maximization problem (32).

According to (8), the SINR maximization problem (33) is equivalent to

$$\min \ \boldsymbol{a}^H \Sigma^{-1} \boldsymbol{a} \qquad \text{s.t.} \ \ (\boldsymbol{a}, \Sigma) \in \mathcal{U}. \tag{35}$$

By expanding the real and imaginary parts of $\boldsymbol{a}$ and Σ, we have

$$\boldsymbol{a}^H \Sigma^{-1} \boldsymbol{a} = \boldsymbol{z}^T \mathbf{R}^{-1} \boldsymbol{z}, \tag{36}$$

where

$$\begin{aligned} \boldsymbol{z} &= \begin{bmatrix} \mathrm{Re}\{\boldsymbol{a}\} \\ \mathrm{Im}\{\boldsymbol{a}\} \end{bmatrix} \in \mathbb{R}^{2M}, \\ \mathbf{R} &= \begin{bmatrix} \mathrm{Re}\{\Sigma\} & -\mathrm{Im}\{\Sigma\} \\ \mathrm{Im}\{\Sigma\} & \mathrm{Re}\{\Sigma\} \end{bmatrix} \in \mathbb{R}^{2M\times 2M}. \end{aligned} \tag{37}$$

Therefore, problem (35) is equivalent to

$$\min \ \boldsymbol{z}^T \mathbf{R}^{-1} \boldsymbol{z} \qquad \text{s.t.} \ \ (\boldsymbol{z}, \mathbf{R}) \in \mathcal{V}, \tag{38}$$

where, $\mathcal{V}$ is a subset of $\mathbb{R}^{2M} \times \mathbb{S}^{2M}_{++}$

$$\mathcal{V} = \left\{ \left(\begin{bmatrix} \mathrm{Re}\{\boldsymbol{a}\} \\ \mathrm{Im}\{\boldsymbol{a}\} \end{bmatrix}, \begin{bmatrix} \mathrm{Re}\{\Sigma\} & -\mathrm{Im}\{\Sigma\} \\ \mathrm{Im}\{\Sigma\} & \mathrm{Re}\{\Sigma\} \end{bmatrix} \right) |(\boldsymbol{a}, \Sigma) \in \mathcal{U} \right\}. \tag{39}$$

Here, $\mathbb{S}_{++}^{2M}$ denote the set of all $2M \times 2M$ symmetric positive definite matrices. The objective function in (38) is a matrix fractional function and is convex. $\mathcal{V}$ is convex since $\mathcal{U}$ is. Using the Schur complement, the convex problem (38) can be re-written as the following SDP problem

$$\min \ t \quad \text{s.t.} \begin{bmatrix} \mathbf{R} & \boldsymbol{z} \\ \boldsymbol{z}^T & t \end{bmatrix} \succeq= 0, \quad (\boldsymbol{z}, \mathbf{R}) \in \mathcal{V}. \tag{40}$$

5 Uncertainty in Steering Vector: Probabilistic Constraints

The idea of probability constraint in the beamforming is to model the steering vector statistically rather than in a deterministic way. And the beamforming maintains the distortionless response in a high probability rather than for all mismatch conditions in the uncertainty set.

The probability-constrained robust beamforming problem can be written as

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^H \hat{\mathbf{R}} \boldsymbol{w} \quad \text{s.t.} \quad \Pr\left\{ \left| \boldsymbol{w}^H \tilde{\mathbf{a}} \right| \geq 1 \right\} \geq p, \tag{41}$$

where p is a certain probability value. According to (9), with triangle inequality,

$$\left| \boldsymbol{w}^H (\mathbf{a} + \Delta) \right| \geq \left| \boldsymbol{w}^H \mathbf{a} \right| - \left| \boldsymbol{w}^H \Delta \right|. \tag{42}$$

Therefore, problem (41) can be written as

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^H \hat{\mathbf{R}} \boldsymbol{w} \quad \text{s.t.} \quad \Pr\left\{ \left| \boldsymbol{w}^H \Delta \right| \leq \left| \boldsymbol{w}^H \mathbf{a} \right| - 1 \right\} \geq p. \tag{43}$$

5.1 Gaussian Mismatch Case

Let Δ follow a complex circularly symmetric Gaussian distribution with zero mean and covariance matrix $\mathbf{C}_\Delta$, i.e., $\Delta \sim \mathcal{N}_\mathcal{C}(0_M, \mathbf{C}_\Delta)$. The covariance matrix $\mathbf{C}_\Delta$ captures the second order statistics of the uncertainties in the steering vector. It is easy to show that the random variable $\boldsymbol{w}^H(\mathbf{a} + \Delta)$ has the following complex Gaussian distribution: $\boldsymbol{w}^H(\mathbf{a} + \Delta) \sim \mathcal{N}_\mathcal{C}\left(\boldsymbol{w}^H \mathbf{a}, \left\| \mathbf{C}_\Delta^{1/2} \boldsymbol{w} \right\|^2\right)$. As $\boldsymbol{w}^H \Delta$ is circular zero mean complex Gaussian, its real and imaginary parts are real independent identically distributed Gaussian

$$\mathrm{Re}\left\{ \boldsymbol{w}^H \Delta \right\} \sim \mathcal{N}_\mathcal{R}\left(0, \left\| \mathbf{C}_\Delta^{1/2} \boldsymbol{w} \right\|^2 \Big/ 2\right)$$
$$\mathrm{Im}\left\{ \boldsymbol{w}^H \Delta \right\} \sim \mathcal{N}_\mathcal{R}\left(0, \left\| \mathbf{C}_\Delta^{1/2} \boldsymbol{w} \right\|^2 \Big/ 2\right).$$

If x and y are two real i.i.d. zero mean Gaussian random variables with variance as δ^2, then $z = \sqrt{x^2 + y^2}$ is Rayleigh distributed with cumulative density function (CDF) as

$$F(z) = 1 - e^{-z^2/2\delta^2}.$$

Thus, the constraint in (43) can be written as

$$\Pr\left\{\left|\boldsymbol{w}^{\mathrm{H}}\Delta\right| \leq \left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right| - 1\right\} = 1 - \exp\left(-\frac{\left(\left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right| - 1\right)^2}{\left\|\mathbf{C}_{\Delta}^{1/2}\boldsymbol{w}\right\|^2}\right) \geq p. \tag{44}$$

The inequality in the right hand side of (44) can be equivalently written as

$$\left\|\mathbf{C}_{\Delta}^{1/2}\boldsymbol{w}\right\| \leq \frac{1}{\sqrt{-\ln(1-p)}}\left(\left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right| - 1\right). \tag{45}$$

Since the cost function in (43) is unchanged when $\boldsymbol{w}$ undergoes an arbitrary phase rotation, we can choose $\boldsymbol{w}$ such that $\mathrm{Re}\left\{\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right\} \geq 0$, $\mathrm{Im}\left\{\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right\} = 0$. The robust beamforming problem can be approximated by the following SOCP problem

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^{\mathrm{H}}\hat{\mathbf{R}}\boldsymbol{w} \ \text{s.t.} \ \left\|\mathbf{C}_{\Delta}^{1/2}\boldsymbol{w}\right\| \leq \frac{1}{\sqrt{-\ln(1-p)}}\left(\boldsymbol{w}^{\mathrm{H}}\mathbf{a} - 1\right). \tag{46}$$

5.2 Worst Mismatch Distribution Case

Here, the case that the distribution of the steering vector mismatch Δ is unknown is considered. Assume that Δ is zero mean with given covariance. The assumptions $\mathrm{Re}\left\{\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right\} \geq 0$, $\mathrm{Im}\left\{\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right\} = 0$ hold. Thus, the constraint in (43) can be written as

$$\Pr\left\{\left|\boldsymbol{w}^{\mathrm{H}}\Delta\right| < \boldsymbol{w}^{\mathrm{H}}\mathbf{a} - 1\right\} \geq p. \tag{47}$$

The Chebyshev inequality states that for any zero mean random variable ξ with variance σ_{ξ}^2 and any positive real number α,

$$\Pr\{|\xi| \geq \alpha\} \leq \frac{\sigma_{\xi}^2}{\alpha^2}. \tag{48}$$

The constraint (47) can be lower bounded as

$$\Pr\left\{\left|\boldsymbol{w}^{\mathrm{H}}\Delta\right| < \boldsymbol{w}^{\mathrm{H}}\mathbf{a} - 1\right\} \geq 1 - \frac{\left\|\mathbf{C}_{\Delta}^{1/2}\boldsymbol{w}\right\|^2}{\left(\boldsymbol{w}^{\mathrm{H}}\mathbf{a} - 1\right)^2} \tag{49}$$

for any distribution of Δ. The above inequality holds if $\left\|\mathbf{C}_{\Delta}^{1/2}\boldsymbol{w}\right\|^2 < \left(\left|\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right| - 1\right)^2$. The deterministic constraint is obtain as below

$$\left\|\mathbf{C}_{\Delta}^{1/2}\boldsymbol{w}\right\| \leq \sqrt{1-p}\left(\boldsymbol{w}^{\mathrm{H}}\mathbf{a} - 1\right). \tag{50}$$

So the robust beamforming problem can be approximated by the following SOCP problem

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^{\mathrm{H}}\hat{\mathbf{R}}\boldsymbol{w} \qquad \text{s.t.} \ \left\|\mathbf{C}_{\Delta}^{1/2}\boldsymbol{w}\right\| \leq \sqrt{1-p}\left(\boldsymbol{w}^{\mathrm{H}}\mathbf{a} - 1\right). \tag{51}$$

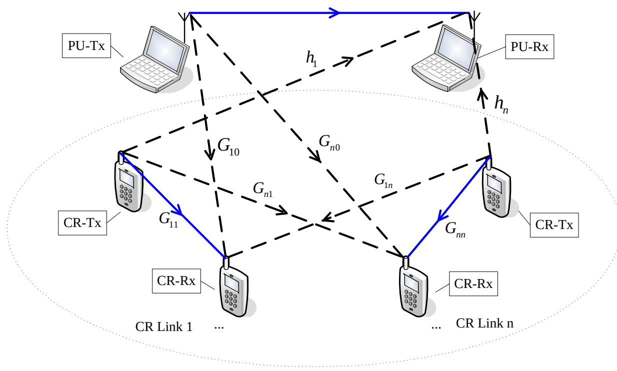

Figure 2: A cognitive radio network is composed of M cognitive links randomly allocated in a cluster area. The dashed lines denote the interference channels.

6 Robust Power Control in Cognitive Radio Networks

In this section, we study the robust power control problems in cognitive radio networks considering the uncertainties in the wireless channel gains. In practice, the wireless channel gains are imperfectly known or time varying. Without considering the uncertainty of these parameters, the solutions of nominal problems are not robust.

Consider an ad hoc cognitive radio network, i.e., there is no central control node. Assume that there are M cognitive user links randomly allocated in a cluster area, as shown in Figure 2. We assume that all the cognitive users can share the same frequency resource with primary user under the constraint that the total interference generated by cognitive users does not exceed a threshold that PU-Rx can tolerate, i.e.,

$$\sum_{i=1}^{M} h_i p_i \leq I, \tag{52}$$

where h_i denotes the channel gain between the cognitive transmitter (CR-Tx) of link i to the PU-Rx. p_i represents the transmitter power of the CR-Tx of link i. I is the maximum interference level that the PU-Rx can tolerate.

The signal-to-interference-noise-ratio (SINR) at the cognitive receiver (CR-Rx) of link i is

$$\mathrm{SINR}_i = \frac{G_{ii} p_i}{\sum\limits_{j \neq i}^{M} G_{ij} p_j + G_{i0} p_0 + n_i}, \tag{53}$$

where G_{ii} represents the channel gain between the CR-Tx and the CR-Rx of link i. G_{ij} is the channel gain between the CR-Tx of link j and the CR-Rx of link i. G_{i0} denotes the channel gain between the primary transmitter (PU-Tx) and the CR-Rx of link i. p_0 is the transmit power of PU-Tx and n_i is the background noise at the CR-Rx of link i.

To guarantee the QoS requirements of cognitive users, i.e., the packet delay constraint or minimum transmit rate, the SINR at each CR-Rx should be larger than a threshold, i.e., we have

$$\mathsf{SINR}_i \geq \gamma_i, \quad \forall i, \tag{54}$$

where γ_i is the SINR requirement at the CR-Rx of link i.

Constraint (54) can be equivalently written as

$$(\mathbf{D} - \mathbf{F})\,\mathbf{p} \geq \mathbf{m}, \tag{55}$$

where $\mathbf{D}$ is a $M \times M$ diagonal matrix with diagonal elements being $\frac{1}{\gamma_1}, ..., \frac{1}{\gamma_M}$. Vector $\mathbf{m} = \left[\frac{\mathbf{N}_1}{\mathbf{G}_{11}}, ..., \frac{\mathbf{N}_M}{\mathbf{G}_{MM}}\right]^{\mathbf{T}}$, where $N_i = G_{i0}p_0 + n_i$. $F = [F_{ij}]$ with

$$F_{ij} = \begin{cases} \frac{G_{ij}}{G_{ii}}, & i \neq j, \\ 0, & i = j. \end{cases} \tag{56}$$

The optimization objective is to minimize the total transmit power of all CR-Txs under both the PU-Rx interference constraint and the SINR constraint of each CR-Rx. Mathematically, the problem is formulated as follows:

Nominal power control problem (P1) :

$$\begin{aligned} \min \ & \sum_{i=1}^{M} p_i \\ \text{s.t.} \ & \sum_{i=1}^{M} h_i p_i \leq I \\ & \mathsf{SINR}_i \geq \gamma_i, \quad \forall i, \end{aligned} \tag{57}$$

where the variables are $0 \leq p_i \leq P_{\max}$ for all i, and $P_{\max}$ is the maximum transmit power for each CR-Tx.

We use ellipsoid sets to describe the uncertainty. Under the worst case, the robust power control problem can be formulated as a SOCP problem.

Let $\mathcal{F}_i$ denote the uncertainty set of the ith row of matrix $\mathbf{F}$. $\mathcal{F}_i$ describes the perturbation of interfering channel gains relevant to channel gain of link i. We use an ellipsoid to describe set $\mathcal{F}_i$, for all i. The same method is also used in [6]. Denote the normalized channel gain from the CR-Tx of link j to the CR-Rx of link i as $F_{ij} = \bar{F}_{ij} + \Delta F_{ij}$, where $\bar{F}_{ij}$ is the nominal value, and ΔF_{ij} is the perturbation part. Denote the ith row of $\bar{\mathbf{F}}$ as $\bar{\mathbf{F}}_i$, and the corresponding perturbation part as $\Delta\mathbf{F}_i$. Under ellipsoid approximation, the uncertainty set of $\mathcal{F}_i$ for $\bar{\mathbf{F}}_i$ can be written as

$$\mathcal{F}_i = \left\{ \bar{\mathbf{F}}_i + \Delta\mathbf{F}_i : \sum_{j \neq i} |\Delta F_{ij}|^2 \leq \epsilon_i^2 \right\}, \tag{58}$$

where $\epsilon_i \geq 0$ is the maximum deviation of each entry in $\mathbf{F}_i$. Since the channel gain vector at each CR-Rx varies independently, the row-wise structure of $\mathcal{F}_i$ can capture the uncertainty well. Besides, the row-wise uncertainty sets of channel gains have a decomposition structure which is important in implementing the distributed algorithm.

Denote the channel gain between the CR-Tx of link i and the PU-Rx as $h_i = \bar{h}_i + \Delta h_i$, where $\bar{h}_i$ is the nominal value, and Δh_i is the corresponding deviation part. Let $\mathcal{H}$ be the uncertainty set of channel gain vector $\mathbf{h}$, where $\mathbf{h} = [\mathbf{h_1}, ..., \mathbf{h_M}]^{\mathbf{T}}$. Use an ellipsoid to describe the uncertainty set $\mathcal{H}$ as follows

$$\mathcal{H} = \left\{ \bar{\mathbf{h}} + \Delta \mathbf{h} : \sum_i |\Delta h_i|^2 \leq \epsilon_0^2 \right\}, \tag{59}$$

where ϵ_0 is the maximum deviation of each entry in $\mathbf{h}$.

The robust power control problem that considers the channel uncertainty can be written as

Robust power control problem (P2) :

$$\begin{aligned}
\min \ & \sum_i p_i \\
\text{s.t.} \ & \sum_i \left(\bar{h}_i + \Delta h_i\right) p_i \leq I \\
& \frac{\sum_{j \neq i} \left(\bar{F}_{ij} + \Delta F_{ij}\right) p_j}{p_i} + \frac{N_i}{G_{ii} p_i} \leq \frac{1}{\gamma_i}, \ \forall i \\
& \sum_{j \neq i} |\Delta F_{ij}|^2 \leq \epsilon_i^2, \ \forall i \\
& \sum_i |\Delta h_i|^2 \leq \epsilon_0^2.
\end{aligned} \tag{60}$$

Since problem **P2** is subject to an infinite number of constraints with respect to the sets $\mathcal{F}_i(\epsilon_i)$, for all i and $\mathcal{H}(\epsilon_0)$, problem **P2** is a SIP problem. A natural method to solve the SIP problem is to transform it into an equivalent problem with finite constraints.

We transform the SIP problem into an equivalent problem by considering the worst case in the constraints of problem **P2**. With Cauchy-Schwartz inequality, we have

$$\max_{\mathbf{F}_i \in \mathcal{F}_i} \left\{ \sum_{j \neq i} \left(F_{ij} - \bar{F}_{ij}\right) p_j \right\} = \epsilon_i \sqrt{\sum_{j \neq i} p_j^2}, \tag{61}$$

$$\max_{\mathbf{h} \in \mathcal{H}} \left\{ \sum_i \left(h_i - \bar{h}_i\right) p_i \right\} = \epsilon_0 \sqrt{\sum_i p_i^2}. \tag{62}$$

Thus, under the worst case, the problem **P2** is transformed as a SOCP problem with the standard form as follows

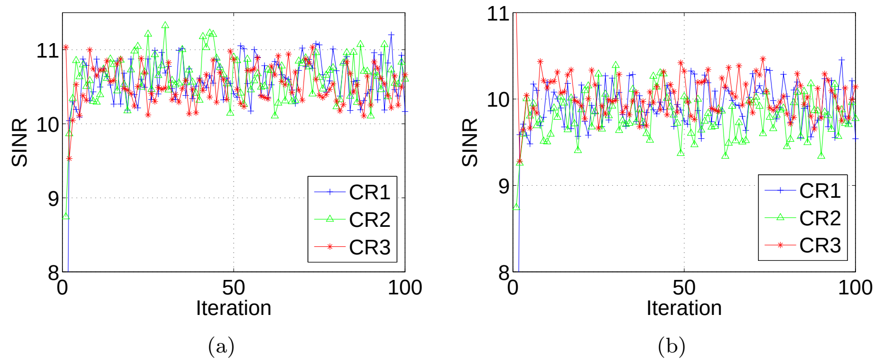

Figure 3: Performance comparisons: (a) SINR under robust algorithm; (b) SINR under non-robust algorithms.

Equivalent robust problem (P3) :

$$\begin{aligned} &\min \ \mathbf{1}^T\mathbf{p} \\ &\text{s.t.} \ \epsilon_0\|\mathbf{p}\|_2 \le -\bar{\mathbf{h}}^T\mathbf{p} + I \\ &\qquad \epsilon_i \left\| \left(\mathbf{I} - \mathbf{e}_i\mathbf{e}_i^T\right)^{1/2}\mathbf{p}\right\|_2 \le \left(\frac{1}{\gamma_i}\mathbf{e}_i - \bar{\mathbf{F}}_i^T\right)^T \mathbf{p} - \frac{N_i}{G_{ii}}, \ \forall i, \end{aligned} \tag{63}$$

where $\mathbf{e}_i$ represents the ith standard basic vector, and the elements in $\mathbf{p}$ should satisfy $0 \le p_i \le P_{\max}$, for all i.

The distributed algorithm is then developed to solve the SOCP problem in a distributed way. In addition, an asynchronous iterative algorithm is further proposed along with a proof of the convergence in [6]. Figure 3 gives an simulation example. Assume the SINR requirement is 10dB. It can be found that when there are channel uncertainties, under robust algorithm the SINR requirement is guaranteed while the outage events happen frequently under non-robust algorithms.

References

[1] S. A. Vorobyov, A. B. Gershman and Z. Q. Luo,"Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem," *IEEE Trans. on Signal Processing*, vol. 51, no.2, pp. 313-324, 2003.

[2] S. J. Kim, A. Magnani, A. Mutapcic, S. Boyd and Z. Q. Luo, "Robust beamforming via worst-case SINR maximization," *IEEE Trans. on Signal Processing*, vol. 56, no.4, pp. 1539-1547, 2008.

[3] R. G. Lorenz and S. Boyd, "Robust minimum variance beamforming," *IEEE Trans. on Signal Processing*, vol. 53, no.5, pp. 1684-1696, 2005.

[4] A. B. Gershman, N. D. Sidiropoulos, S. Shahbazpanahi, M. Bengtsson and B. Ottersten, "Convex optimization-based beamforming," *IEEE Signal Processing Magazine*, vol. 27, no. 3, pp. 62-74, 2010.

[5] S. A. Vorobyov, H. Chen and A. B. Gershman, "On the relationship between robust minimum variance beamformers with probabilistic and worst-case distortionless response constraints," *IEEE Trans. on Signal Processing*, vol. 56, no.11, pp. 5719-5724, 2008.

[6] S. Sun, W. Ni and Y. Zhu, "Robust power control in cognitive radio networks: A distributed way," in *Proc. IEEE International Conference on Communications (ICC)*, Kyoto, Japan, June, 2011.

[7] A. Ben-Tal, L. E. Ghaoui, and A. Nemirovski, *Robust Optimization.* Princeton University Press, 2009.