13.1.1 Lagrange Multipliers for Q^2P

We now define the (inequality constrained) $\mathrm{Q}^2\mathrm{P}$ in x. (Though our notation does differ slightly in the separate parts (sections) of this chapter.)

$$(\mathrm{Q}^2\mathrm{P}_x) \quad \begin{array}{rll} q^* := & \min & q_0(x) := x^T q_0 x + 2g_0^T x + \alpha_0 \\ & \text{subject to} & q_k(x) := x^T Q_k x + 2g_k^T x + \alpha_k \leq 0 \\ & & k \in \mathcal{I} := \{1, \ldots, m\} \\ & & x \in \Re^n, \end{array}$$

where the matrices Q_k are symmetric. The Lagrangian of $\mathrm{Q}^2\mathrm{P}_x$ is

$$L(x, \lambda) := q_0(x) + \sum_{k \in \mathcal{I}} \lambda_k q_k(x),$$

where $\lambda = (\lambda_k) \geq 0$ are nonnegative Lagrange multipliers.

Lagrange multipliers can be used in two ways. First, if a constraint qualification holds for $\mathrm{Q}^2\mathrm{P}$ at the optimum $\bar{x}$ (e.g. the Mangasarian-Fromovitz constraint qualification), then the Karush-Kuhn-Tucker necessary conditions for optimality hold, i.e.

$$\nabla L(x, \lambda) = 0, \text{ and } \lambda_k q_k(x) = 0, \forall k \in \mathcal{I}.$$

Therefore, the optimum $\bar{x}$ can be searched among the points satisfying stationarity of the Lagrangian and complementary slackness. Moreover, if the Lagrangian is also convex, then this is a sufficient condition for optimality.

Lagrange multipliers can also be used to derive the Lagrangian dual (or relaxation) of $\mathrm{Q}^2\mathrm{P}_x$

$$(DQ^2P_x) \qquad q^* \geq d^* := \max_{\substack{\mu \\ \lambda \geq 0}} \min_x q_0(x) + \sum_{k \in \mathcal{I}} \lambda_k q_k(x).$$

A zero duality gap holds if $q^* = d^*$. This can fail in the nonconvex case. Strong duality holds if $q^* = d^*$ and also d^* is attained. Moreover, d^* can be efficiently evaluated using SDP.

13.2 GLOBAL QUADRATIC OPTIMIZATION VIA CONIC RELAXATION

Yuri Nesterov

Starting from the pioneering paper [285], there were obtained several results [574, 861, 577], which show that a solution of an indefinite quadratic maximization problem with some linear constraints on the squared variables can be approximated with a constant relative accuracy. In this Section 13.2 we present some improvements and extensions of the results [577].

In Section 13.2.1 we consider a problem of maximizing an indefinite quadratic form subject to arbitrary convex constraints on the squared variables. For convenience of the dual description we use a conic representation of these constraints. We introduce a convex conic relaxation for that problem and prove that it provides us at least with an approximation of $\frac{\pi-2}{6-\pi}$ relative accuracy. In Section 13.2.2 we show how to improve this approximation using the diagonal elements of the quadratic objective function. The relative accuracy, which we can get in this case, is $\frac{12}{37}$. In Section 13.2.3 we extend the results of Section 13.2.1, 13.2.2 onto the case of general convex constraints on squared variables. We conclude the first part of the section with a discussion of the difficulties which arise in the problems with linear equality constraints on the initial variables (Section 13.2.4).

In the second part of the section, which starts from Section 13.2.5, we study another way of deriving the conic relaxation. This approach can be applied only to a small number of sets (l_p-balls, $p \geq 2$), but it allows to treat also the linear equations. We prove that for such problems the conic relaxation gives $(1 - \frac{2}{p})$ relative accuracy. In Section 13.2.6 we apply these results to a problem of maximizing a quadratic function over a unit box subject to a system of homogeneous linear equalities. We show that it is possible to compute in polynomial time a $(1 - \frac{1}{e \ln n})$-solution of that problem. We conclude the section with a discussion of the results.

We first recall some of the notation we use. For two vectors $x,\ y \in R^n$ we denote $\langle x, y \rangle$ the standard inner product:

$$\langle x, y \rangle = \sum_{i=1}^{n} x^{(i)} y^{(i)}.$$

Then $\| x \| = \langle x, x \rangle^{1/2}$. Since we work in several finite-dimensional spaces, the meaning of this notation is defined by the spaces of the arguments. For example, for two $(m \times n)$-matrices X and Y we have

$$\langle X, Y \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} Y_{ij}.$$

We use the standard notation for l_p-norms:

$$\| x \|_p = \left[\sum_{i=1}^{n} | x^{(i)} |^p \right]^{1/p}, \quad x \in R^n,\ p \geq 1.$$

Again, the meaning of the notation depends on the dimension of space of the argument. Recall, that for $p = \infty$ we have $\| x \|_\infty = \max_{1 \leq i \leq n} | x^{(i)} |$. The norm dual to $\| \cdot \|_p$ is $\| \cdot \|_{p^*}$ with $p^* = \frac{p}{p-1}$:

$$\| y \|_{p^*} = \max\{\langle y, x\rangle : \| x \|_p \leq 1\}.$$

For a symmetric matrix A we write $A \succeq 0$ if A is positive semidefinite. Notation $B \succeq A$ means that $B - A \succeq 0$. For $x \in R^n$ we denote $\mathrm{Diag}\,(x)$ the diagonal $(n \times n)$-matrix with diagonal entries $x^{(i)}$. Conversely, $\mathrm{diag}(X) \in R^n$ denotes the diagonal of an $(n \times n)$-matrix X. Notation e_i is used for the ith coordinate vector of R^n and $1_n \in R^n$ stands for the vector of all ones. Thus, $I_n = \mathrm{Diag}\,(1_n)$ is a unit matrix. Notation 0_n is used for the zero vector in R^n.

We use square brackets in order to indicate the component-wise operations with the vectors. For example, notation $[x \cdot y]$ stands for the vector with components $x^{(i)}y^{(i)}$, x, $y \in R^n$. Notation $[x]^2$ is used for the vector with the components $(x^{(i)})^2$. If $f(\tau)$ is a univariate function, we denote $f[x]$ the vector with the components $f(x^{(i)})$. In order to indicate the partial ordering in R^n we use the usual inequality signs. Thus, $x \geq y$ for x and y from R^n means that $x^{(i)} \geq y^{(i)}$, $i = 1, \ldots, n$.

Finally, $[\alpha, \beta]^n$ denotes a continuous box in R^n, that is $\{x \in R^n : \alpha 1_n \leq x \leq \beta 1_n\}$. For a boolean box $\{x \in R^n : x^{(i)} = (\alpha \text{ or } \beta)\}$ we use notation $\{\alpha, \beta\}^n$.

13.2.1 Convex conic constraints on squared variables

Let Q be an arbitrary symmetric $(n \times n)$-matrix. Consider the following pair of optimization problems:

$$\begin{aligned} \text{find } \phi^* = &\quad \max\{\langle Qx, x\rangle : [x]^2 \in \mathcal{F}\}, \\ \text{find } \phi_* = &\quad \min\{\langle Qx, x\rangle : [x]^2 \in \mathcal{F}\}. \end{aligned} \tag{13.2.1}$$

where $\mathcal{F}$ is a closed convex set. Our main assumption on the problem (13.2.1) is as follows.

Assumption 13.2.1 *1). The set $\mathcal{F}$ is bounded. 2). There exists a strictly positive $v \in \mathcal{F}$.*

In order to simplify the dual analysis, in this section we assume that the feasible set $\mathcal{F}$ is presented in a conic form:

$$\mathcal{F} = \{v \in K : Av = b\}, \tag{13.2.2}$$

where K is a convex closed pointed cone in R^n with non-empty interior, A is an $(m \times n)$-matrix and $b \neq 0_m$. Our additional assumption on the set $\mathcal{F}$ is as follows.

Assumption 13.2.2 $\{v \in \mathrm{int}\, K : Av = b\} \neq \emptyset$.

Note that the form (13.2.2) is quite general, since any bounded convex set can be written in this way (see [585] for details). At the same time, in Section

13.2.3 we will show how to transform our result on the case of a general convex feasible set $\mathcal{F}$.

Using the same technique as in [577], we can rewrite the pair of problems (13.2.1) in a trigonometric form.

Lemma 13.2.1

$$\begin{aligned} \phi^* &= \max_{\substack{X \succeq 0,\, \mathrm{diag}(X) = 1_n,\\ d \geq 0,\, [d]^2 \in \mathcal{F},}} \tfrac{2}{\pi}\langle Q, \mathrm{Diag}\,(d) \arcsin[X] \mathrm{Diag}\,(d)\rangle, \\ \phi_* &= \min_{\substack{X \succeq 0,\, \mathrm{diag}(X) = 1_n,\\ d \geq 0,\, [d]^2 \in \mathcal{F}.}} \tfrac{2}{\pi}\langle Q, \mathrm{Diag}\,(d) \arcsin[X] \mathrm{Diag}\,(d)\rangle, \end{aligned} \tag{13.2.3}$$

Proof.
Indeed, let us represent a vector $x \in R^n$ as follows:

$$x = [d \cdot \sigma], \quad d \geq 0 \in R^n,\ \sigma \in \{-1, 1\}^n.$$

Note that $[x]^2 = [d]^2$. Therefore $\phi^* = \max\limits_{d}\{\Phi(d) : d \geq 0,\ [d]^2 \in \mathcal{F}\}$ with

$$\Phi(d) = \max\{\langle \mathrm{Diag}\,(d) Q \mathrm{Diag}\,(d)\sigma, \sigma\rangle :\ \sigma \in \{-1, 1\}^n\}.$$

Using Theorem 2.3 [577], we can represent $\Phi(d)$ in the following form:

$$\Phi(d) = \max\{\tfrac{2}{\pi}\langle \mathrm{Diag}\,(d) Q \mathrm{Diag}\,(d), \arcsin[X]\rangle :\ X \succeq 0,\ \mathrm{diag}(X) = 1_n\}.$$

Inserting this representation in the above expression for ϕ^* we get the first statement of the lemma. The second one can be obtained in a similar way. ■

Note that in general none of the problems (13.2.1) is convex in x. Therefore, in order to estimate their optimal values, we need to use a kind of convex relaxation. Let us define the *conic relaxation* of problems (13.2.1):

$$\begin{aligned} \psi^* &= \max\{\langle Q, X\rangle :\ \mathrm{diag}(X) \in \mathcal{F},\ X \succeq 0\}, \\ \psi_* &= \min\{\langle Q, X\rangle :\ \mathrm{diag}(X) \in \mathcal{F},\ X \succeq 0\}. \end{aligned} \tag{13.2.4}$$

Sometimes it is convenient to use a dual form of these relaxations. Recall that for a convex cone $K \subseteq R^n$ the dual cone K^* is defined as follows:

$$K^* = \{u \in R^n :\ \langle u, v\rangle \geq 0,\ \forall v \in K\}.$$

Lemma 13.2.2

$$\begin{aligned} \psi^* &= \min_{y \in R^m, u \in R^n} \{\langle b, y\rangle :\ Q + \mathrm{Diag}\,(u) \preceq \mathrm{Diag}\,(A^T y),\ u \in K^*\}, \\ \psi_* &= \max_{y \in R^m, u \in R^n} \{\langle b, y\rangle :\ Q \succeq \mathrm{Diag}\,(u) + \mathrm{Diag}\,(A^T y),\ u \in K^*\}. \end{aligned} \tag{13.2.5}$$

Proof.
In view of Assumptions 13.2.1, 13.2.2, we can get a dual representation of the upper relaxation ψ^* as follows:

$$\psi^* = \max_{X,v}\{\langle Q, X\rangle : \ A\mathrm{diag}(X) = b,\ \mathrm{diag}(X) = v,\ X \succeq 0,\ v \in K\}$$

$$= \max_{X\succeq 0, v\in K} \ \min_{y\in R^m, u\in R^n} \{\langle Q, X\rangle + \langle y, b - A\mathrm{diag}(X)\rangle + \langle u, \mathrm{diag}(X) - v\rangle\}$$

$$= \min_{y\in R^m, u\in R^n} \max_{X,v}\{\langle Q + \mathrm{Diag}\,(u - A^T y), X\rangle + \langle b, y\rangle - \langle u, v\rangle : ||X \succeq 0,\ v\in K\}$$

$$= \min_{y\in R^m, u\in R^n}\{\langle b, y\rangle : \ Q + \mathrm{Diag}\,(u) \preceq \mathrm{Diag}\,(A^T y),\ u \in K^*\}.$$

Similarly, for the lower relaxation we get the following:

$$\psi \min_{X,v}\{\langle Q, X\rangle : \ A\mathrm{diag}(X) = b,\ \mathrm{diag}(X) = v,\ X \succeq 0,\ v \in K\}$$

$$= \min_{X\succeq 0, v\in K} \ \max_{y\in R^m, u\in R^n} \{\langle Q, X\rangle + \langle y, b - A\mathrm{diag}(X)\rangle + \langle u, v - \mathrm{diag}(X)\rangle\}$$

$$= \max_{y\in R^m, u\in R^n} \min_{X,v}\{\langle Q - \mathrm{Diag}\,(u + A^T y), X\rangle + \langle b, y\rangle + \langle u, v\rangle : ||X \succeq 0,\ v \in K\}$$

$$= \max_{y\in R^m, u\in R^n}\{\langle b, y\rangle : \ Q \succeq \mathrm{Diag}\,(u) + \mathrm{Diag}\,(A^T y),\ u \in K^*\}.$$

■

Let us establish some relations between the relaxations (13.2.4) and the optimal values of the problems (13.2.1). Denote

$$\psi(\alpha) = \alpha\psi^* + (1-\alpha)\psi_*. \tag{13.2.6}$$

The proof of the following theorem is similar to that of Theorem 3.3 [577].

Theorem 13.2.1

$$\psi_* \le \phi_* \le \psi(1 - \tfrac{2}{\pi}) \le \psi(\tfrac{2}{\pi}) \le \phi^* \le \psi^*. \tag{13.2.7}$$

Proof.
Note that $\psi_* \le \psi^*$ by definition. So, the middle inequality in (13.2.7) is correct. Further, if $[x]^2 \in \mathcal{F}$ then the matrix $X = xx^T$ is feasible for both relaxation problems (13.2.4) since $\mathrm{diag}(X) = [x]^2$. Moreover, $\langle Q, X\rangle = \langle Qx, x\rangle$. Thus, both bounding inequalities in the chain (13.2.7) are valid. Let us prove now two remaining inequalities.

Let us choose arbitrary $u \in K$ and $y \in R^m$, which satisfy the constraints of the dual form (13.2.5) of the lower relaxation ψ_*:

$$(u, y) \in \mathcal{F}_d = \{(u, y) \in K^* \times R^m : \ Q \succeq \text{Diag}\,(u) + \text{Diag}\,(A^T y)\}. \qquad (13.2.8)$$

Consider a pair (X, d), which satisfies the constraints of the trigonometric representation (13.2.3) for ϕ^*:

$$X \succeq 0, \quad \text{diag}(X) = 1_n, \quad d \geq 0, \quad A[d]^2 = b, \quad [d]^2 \in K. \qquad (13.2.9)$$

Since $X \succeq 0$ and $| X_{ij} | \leq 1$ we have $\arcsin[X] \succeq X$ in view of Corollary 3.2 [577]. Therefore, using Lemma 13.2.1 we get the following:

$$\begin{aligned}
\phi^* &\geq \tfrac{2}{\pi}\langle \text{Diag}\,(d) Q \text{Diag}\,(d), \arcsin[X]\rangle \\
&= \tfrac{2}{\pi}\langle \text{Diag}\,(d)(Q - \text{Diag}\,(u) - \text{Diag}\,(A^T y))\text{Diag}\,(d), \arcsin[X]\rangle \\
&\qquad\qquad + \langle u + A^T y, [d]^2\rangle \\
&\geq \tfrac{2}{\pi}\langle \text{Diag}\,(d)(Q - \text{Diag}\,(u) - \text{Diag}\,(A^T y))\text{Diag}\,(d), X\rangle + \langle u + A^T y, [d]^2\rangle \\
&= \tfrac{2}{\pi}\langle Q, \text{Diag}\,(d) X \text{Diag}\,(d)\rangle + (1 - \tfrac{2}{\pi})\langle u + A^T y, [d]^2\rangle.
\end{aligned}$$

Note that $u \in K^*$ and $[d]^2 \in K$. Therefore $\langle u, [d]^2\rangle \geq 0$. In view of (13.2.9) we have

$$\langle A^T y, [d]^2\rangle = \langle A[d]^2, y\rangle = \langle b, y\rangle.$$

Finally, for any pair (X, d), which satisfy (13.2.9) we have $Y = \text{Diag}\,(d) X \text{Diag}\,(d)$ feasible for the primal relaxation problems:

$$Y \in \mathcal{F}_p = \{Y : \ Y \succeq 0, \ A\text{diag}(Y) = b, \ \text{diag}(Y) \in K\}.$$

On the other hand, any $Y \in \mathcal{F}_p$ can be represented as $Y = \text{Diag}\,(d) X \text{Diag}\,(d)$ with X and d, which satisfy (13.2.9). Therefore, we conclude that

$$\phi^* \geq \tfrac{2}{\pi}\langle Q, Y\rangle + (1 - \tfrac{2}{\pi})\langle b, y\rangle, \quad \forall Y \in \mathcal{F}_p, \ (u, y) \in \mathcal{F}_d.$$

This proves the forth inequality in the chain (13.2.7). The remaining inequality can be proved in a similar way. ■

Definition 13.2.1 *We say that the value* ψ *approximates* ϕ^* *with a relative accuracy* $\mu \in [0, 1]$ *if* $| \psi - \phi^* | \leq \mu(\phi^* - \phi_*)$. *We call this approximation implementable if* $\psi \leq \phi^*$.

Corollary 13.2.1 *1. Let* $\alpha = \frac{2}{\pi}$. *Then the value* $\psi(\alpha)$ *is an implementable approximation of* ϕ^* *with the relative accuracy* $\mu = \frac{\pi}{2} - 1 < \frac{4}{7}$.

2. Let $\beta = \frac{(1+\alpha)^2 - 2}{3\alpha - 1}$. *Then the value* $\psi(\beta)$ *approximates* ϕ^* *with the relative accuracy* $\mu = \frac{\pi - 2}{6 - \pi} < \frac{2}{5}$.

The proof of that statement is exactly the same as that of Corollary 3.4 in [577].

13.2.2 Using additional information

In this section it is shown how to improve the quality of our bounds by taking into account some additional information. Define

$$\begin{aligned} \tau^* &= \max\{\langle \operatorname{diag}(Q), v\rangle : \ v \geq 0, \ v \in \mathcal{F}\}, \\ \tau_* &= \min\{\langle \operatorname{diag}(Q), v\rangle : \ v \geq 0, \ v \in \mathcal{F}\}. \end{aligned} \tag{13.2.10}$$

Note that these values are computable in polynomial time. In view of Lemma 13.2.1 we have

$$\phi_* \leq \tau_* \leq \tau^* \leq \phi^*. \tag{13.2.11}$$

Hence,

$$\beta^* \equiv \tfrac{\psi^*-\tau^*}{\psi^*-\psi_*} \in [0,1], \quad \beta_* \equiv \tfrac{\tau_*-\psi_*}{\psi^*-\psi_*} \in [0,1].$$

Using these values we can express τ^* and τ_* as follows:

$$\begin{aligned} \tau^* &= \psi^* - \beta^*(\psi^* - \psi_*) &= \psi(1-\beta^*), \\ \tau_* &= \psi_* + \beta_*(\psi^* - \psi_*) &= \psi(\beta_*). \end{aligned}$$

Denote $\omega(\beta) = \beta \arcsin(\beta) + \sqrt{1-\beta^2} \equiv 1 + \int\limits_0^\beta \arcsin(\tau) d\tau$, $\beta \in [0,1]$. This function is increasing and convex with $\omega(0) = 1$ and $\omega(1) = \frac{\pi}{2}$. In what follows we denote $\bar{\beta}$ the unique root of the following equation:

$$\tfrac{2}{\pi}\omega(\beta) = 1 - \beta, \quad \beta \in [0,1].$$

It can be shown that $\frac{23}{70} < \bar{\beta} < \frac{24}{73}$.

Theorem 13.2.2 *1. Denote*

$$\begin{aligned} \alpha^* &= \max\{\tfrac{2}{\pi}\omega(\beta_*), 1-\beta^*\}, \\ \alpha_* &= \min\{1 - \tfrac{2}{\pi}\omega(\beta^*), \beta_*\}. \end{aligned}$$

The optimal values of the problems (13.2.1) satisfy the following relations:

$$\psi^* \geq \ \phi^* \ \geq \psi(\alpha^*), \tag{13.2.12}$$

$$\psi_* \leq \ \phi_* \ \leq \psi(\alpha_*). \tag{13.2.13}$$

2. The value $\psi(\alpha^*)$ is an implementable approximation of ϕ^* with relative accuracy

$$\mu = \tfrac{1-\alpha^*}{1-\alpha_*} \leq \tfrac{\bar{\beta}}{1-\bar{\beta}} < \tfrac{24}{49}.$$

3. Denote $\bar{\alpha} = \frac{\alpha^*(2-\alpha_*)-\alpha_*}{1+\alpha^*-2\alpha_*}$. The value $\psi(\bar{\alpha})$ is a μ-approximation of ϕ^* with

$$\mu = \tfrac{1-\alpha^*}{1+\alpha^*-2\alpha_*} \leq \tfrac{\bar{\beta}}{2-3\bar{\beta}} < \tfrac{12}{37}.$$

In Items 2 and 3 the upper bounds are achieved for $\beta^* = \beta_* = \bar{\beta}$.

Proof.
Let $X \succeq 0$ and $d \geq 0$ be feasible for the trigonometric form of the upper relaxation (13.2.3):

$$\operatorname{diag}(X) = 1_n, \quad A[d]^2 = b, \quad [d]^2 \in K.$$

Consider the matrices $X_\gamma = \gamma X + (1-\gamma) I_n$, $\gamma \in [0,1]$. Then

$$\arcsin[X_\gamma] = \arcsin[\gamma X] + (\tfrac{\pi}{2} - \arcsin(\gamma)) I_n.$$

Therefore

$$\begin{aligned} \phi^* &\geq \tfrac{2}{\pi}\langle Q, \operatorname{Diag}(d) \arcsin[X_\gamma] \operatorname{Diag}(d)\rangle \\ &= \tfrac{2}{\pi}\langle Q, \operatorname{Diag}(d)\arcsin[\gamma X]\operatorname{Diag}(d)\rangle + \left(1 - \tfrac{2}{\pi}\arcsin(\gamma)\right)\langle \operatorname{diag}(Q), [d]^2\rangle. \end{aligned} \tag{13.2.14}$$

Let us choose now arbitrary $u \in K$ and $y \in R^m$ which satisfy the constraints of the dual form (13.2.5) of the lower relaxation ψ_*:

$$(u, y) \in \mathcal{F}_d = \{(u,y) \in K^* \times R^m : \ Q \succeq \operatorname{Diag}(u) + \operatorname{Diag}(A^T y)\}.$$

Then, in view of Corollary 3.2 [577] we have:

$$\begin{aligned} &\langle Q, \operatorname{Diag}(d)\arcsin[\gamma X]\operatorname{Diag}(d)\rangle \\ &\quad = \langle Q - \operatorname{Diag}(u) - \operatorname{Diag}(A^T y), \operatorname{Diag}(d)\arcsin[\gamma X]\operatorname{Diag}(d)\rangle \\ &\qquad + \langle \operatorname{Diag}(u + A^T y), \operatorname{Diag}(d)\arcsin[\gamma X]\operatorname{Diag}(d)\rangle \\ &\quad \geq \gamma\langle Q - \operatorname{Diag}(u) - \operatorname{Diag}(A^T y), \operatorname{Diag}(d) X \operatorname{Diag}(d)\rangle \\ &\qquad + \arcsin(\gamma)\langle u + A^T y, [d]^2\rangle \\ &\quad = \gamma\langle Q, \operatorname{Diag}(d) X \operatorname{Diag}(d)\rangle + (\arcsin(\gamma) - \gamma)\langle u + A^T y, [d]^2\rangle. \end{aligned}$$

Note that $\arcsin(\gamma) \geq \gamma$ for $\gamma \in [0,1]$. At the same time $u \in K^*$ and $[d]^2 \in K$. Therefore $\langle u, [d]^2\rangle \geq 0$. Finally, $\langle A^T y, [d]^2\rangle = \langle A[d]^2, y\rangle = \langle b, y\rangle$. Thus,

$$\begin{aligned} \langle Q, \operatorname{Diag}(d)\arcsin[\gamma X]\operatorname{Diag}(d)\rangle \geq \gamma\langle Q, \operatorname{Diag}(d) X \operatorname{Diag}(d)\rangle \\ +(\arcsin(\gamma) - \gamma)\langle b, y\rangle. \end{aligned}$$

Substituting this inequality in (13.2.14) we get the following:

$$\begin{aligned} \phi^* &\geq \tfrac{2}{\pi}\left(\gamma\langle Q, \operatorname{Diag}(d) X \operatorname{Diag}(d)\rangle + (\arcsin(\gamma) - \gamma)\langle b, y\rangle\right) \\ &\qquad + \left(1 - \tfrac{2}{\pi}\arcsin(\gamma)\right)\langle \operatorname{diag}(Q), [d]^2\rangle \\ &\geq \tfrac{2}{\pi}\left(\gamma\langle Q, \operatorname{Diag}(d) X \operatorname{Diag}(d)\rangle + (\arcsin(\gamma) - \gamma)\langle b, y\rangle\right) \\ &\qquad + \left(1 - \tfrac{2}{\pi}\arcsin(\gamma)\right)\tau_*. \end{aligned}$$

Using the same reasoning as in Theorem 13.2.1, we conclude that

$$\begin{aligned}\phi^* &\geq \tfrac{2}{\pi}\gamma\psi^* + \tfrac{2}{\pi}(\arcsin(\gamma) - \gamma)\psi_* + \left(1 - \tfrac{2}{\pi}\arcsin(\gamma)\right)\tau_* \\ &= \tfrac{2}{\pi}\arcsin(\gamma)\psi\left(\tfrac{\gamma}{\arcsin(\gamma)}\right) + \left(1 - \tfrac{2}{\pi}\arcsin(\gamma)\right)\psi(\beta_*) \\ &= \psi\left(\tfrac{2\gamma}{\pi} + \left(1 - \tfrac{2}{\pi}\arcsin(\gamma)\right)\beta_*\right).\end{aligned}$$

The right-hand side of the above inequality is maximal for $\gamma^* = \sqrt{1-\beta_*^2}$. Then

$$\tfrac{2\gamma^*}{\pi} + \left(1 - \tfrac{2}{\pi}\arcsin(\gamma^*)\right)\beta_* = \tfrac{2}{\pi}\left(\sqrt{1-\beta_*^2} + \beta_*\arccos(\gamma^*)\right) = \tfrac{2}{\pi}\omega(\beta_*).$$

Thus, $\phi^* \geq \psi(\frac{2}{\pi}\omega(\beta_*))$. Combining this inequality with (13.2.11), we get the lower bound in (13.2.12). The relations (13.2.13) for ψ_* can be obtained in a similar way.

Let us prove now Item 2 of the theorem. In view of (13.2.12) and (13.2.13) we have

$$0 \leq \frac{\phi^* - \psi(\alpha^*)}{\phi^* - \phi_*} \leq \frac{\psi^* - \psi(\alpha^*)}{\psi^* - \phi_*} \leq \frac{\psi^* - \psi(\alpha^*)}{\psi^* - \psi(\alpha_*)} = \frac{\psi(1) - \psi(\alpha^*)}{\psi(1) - \psi(\alpha_*)} = \frac{1-\alpha^*}{1-\alpha_*}. \tag{13.2.15}$$

Note that

$$\begin{aligned}1-\alpha^* &= 1 - \max\{\tfrac{2}{\pi}\omega(\beta_*), 1-\beta^*\} &&= \min\{1 - \tfrac{2}{\pi}\omega(\beta_*), \beta^*\}, \\ 1-\alpha_* &= 1 - \min\{1 - \tfrac{2}{\pi}\omega(\beta^*), \beta_*\} &&= \max\{\tfrac{2}{\pi}\omega(\beta^*), 1-\beta_*\}.\end{aligned}$$

Thus, we need to find an upper bound for the ratio

$$\rho(\beta_1,\beta_2) = \frac{\min\{1 - \frac{2}{\pi}\omega(\beta_2), \beta_1\}}{\max\{\frac{2}{\pi}\omega(\beta_1), 1-\beta_2\}}, \quad 0 \leq \beta_1, \beta_2 \leq 1.$$

Lemma 13.2.3

$$\max\{\rho(\beta_1,\beta_2) : \ 0 \leq \beta_1,\beta_2 \leq 1\} = \frac{\bar\beta}{1-\bar\beta}.$$

Proof.
We need to prove that

$$(1-\bar\beta)\min\{1 - \tfrac{2}{\pi}\omega(\beta_2), \beta_1\} \leq \bar\beta\max\{\tfrac{2}{\pi}\omega(\beta_1), 1-\beta_2\}, \quad 0 \leq \beta_1,\beta_2 \leq 1.$$

This is equivalent to the statement that the convex function

$$g(\beta_1,\beta_2) = \bar\beta\max\{\tfrac{2}{\pi}\omega(\beta_1), 1-\beta_2\} + (1-\bar\beta)\max\{\tfrac{2}{\pi}\omega(\beta_2) - 1, -\beta_1\}$$

is non-negative for $0 \leq \beta_1, \beta_2 \leq 1$.

Note that $g(\bar{\beta},\bar{\beta}) = 0$ in view of the definition of $\bar{\beta}$. The subdifferential of the function $g(\cdot)$ at this point is as follows:

$$\partial g(\bar{\beta},\bar{\beta}) = \bar{\beta}\text{Conv}\,\{(\tfrac{2}{\pi}\omega'(\bar{\beta}),0);\ (0,-1)\} + (1-\bar{\beta})\text{Conv}\,\{(0,\tfrac{2}{\pi}\omega'(\bar{\beta}));\ (-1,0)\}.$$

Thus, this set contains the following points:

$$(\tfrac{2}{\pi}\bar{\beta}\omega'(\bar{\beta}) - 1 + \bar{\beta}, 0), \quad (0, \tfrac{2}{\pi}(1-\bar{\beta})\omega'(\bar{\beta}) - \bar{\beta}), \quad (\tfrac{2}{\pi}\bar{\beta}\omega'(\bar{\beta}), \tfrac{2}{\pi}(1-\bar{\beta})\omega'(\bar{\beta})).$$

Note that

$$\tfrac{2}{\pi}\omega'(\bar{\beta}) < \omega'(\bar{\beta}) < \tfrac{\bar{\beta}}{1-\bar{\beta}} < \tfrac{1-\bar{\beta}}{\bar{\beta}}.$$

Therefore the first coordinate of the first point and the second coordinate of the second point are negative. Since both coordinates of the third point are positive, we conclude that $0 \in \text{int}\,\partial g(\bar{\beta},\bar{\beta})$. ■

Applying Lemma 13.2.3 to (13.2.15) we prove the statement of Item 2.

In order to prove Item 3 note that in view of inequalities (13.2.12) and (13.2.13), for any $\alpha \in [0,1]$ we have

$$\begin{aligned} \tfrac{|\psi(\alpha)-\phi^*|}{\phi^*-\phi_*} &\leq \tfrac{|\psi(\alpha)-\phi^*|}{\phi^*-\psi(\alpha_*)} = \max\left\{\tfrac{\psi(\alpha)-\phi^*}{\phi^*-\psi(\alpha_*)}, \tfrac{\phi^*-\psi(\alpha)}{\phi^*-\psi(\alpha_*)}\right\} \\ &\leq \max\left\{\tfrac{\psi(\alpha)-\psi(\alpha^*)}{\psi(\alpha^*)-\psi(\alpha_*)}, \tfrac{\psi^*-\psi(\alpha)}{\psi^*-\psi(\alpha_*)}\right\} = \max\left\{\tfrac{\alpha-\alpha^*}{\alpha^*-\alpha_*}, \tfrac{1-\alpha}{1-\alpha_*}\right\} \equiv r(\alpha). \end{aligned}$$

The minimum $\bar{\alpha}$ of the function $r(\alpha)$ is a solution of the following equation:

$$(\alpha - \alpha^*)(1-\alpha_*) = (1-\alpha)(\alpha^* - \alpha_*).$$

That is $\bar{\alpha} = \frac{\alpha^*(2-\alpha_*)-\alpha_*}{1+\alpha^*-2\alpha_*}$. Using Lemma 13.2.3 we can estimate the optimal value $r(\bar{\alpha})$ as follows:

$$r(\bar{\alpha}) = \tfrac{1}{1-\alpha_*}\left(1 - \tfrac{\alpha^*(2-\alpha_*)-\alpha_*}{1+\alpha^*-2\alpha_*}\right) = \tfrac{1-\alpha^*}{1+\alpha^*-2\alpha_*} = \tfrac{\rho(\beta^*,\beta_*)}{2-\rho(\beta^*,\beta_*)} \leq \tfrac{\bar{\beta}}{2-3\bar{\beta}}.$$

■

13.2.3 General constraints on squared variables

Let us consider now the quadratic optimization problems in the following form:

$$\begin{aligned} \text{find } \phi^* &= \max\{\langle Qx, x\rangle :\ [x]^2 \in \mathcal{F}\}, \\ \text{find } \phi_* &= \min\{\langle Qx, x\rangle :\ [x]^2 \in \mathcal{F}\}. \end{aligned} \tag{13.2.16}$$

where $\mathcal{F}$ is a closed convex set, which satisfies Assumption 13.2.1. Let us show that all results of Sections 13.2.1, 13.2.2 can be easily applied to the problem (13.2.16). Denote by $\xi(u)$ the support function of the set $\mathcal{F}$:

$$\xi(u) = \max\{\langle u, v\rangle :\ v \in \mathcal{F}\}.$$

Theorem 13.2.3 *The statements of Theorems 13.2.1, 13.2.2 are valid for the problem (13.2.16) with the relaxation values* ψ^*, ψ_* *and* τ^*, τ_* *defined as follows:*

$$\begin{aligned} \psi^* &= \min_u \{\xi(u) : \text{ Diag}\,(u) \succeq Q\}, \\ \psi_* &= \max_u \{-\xi(u) : \ Q + \text{Diag}\,(u) \succeq 0)\}, \\ \tau^* &= \xi(\text{diag}(Q)), \quad \tau_* = -\xi(-\text{diag}(Q)). \end{aligned} \tag{13.2.17}$$

Proof.
In order to prove the theorem we need to rewrite the problem (13.2.16) in a conic form. Note that in view of Assumption 13.2.1 the set $\mathcal{F}$ can be represented in the following form:

$$\mathcal{F} = \{v \in S : \ Bv = d\},$$

where S is a bounded convex set with non-empty interior, B is a non-degenerate $(m \times n)$-matrix and $d \in R^m$. Without loss of generality we can assume that

$$\{v \in \text{int}\, S : \ Bv = d\} \neq \emptyset.$$

We allow also $B = 0$; in this case $d = 0$.

Let us consider a conic hull of the set S:

$$K = \{(v, \tau) : \ \tau > 0, \ \tfrac{1}{\tau} v \in S\} \bigcup \{0\}.$$

In view of our assumptions K is a closed convex cone with non-empty interior. The cone dual to K can be represented as follows (see, for example, [345]):

$$K^* = \{\hat{u} = (u, \mu) : \ \mu \geq \xi_S(-u)\},$$

where $\xi_S(\cdot)$ is a support function of the set S.

Now we can rewrite the problem (13.2.16) in the following form:

$$\begin{aligned} \text{find } \phi^* = &\quad \max\{\langle \hat{Q}\hat{x}, \hat{x}\rangle : \ [\hat{x}]^2 \in \hat{\mathcal{F}}\}, \\ \text{find } \phi_* = &\quad \min\{\langle \hat{Q}\hat{x}, \hat{x}\rangle : \ [\hat{x}]^2 \in \hat{\mathcal{F}}\}, \end{aligned} \tag{13.2.18}$$

where $\hat{x} \in R^{n+1}$, $\hat{Q} = \begin{pmatrix} Q & 0_n \\ 0_n^T & 0 \end{pmatrix}$, $\hat{\mathcal{F}} = \{z = (v, \tau) \in K : \ \hat{A}z = b\}$ and

$$\hat{A} = \begin{pmatrix} B & -d \\ 0_n^T & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0_m \\ 1 \end{pmatrix}.$$

Note that the problems in (13.2.18) satisfy Assumptions 13.2.1, 13.2.2. Therefore for their relaxation values ψ^* and ψ_* all statements of Theorems

13.2.1, 13.2.2 are valid. Let us find the expressions for ψ^*, ψ_*, τ^* and τ_* in terms of the initial objects of the problem (13.2.16). It is clear that

$$\begin{aligned}
\tau^* &= \max\{\langle \operatorname{diag}(\hat{Q}), z\rangle :\ z \in \hat{\mathcal{F}}\} \\
&= \max\{\langle \operatorname{diag}(Q), v\rangle :\ Bv = \tau d,\ \tau = 1,\ v/\tau \in S\} = \xi(\operatorname{diag}(Q)), \\
\tau_* &= \min\{\langle \operatorname{diag}(\hat{Q}), z\rangle :\ z \in \hat{\mathcal{F}}\} \\
&= \min\{\langle \operatorname{diag}(Q), v\rangle :\ Bv = \tau d,\ \tau = 1,\ v/\tau \in S\} = -\xi(-\operatorname{diag}(Q)).
\end{aligned}$$

Further, in view of Lemma 13.2.2 the upper relaxation value ψ^* can be represented as follows:

$$\begin{aligned}
\psi^* &= \min_{\hat{y} \in R^{m+1}, \hat{u} \in R^{n+1}} \{\langle b, \hat{y}\rangle :\ \hat{Q} + \operatorname{Diag}(\hat{u}) \preceq \operatorname{Diag}(\hat{A}^T \hat{y}),\ \hat{u} \in K^*\} \\
&= \min_{(y,\gamma) \in R^{m+1}, (u,\mu) \in R^{n+1}} \{\gamma :\ Q + \operatorname{Diag}(u) \preceq \operatorname{Diag}(B^T y),\ \mu \\
&\qquad \leq \gamma - \langle d, y\rangle,\ \mu \geq \xi_S(-u)\} \\
&= \min_{u,y} \{\xi_S(-u) + \langle d, y\rangle :\ \operatorname{Diag}(B^T y - u) \succeq Q\} \\
&= \min_{u,y} \{\xi_S(u - B^T y) + \langle d, y\rangle :\ \operatorname{Diag}(u) \succeq Q\}.
\end{aligned}$$

Note that in the last expression y does not enter the constraints. Therefore we can replace the objective function of this problem by its minimum in y. That is

$$\begin{aligned}
\min_y \{\xi_S(u - B^T y) + \langle d, y\rangle\} &= \min_y \max_{v \in S} \{\langle u - B^T y, v\rangle + \langle d, y\rangle\} \\
&= \max_{v \in S} \min_y \{\langle u, v\rangle + \langle d - Bv, y\rangle\} \\
&= \max_{v \in S} \{\langle u, v\rangle :\ Bv = d\} = \xi(u).
\end{aligned}$$

Thus, we get the representation (13.2.17) for ψ^*. The representation of ψ_* can be obtained in a similar way:

$$\begin{aligned}\psi^* &= \max_{\hat{y}\in R^{m+1},\hat{u}\in R^{n+1}} \{\langle b,\hat{y}\rangle :\ \hat{Q}\succeq \operatorname{Diag}(\hat{u}) + \operatorname{Diag}(\hat{A}^T\hat{y}),\ \hat{u}\in K^*\}\\
&= \max_{(y,\gamma)\in R^{m+1},(u,\mu)\in R^{n+1}} \{\gamma :\ Q\succeq \operatorname{Diag}(u) + \operatorname{Diag}(B^T y),\\
&\qquad 0\geq \mu+\gamma-\langle d,y\rangle,\ \mu\geq \xi_S(-u)\}\\
&= \max_{u,y}\{-\xi_S(-u)+\langle d,y\rangle :\ Q\succeq \operatorname{Diag}(u+B^T y)\}\\
&= \max_{u,y}\{-\xi_S(B^T y+u)+\langle d,y\rangle :\ Q+\operatorname{Diag}(u)\succeq 0\}\\
&= \max_{u}\{-\min_{y}\{\xi_S(B^T y+u)-\langle d,y\rangle\} :\ Q+\operatorname{Diag}(u)\succeq 0\}\\
&= \max_{u}\{-\xi(u) :\ Q+\operatorname{Diag}(u)\succeq 0\}.\end{aligned}$$

■

Let us present an example of application of Theorems 13.2.3, 13.2.2. Consider the following problem:

$$\begin{aligned}\text{find } \phi^* &= \max\{\langle Qx_1,x_2\rangle :\ [(x_1,x_2)]^2\in\mathcal{F}\},\\
\text{find } \phi_* &= \min\{\langle Qx_1,x_2\rangle :\ [(x_1,x_2)]^2\in\mathcal{F}\},\end{aligned}\tag{13.2.19}$$

where Q is a $(k\times n)$-matrix, $x_1\in R^k$, $x_2\in R^n$ and $\mathcal{F}$ is a closed convex set, which satisfies Assumption 13.2.1. Since the quadratic objective function in this problem is bilinear, we conclude that $\phi_* = -\phi^*$ and $\tau^* = \tau_* = 0$.

The conic relaxation for this problem is defined as follows:

$$\begin{aligned}\psi^* &= \min_{u=(u_1,u_2)}\left\{\xi(u) :\ \begin{pmatrix}\operatorname{Diag}(u_1) & -Q^T\\ -Q & \operatorname{Diag}(u_2)\end{pmatrix}\succeq 0\right\},\\
\psi_* &= \max_{u=(u_1,u_2)}\left\{-\xi(u) :\ \begin{pmatrix}\operatorname{Diag}(u_1) & Q^T\\ Q & \operatorname{Diag}(u_2)\end{pmatrix}\succeq 0\right\}.\end{aligned}$$

It is clear that $\psi_* = -\psi^*$. At the same time, $\beta^* = \beta_* = \frac{1}{2}$. Therefore,

$$\alpha^* = \max\{\tfrac{2}{\pi}\omega(\beta_*), 1-\beta^*\} = \tfrac{2}{\pi}\omega(\tfrac{1}{2}).$$

Therefore, in view of Theorem 13.2.2 we have:

$$\psi^*\geq\phi^*\geq\psi(\alpha^*) = (2\alpha^*-1)\psi^*.$$

Note that $\alpha^* = \frac{2}{\pi}(\frac{1}{2}\arcsin\frac{1}{2}+\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{\pi}+\frac{1}{6}$. Thus, we have proved the following theorem.

Theorem 13.2.4 *In the problem (13.2.19) the optimal and relaxation values are related as follows:*

$$\psi^* \geq \phi^* \geq \gamma\psi^*$$

with $\gamma = \frac{2\sqrt{3}}{\pi} - \frac{2}{3} > 0.43$.

13.2.4 Why the linear constraints are difficult?

In the previous sections we have got a constant relative accuracy estimates for a quadratic maximization problem with convex constraints on *squared* variables. Such type of constraints are rather specific. Therefore it is natural to try to extend the results onto the problems with convex constraints on the variables of the quadratic form. However, it appears that this is not trivial. In this section we show that even a single linear constraint can make a quadratic problem completely intractable by the presented technique.

Consider the following optimization problem:

$$\begin{aligned} \phi^* = \quad & \max \quad \langle Qx, x\rangle, \\ \text{s.t.} \quad & x \in \{-1, 1\}^n, \\ & \langle c, x\rangle = \beta, \end{aligned} \tag{13.2.20}$$

where Q is an $(n \times n)$-matrix, $c \in R^n$ and $\beta > 0$. Define ϕ_* as a minimal value of the objective function in (13.2.20). A natural relaxation for this problem is as follows:

$$\psi^* = \max\{\langle Q, X\rangle : \langle Xc, c\rangle = \beta^2, \ \mathrm{diag}(X) = 1_n, \ X \succeq 0\}. \tag{13.2.21}$$

Let us show that this relaxation can be arbitrary bad in terms of relative accuracy.

Denote by v_i, $i = 1, \ldots, 2^n$ the nodes of the boolean unit box $\{-1, 1\}^n$. Let us assume that there exists only one node v_*, which satisfies the linear constraint of the problem (13.2.20). Moreover, let us assume that there are two other nodes, v_+ and v_- such that

$$0 < \langle c, v_-\rangle < \beta < \langle c, v_+\rangle. \tag{13.2.22}$$

Note that in view of our assumption we have $\phi^* = \phi_*$ independently on our choice of the matrix Q.

Let us define a convex polytope $\mathcal{P}_n$ of positive semidefinite $(n \times n)$-matrices:

$$\mathcal{P}_n = \mathrm{Conv}\,\{V_i = v_i v_i^T, \ i = 1, \ldots, 2^n\}.$$

Lemma 13.2.4 *Any* V_i *is an extreme point of* $\mathcal{P}_n$. *Any pair of nodes* V_i, V_j *is connected by an exposed edge.*

Proof.

Since V_i is a rank-one matrix, the first statement is evident. In order to prove the second statement note that the edge $[V_i, V_j]$ is not exposed if and only if there exist some coefficients $\lambda_k > 0$, $k \in \mathcal{I}$, $i, j \notin \mathcal{I}$ such that

$$\alpha V_i + (1-\alpha)V_j = \sum_{k \in \mathcal{I}} \lambda_k V_k, \quad \sum_{k \in \mathcal{I}} \lambda_k = 1,$$

for some $\alpha \in (0,1)$. Since all nodes of $\mathcal{P}_n$ are positive semidefinite rank-one matrices, we conclude that

$$v_k \in \{v : \ v = \alpha v_i + \beta v_j, \ (\alpha, \beta) \in R^2\}, \quad \forall k \in \mathcal{I}.$$

A simple calculation shows that it is possible only for $v_k = \pm v_i$ or $v_k = \pm v_j$. ■

Note that in view of our assumption (13.2.22) there exists a matrix $\tilde{V} \in \mathcal{P}_n$ such that

$$\tilde{V} = \alpha v_- v_-^T + (1-\alpha) v_+ v_+^T, \ \alpha \in (0,1), \quad \langle \tilde{V} c, c \rangle = \beta^2.$$

Let us choose now $Q = \tilde{V} - v_* v_*^T$. Note that the feasible set of the relaxation problem (13.2.21) contains $\mathcal{P}_n$. Therefore

$$\psi^* \geq \langle Q, \tilde{V} \rangle > \langle Q, v_* v_*^T \rangle = \phi^*.$$

The lower relaxation value ψ_* never exceed $\phi_* = \phi^*$. Therefore, for our example the value $\psi^* - \psi_*$ is strictly positive. This means that the relative accuracy of the value ψ^* is infinitely bad.

Note that the main source of our troubles in the above example is that the linear constraint $\langle Xc, c \rangle = \beta^2$ intersects an edge of the matrix polytope $\mathcal{P}_n$. That can happen with any value of β except $\beta = 0$. Thus, we still can hope that for the problems with homogeneous linear constraints the conic relaxation can work. In the next sections we will see some problems, for which it is true.

13.2.5 Maximization with a smooth constraint

In the previous section we have established some constant bounds on relative accuracy of the conic relaxations (13.2.4) for a quadratic maximization problem with convex constraints for the squared variables. At the same time, in Section 13.2.4 we have seen that some linear constraints on the initial variables can make the problem intractable in terms of relative accuracy. In this section we present another approach for deriving the conic relaxations. This approach is based on the standard second order optimality conditions and it allows to treat the quadratic maximization problems over l_p-boxes, $p \geq 2$, with homogeneous linear equality constraints (see Section 13.2.6). However, the quality of relaxation in this framework becomes dependent on p.

Let $f(y)$, $y \in R^m$, be a homogeneous function of degree p:

$$f(\tau y) = \tau^p f(y), \quad y \in R^m, \ \tau \geq 0. \tag{13.2.23}$$

We assume that $f(y)$ is non-negative and twice continuously differentiable at any non-zero point of R^m (notation $f \in H_p$). Recall, that for homogeneous functions we have the following simple relations.

Lemma 13.2.5 *If $f(y)$ is homogeneous of degree p then for any $y \in R^m$ and $\tau \geq 0$ we have*

$$\begin{aligned} f'(\tau y) &= \tau^{p-1} f'(y), & (13.2.24)\\ f''(y)y &= (p-1)f'(y), & (13.2.25)\\ \langle f'(y), y\rangle &= pf(y), & (13.2.26)\\ \langle f''(y)y, y\rangle &= p(p-1)f(y). & (13.2.27) \end{aligned}$$

Proof.
Indeed, if we differentiate (13.2.23) in y we get (13.2.24). If we differentiate (13.2.24) in τ and take $\tau = 1$ we get (13.2.25). In order to get (13.2.26) we differentiate (13.2.23) in τ and take $\tau = 1$. Finally, (13.2.25) and (13.2.26) give (13.2.27). ■

Let Q be a symmetric $(m \times m)$-matrix. Consider the following maximization problem:

$$\text{find } \phi^*(Q) = \max\{\langle Qy, y\rangle : \ f(y) \leq 1\}. \qquad (13.2.28)$$

If $Q \preceq 0$ then (13.2.28) is a concave maximization problem and $\phi^*(Q) = 0$. In the other cases we need some necessary conditions to characterize the local solutions of the problem (13.2.28).

Lemma 13.2.6 *Let $f \in H_p$ with $p > 0$. Then for any local maximum y_* of the problem (13.2.28) with $\langle Qy_*, y_*\rangle > 0$ we have $f(y_*) = 1$. Moreover, there exists a value $\lambda = \lambda(y_*) > 0$ such that*

$$\begin{aligned} \langle Qy_*, y_*\rangle &= p\lambda, & (13.2.29)\\ Qy_* &= \lambda f'(y_*), & (13.2.30)\\ Q &\preceq \lambda\left[f''(y_*) - \frac{p-2}{p} f'(y_*)f'(y_*)^T\right]. & (13.2.31) \end{aligned}$$

Proof.
Since $\langle Qy_*, y_*\rangle > 0$ and $f(y)$ is a homogeneous function of positive degree, we necessarily have $f(y_*) = 1$. Let us write down a Lagrangean for this problem:

$$\mathcal{L}(y, \lambda) = \tfrac{1}{2}\langle Qy, y\rangle - \lambda[f(y) - 1].$$

Then, the second order necessary conditions for the problem (13.2.28) can be written as follows:

$$\begin{aligned} \mathcal{L}'_y(y_*, \lambda) &= 0, & (13.2.32)\\ \langle \mathcal{L}''_{yy}(y_*, \lambda)h, h\rangle &\preceq 0, \quad \forall h : \ \langle f'(y_*), h\rangle = 0, & (13.2.33) \end{aligned}$$

with some $\lambda \in R$. Equation (13.2.32) is exactly (13.2.30). Multiplying (13.2.30) by y_* and using (13.2.26) we get

$$\langle Qy_*, y_* \rangle = \lambda \langle f'(y_*), y_* \rangle = \lambda p f(y_*) \lambda p > 0,$$

and that is (13.2.29). Finally, since $\langle f'(y_*), y_* \rangle = p > 0$, any $h \in R^m$ such that $\langle f'(y_*), h \rangle = 0$ can be represented in the form

$$h = \left(I - \tfrac{1}{p} y_* f'(y_*)^T \right) u, \quad u \in R^n.$$

Therefore the condition (13.2.33) can be rewritten as

$$(I - \tfrac{1}{p} f'(y_*) y_*^T) \mathcal{L}''_{yy}(y_*, \lambda)(I - \tfrac{1}{p} y_* f'(y_*)^T) \preceq 0. \tag{13.2.34}$$

Note that $\mathcal{L}''_{yy}(y_*, \lambda) = Q - \lambda f''(y_*)$ and

$$\begin{aligned}
&(I - \tfrac{1}{p} f'(y_*) y_*^T) Q (I - \tfrac{1}{p} y_* f'(y_*)^T) \\
&= Q - \tfrac{1}{p} f'(y_*) y_*^T Q - \tfrac{1}{p} Q y_* f'(y_*)^T + \tfrac{1}{p^2} \langle Qy_*, y_* \rangle f'(y_*) f'(y_*)^T \\
&= Q - \tfrac{\lambda}{p} f'(y_*) f'(y_*)^T
\end{aligned}$$

in view of (13.2.30) and (13.2.29). Similarly, since $f(y_*) = 1$ we have

$$\begin{aligned}
(I- &\tfrac{1}{p} f'(y_*) y_*^T) f''(y_*) (I - \tfrac{1}{p} y_* f'(y_*)^T) \\
&= f''(y_*) - \tfrac{1}{p} f'(y_*) y_*^T f''(y_*) - \tfrac{1}{p} f''(y_*) y_* f'(y_*)^T \\
&+ \tfrac{1}{p^2} \langle f''(y_*) y_*, y_* \rangle f'(y_*) f'(y_*)^T \\
&= f''(y_*) - 2 \tfrac{p-1}{p} f'(y_*) f'(y_*)^T + \tfrac{p(p-1)}{p^2} f'(y_*) f'(y_*)^T \\
&= f''(y_*) - \tfrac{p-1}{p} f'(y_*) f'(y_*)^T,
\end{aligned}$$

in view of (13.2.25) and (13.2.27). Substituting these expressions in (13.2.34) we get

$$\begin{aligned}
Q &\preceq \lambda f''(y_*) + \tfrac{\lambda}{p} f'(y_*) f'(y_*)^T - \lambda \tfrac{p-1}{p} f'(y_*) f'(y_*)^T \\
&= \lambda \left[f''(y_*) - \tfrac{p-2}{p} f'(y_*) f'(y_*)^T \right].
\end{aligned}$$

■

We will use Lemma 13.2.6 in order to estimate the quality of relaxations for some non-convex maximization problems. Let $A = (a_1, \ldots, a_n) \in R^{m \times n}$ be a non-degenerate $(m \times n)$-matrix. Consider the following function:

$$f_A(y) = \sum_{i=1}^{n} | \langle a_i, y \rangle |^p,$$

where $p \geq 2$. The problem we are going to address now is as follows:

$$\text{find } \phi^*(Q, A) = \max\{\langle Qy, y\rangle : \; f_A(y) \leq 1\}. \tag{13.2.35}$$

For this problem we can introduce the following relaxation:

$$\psi_p^*(Q, A) = \min_u\{\| u \|_q \colon \; A\mathrm{Diag}\,(u)A^T \succeq Q\}, \tag{13.2.36}$$

where $q = (\frac{p}{2})^* = \frac{p}{p-2}$ (compare with (13.2.17)). Now we can prove the main result of this section.

Theorem 13.2.5 *Let the feasible set of the problem (13.2.35) be bounded. Then*

$$\frac{1}{p-1}\psi_p^*(Q, A) \leq \phi^*(Q, A) \leq \psi_p^*(Q, A). \tag{13.2.37}$$

Moreover, any local maximum y_* of the problem (13.2.35) with positive value of the objective function satisfies inequality $\langle Qy_*, y_*\rangle \geq \frac{1}{p-1}\psi_p^*(Q, A)$.

Proof.
Indeed, let u be feasible for the problem (13.2.36). Then for any $y \in R^m$ with $f_A(y) \leq 1$ we have

$$\langle Qy, y\rangle \leq \langle A\mathrm{Diag}\,(u)A^T y, y\rangle = \langle u, [A^T y]^2\rangle.$$

At the same time,

$$\| \, [A^T y]^2 \, \|_{p/2}^{p/2} = \sum_{i=1}^{n} | \, \langle a_i, y\rangle \, |^p = f_A(y) \leq 1.$$

Therefore, for any feasible y we have

$$\langle Qy, y\rangle \leq \langle u, [A^T y]^2\rangle \leq \| \, u \, \|_q \cdot \| \, [A^T y]^2 \, \|_{p/2} \leq \| \, u \, \|_q \, .$$

Hence, $\phi^*(Q, A) \leq \psi_p^*(Q, A)$.

On the other hand, let y_* be a local maximum of (13.2.35) with $\langle Qy_*, y_*\rangle > 0$. Then, in view of Lemma 13.2.6 (13.2.31) for $\lambda = \lambda(y_*)$ we have:

$$Q \preceq \lambda f''(y_*) = p(p-1)\lambda \sum_{i=1}^{n} | \, \langle a_i, y_*\rangle \, |^{p-2} \, a_i a_i^T$$

(we have used the condition $p \geq 2$). Thus, the vector $u \in R^n$ with the components

$$u^{(i)} = p(p-1)\lambda \, | \, \langle a_i, y_*\rangle \, |^{p-2}, \; i = 1, \ldots, n,$$

is feasible for the problem (13.2.36). Note that

$$\begin{aligned} \| \, u \, \|_q &= p(p-1)\lambda \left[\sum_{i=1}^{n} | \, \langle a_i, y_*\rangle \, |^{(p-2)q} \right]^{1/q} \\ &= p(p-1)\lambda \left[\sum_{i=1}^{n} | \, \langle a_i, y_*\rangle \, |^{p} \right]^{1/q} \\ &= p(p-1)\lambda [f_A(y_*)]^{1/q} = p(p-1)\lambda. \end{aligned}$$

Hence, in view of (13.2.29) we have

$$\langle Qy_*, y_* \rangle = p\lambda = \frac{\| u \|_q}{p-1} \geq \frac{1}{p-1}\psi_p^*(Q, A).$$

Note that the above proof shows that under assumptions of the theorem the function $\psi_p^*(Q, A)$ is well defined.

Finally, if there is no local maximum of the problem (13.2.35) with $\langle Qy_*, y_* \rangle > 0$, then $Q \preceq 0$ and in this case we have $\psi_p^*(Q, A) = \phi^*(Q, A) = 0$. ■

Let us estimate the relative accuracy of the relaxation (13.2.36). First, we need the following trivial result.

Lemma 13.2.7 *Let for some non-negative values ϕ, ψ and γ we have the following relations:*

$$\gamma\psi \leq \phi \leq \psi.$$

Then, for $\beta = \frac{2\gamma}{1+\gamma}$ we have: $| \beta\psi - \phi | \leq (1-\beta)\phi$.

Define $\phi_*(Q, A) = \min\{\langle Qy, y\rangle : \ f_A(y) \leq 1\}$.

Theorem 13.2.6 *Let $\psi^* = \frac{2}{p}\psi_p^*(Q, A)$. Then*

$$| \phi^*(Q, A) - \psi^* | \leq (1 - \tfrac{2}{p})(\phi^*(Q, A) - \phi_*(Q, A)). \tag{13.2.38}$$

Proof.
Note that $\phi_*(Q, A) \leq 0$. Therefore it is sufficient to prove

$$| \phi^*(Q, A) - \psi^* | \leq (1 - \tfrac{2}{p})\phi^*(Q, A).$$

Let us choose $\gamma = \frac{1}{p-1}$ and $\beta = \frac{2\gamma}{1+\gamma} = \frac{2}{p}$. Then the above inequality follows from Theorem 13.2.5 and Lemma 13.2.7. ■

Let us compare now the relaxation (13.2.36) with the conic relaxation (13.2.17). Of course, we have to choose a problem which can be treated by both approaches. Consider the problem

$$\max\{\langle Qx, x\rangle : \| x \|_p \leq 1\}, \quad p \geq 2.$$

This problem can be presented in the form (13.2.35) with $A = I_n$. On the other hand, it can be written in the form (13.2.16) with

$$\mathcal{F} = \{v : \| v \|_{p/2} \leq 1\}.$$

In this case $\xi(u) = \| u \|_q$ and we can see that (13.2.36) coincides with (13.2.17).

13.2.6 Some applications

Let us show that the results of the previous section can be extended onto the problems with linear equality constraints. Consider the following quadratic maximization problem:

$$\begin{aligned} \text{find } \phi_p^* = \quad & \max_{x \in R^n} \quad \langle Cx, x \rangle, \\ \text{s.t.} \quad & \| x \|_p \leq 1, \\ & Bx = 0, \end{aligned} \tag{13.2.39}$$

where C is an arbitrary $(n \times n)$-matrix, $p \geq 2$ and B is a non-degenerate $((n-m) \times n)$-matrix with $n > m$. Let the rows of some $(m \times n)$-matrix A span the null space of the matrix B:

$$BA^T y = 0, \quad \forall y \in R^m.$$

Then we can change variables $x = A^T y$ and obtain a problem, which is equivalent to (13.2.39):

$$\phi_p^* = \max_{y \in R^m} \{ \langle ACA^T y, y \rangle : \ f_A(y) \leq 1 \} = \phi^*(ACA^T, A).$$

Thus, in view of Theorem 13.2.5 and Lemma 13.2.7 we get the following result.

Theorem 13.2.7 *For any* $p \geq 2$ *we have*

$$\frac{1}{p-1} \psi_p^*(ACA^T, A) \leq \phi_p^* \leq \psi_p^*(ACA^T, A).$$

The value $\psi^* = \frac{2}{p} \psi_p^*(ACA^T, A)$ *approximates the solution of the problem (13.2.39) with* $(1 - \frac{2}{p})$ *relative accuracy.*

Now, let us consider the case when the objective function of the problem (13.2.39) has a non-zero linear term:

$$\begin{aligned} \text{find } \hat{\phi}_p^* = \quad & \max_{x \in R^n} \quad \langle Cx, x \rangle + 2\langle c, x \rangle, \\ \text{s.t.} \quad & \| x \|_p \leq 1, \\ & Bx = 0. \end{aligned} \tag{13.2.40}$$

This problem can be homogenized in a standard way:

$$\begin{aligned} & \max_{(x,\tau) \in R^{n+1}} \quad \langle Cx, x \rangle + 2\tau \langle c, x \rangle, \\ \text{s.t.} \quad & \| x \|_p \leq 1, \ | \tau | \leq 1, \\ & Bx = 0. \end{aligned} \tag{13.2.41}$$

Clearly, the optimal value of this problem is $\hat{\phi}_p^*$. However, this problem has two separate constraints for x and τ. Therefore, in order to apply the results of Section 13.2.5 we need to replace them by a single functional inequality. Consider the following problem:

$$\begin{aligned} \text{find } \bar{\phi}_p^* = \quad & \max_{(x,\tau)\in R^{n+1}} \quad \langle Cx, x\rangle + 2\tau\langle c, x\rangle, \\ \text{s.t.} \quad & \| (x,\tau) \|_p \leq 1, \\ & Bx = 0, \end{aligned} \tag{13.2.42}$$

Denote by $\bar{\psi}_p^*$ the value of the conic relaxation for the last problem.

Theorem 13.2.8 *Let $p \geq 2$. For $\psi_a^* = 2^{2/p}\bar{\psi}_p^*$ we have:*

$$\frac{1}{2^{2/p}(p-1)}\psi_a^* \leq \hat{\phi}_p^* \leq \psi_a^*.$$

The value $\psi_r^* = \frac{2}{p+2^{-2/p}-1}\bar{\psi}_p^*$ has at least $(1-\frac{1}{2p})$ relative accuracy.

Proof.
Note that the problems (13.2.41) and (13.2.42) have the same objective function and the same system of linear equations. Denote by $\mathcal{F}_0$ the feasible set of the problem (13.2.42) and by $\mathcal{F}_1$ the feasible set of the problem (13.2.41). Clearly, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset 2^{1/p}\mathcal{F}_0$. Therefore

$$\bar{\phi}_p^* \leq \hat{\phi}_p^* \leq 2^{2/p}\bar{\phi}_p^*.$$

On the other hand, in view of Theorem 13.2.7, we have:

$$\frac{1}{p-1}\bar{\psi}_p^* \leq \bar{\phi}_p^* \leq \bar{\psi}_p^*.$$

Hence, for $\psi_a^* = 2^{2/p}\bar{\psi}_p^*$ we obtain:

$$\psi_a^* = 2^{2/p}\bar{\psi}_p^* \geq 2^{2/p}\bar{\phi}_p^* \geq \hat{\phi}_p^* \geq \bar{\phi}_p^* \geq \tfrac{1}{p-1}\bar{\psi}_p^* = \tfrac{1}{2^{2/p}(p-1)}\psi_a^*.$$

In order to get the statement on the relative accuracy, we take $\psi = \psi_a^*$, $\phi = \hat{\phi}_p^*$, $\gamma = \frac{1}{2^{2/p}(p-1)}$ and apply Lemma 13.2.7. Then the values β and ψ_r^* can be obtained as follow:

$$\begin{aligned} \beta \quad &= \tfrac{2\gamma}{1+\gamma} = \tfrac{2}{1+2^{2/p}(p-1)} \geq \tfrac{1}{2p}, \\ \psi_r^* \quad &= \beta\psi_a^* = \tfrac{2}{p+2^{-2/p}-1}\bar{\psi}_p^*. \end{aligned}$$

■

We see that the quality of conic relaxation decreases as p increase. Therefore, we cannot directly apply the results of Section 13.2.5 to a problem with box constraints. However, at the same time, when p increase the shape of l_p balls becomes very close to the shape of the n-dimensional unit box. Therefore, we can use the values $\psi_p^*(ACA^T, A)$ with p large enough in order to get some bounds for ϕ_∞^*.

Theorem 13.2.9 *Let* $p = 2\ln n$, $\psi_a^* = e\psi_p^*(ACA^T, A)$ *and* $\gamma = \frac{1}{e(2\ln n - 1)}$. *Then*

$$\gamma\psi_a^* \leq \psi_\infty^* \leq \psi_a^*.$$

The value $\psi_r^* = \frac{2\gamma}{1+\gamma}\psi_a^*$ *has at least* $(1 - \frac{1}{e\ln n})$ *relative accuracy.*

Proof.
It is well known that for any two values $p \geq 2$ we have:

$$\frac{1}{n^{1/p}} \parallel x \parallel_p \leq \parallel x \parallel_\infty \leq \parallel x \parallel_p, \quad x \in R^n.$$

Therefore

$$\{x \in R^n : \parallel x \parallel_p \leq 1\} \subset \{x \in R^n : \parallel x \parallel_\infty \leq 1\} \subset \{s \in R^n : \parallel x \parallel_p \leq n^{1/p}\}.$$

Since the objective function of the problem (13.2.39) is homogeneous of degree two, this implies that $\phi_p^* \leq \phi_\infty^* \leq n^{2/p}\phi_p^*$. Thus, using Theorem 13.2.7 we obtain the following:

$$\begin{aligned} \psi_\infty^* &= e\psi_p^*(ACA^T, A) = n^{2/p}\psi_p^*(ACA^T, A) \geq n^{2/p}\phi_p^* \\ &\geq \phi_\infty^* \geq \phi_p^* \geq \tfrac{1}{p-1}\psi_p^*(ACA^T, A) = \tfrac{1}{e(2\ln n - 1)}\psi^*. \end{aligned}$$

In order to get the statement on relative accuracy we apply Lemma 13.2.7 with

$$\beta = \frac{2\gamma}{1+\gamma} = \frac{2}{1 + e(2\ln n - 1)} > \frac{1}{e\ln n}.$$

■

13.2.7 Discussion

In the previous sections we have presented some estimates for the quality of the conic relaxation for different non-convex quadratic maximization problems. The constant bounds of Sections 13.2.1, 13.2.2 can be applied to a quite large class of non-convex problems and we can expect that they can be used in many practical applications. The bounds we get in Section 13.2.5 are not so good. Indeed, they can be applied only to a rather special feasible set, that is an intersection of an l_p-ball, $p \geq 2$, with a linear subspace. Moreover, the quality of these bounds decrease as p increase.

Nevertheless, the results of Section 13.2.5 suggest some interesting conclusions. Firstly, the relative accuracy we get from the relaxation (13.2.36) is $(1 - \frac{2}{p})$. Thus, the accuracy goes to zero as p approaches two. For p small enough the results of Theorem 13.2.5 become even better than the bounds of Section 13.2.1. An important advantage of the estimates (13.2.37) is that we get the separate bounds for the minimal and the maximal value of the problem. The lower estimate for the maximal value remains positive even if the minimal value of the problems is a large negative value.

Secondly, Theorem 13.2.5 tells us that the value of the objective function of the problem (13.2.35) at *any* local solution is not worse than the lower bound we get from the conic relaxation. In fact, this statement is a kind of surprise. Indeed, if we measure a hardness of a problem as a largest ratio of the values of the objective function at the global and a local maximum, it appears that the problem (13.2.35) is not so difficult, at least for p small enough. Usually the general methods of nonlinear optimization are quite efficient in finding a local solution. Since the computational cost of such schemes is much less than that of the schemes of semidefinite programming, we can conclude that for practical applications the traditional schemes look quite attractive.[1]

Finally, in Section 13.2.6 we have shown that the results of Theorem 13.2.5 provides us with some bounds for very difficult problems. Indeed, during last years there were obtained many negative results related to the possibilities to find an approximate solution of an NP-hard problem under hypothesis that $P \neq NP$. The results relevant to the topic of our section can be found in [79]:

> Consider a quadratic optimization problem in the following form:
>
> $$\max\{\langle Cx, x\rangle : \; Bx \leq b, \; 0 \leq x \leq 1_n\}. \tag{13.2.43}$$
>
> Denote by $\widetilde{P}$ the class of languages recognizable in quasi-polynomial time.
> **Theorem 1.2.** *Assume $NP \not\subseteq \widetilde{P}$. Then there is a constant $\delta > 0$ such that the problem (13.2.43) has no polynomial time, $(1 - 2^{-\log^{\delta} n})$-approximation algorithm.*
> **Theorem 1.3.** *Assume $P \neq NP$. Then there is a constant $\mu \in (0, \frac{1}{3})$ such that a μ-approximation of the problem (13.2.43) cannot be found in polynomial time.*

In these statements the μ-approximation is understood in a weak sense. We need to compute an estimate for the value of the objective function only.

Note that using Theorems 13.2.8 and 13.2.9, we can approximate in polynomial time the optimal value of the problem

$$\max\{\langle Cx, x\rangle : \; Bx = \tfrac{1}{2}1_n, \; 0 \leq x \leq 1_n\}. \tag{13.2.44}$$

with $(1 - O(\frac{1}{\ln n}))$ relative accuracy. This result is better than the limiting bound of Theorem 1.2 [79]. At the same time, the optimization problem, which

[1] Of course, in non-convex case we cannot prove any global efficiency estimates. Moreover, in general we cannot guarantee a convergence to a point, which satisfies the necessary second order optimality conditions. This negative result is valid even for the second order methods.

is used in the proof of Theorems 1.2, 1.3 [79], has, in fact, only linear equalities constraint:

$$\max\{\langle Cx, x\rangle : \; Bx = b, \; 0 \leq x \leq 1_n\}. \tag{13.2.45}$$

Thus, the difference in the formulations (13.2.44) and (13.2.45) looks very minor. Indeed, any system of linear equations $Bx = b$ can be rewritten in the following form:

$$\bar{B}x = \tfrac{1}{2}1_n, \quad \langle a, x\rangle = 1,$$

with some matrix $\bar{B}$ and a vector $a \in R^n$. Hence, the feasible set of the problem (13.2.45) differs from the feasible set of the problem (13.2.44) just by a single linear equation, which does not pass through the center of the box. However, it appears that this linear equation makes the problem (13.2.45) completely different.

Let us look at the concrete form of the problem (13.2.45) ([79], p.438). Denote by X and Y two $(n \times n)$-matrices. And let $\phi(X, Y)$ be a bilinear form in X and Y with all non-negative coefficients. Then the problem (13.2.45) is as follows:

$$\begin{aligned} &\max \; \phi(X, Y), \\ \text{s.t.} \quad & X1_n = 1_n, \; Y1_n = 1_n, \\ & 0 \leq X, Y \leq 1_{n\times n}. \end{aligned} \tag{13.2.46}$$

Now we can see the source of our troubles. Indeed, the technique of Section 13.2.5 can be applied only to l_p boxes with $p \geq 2$. However, if we will try to approximate the feasible set of the problem (13.2.46) with the boxes $\mathcal{B}_p = \{x : \| x - \frac{1}{2}1_n \|_p \leq \frac{1}{2}\}$, we need to choose p very large. It is necessary to take $p = O(n \ln n)$ just to have a non-empty intersection of the box $\mathcal{B}_p$ with the system of linear constraints in (13.2.46).

Thus, we conclude that the feasible set of the problem (13.2.46) is too far from the center of the box. On the other hand, it is clear that the box structure in (13.2.46) is quite artificial: the constraint $X, \, Y \leq 1_{n\times n}$ can be eliminated without changing the feasible set of the problem. Note that we can easily rewrite the problem (13.2.46) in a more symmetric form:

$$\begin{aligned} &\max \; \phi(X, Y), \\ \text{s.t.} \quad & \| Xe_i \|_1 \leq 1, \; i = 1, \ldots n, \\ & \| Ye_i \|_1 \leq 1, \; i = 1, \ldots n. \end{aligned} \tag{13.2.47}$$

Since the coefficients of the form $\phi(X, Y)$ are non-negative, the optimal value of the problem (13.2.47) is the same as that of (13.2.43). The polyhedral structure of the feasible set in (13.2.47) can be seen as a combination of l_∞-structure with l_1-structure. However, it appears the latter structure is exactly that one, for which no reasonable bounds for quadratic problems are known.

Thus, the above discussion highlights the following unsolved problem:

Find some bounds for the optimal value of the following quadratic problem:

$$\phi^* = \max\{\langle Qx, x\rangle : \| x \|_p \leq 1,\ x \in R^n\}, \quad 1 \leq p < 2. \tag{13.2.48}$$

For an indefinite Q a trivial bound for ϕ^* is given by its maximal eigenvalue $\lambda_{\max}(Q)$:

$$\lambda_{\max}(Q) \geq \phi^* \geq \lambda_{\max}(Q) \cdot n^{1-\frac{2}{p}}, \quad 1 \leq p \leq 2.$$

For $p = 1$ we can suggest for the problem (13.2.48) a kind of semidefinite relaxation:

$$\begin{aligned} \psi^* &= \max_{X,u}\{\langle Q, X\rangle : \ \mathrm{Diag}\,(u) \succeq X,\ \langle 1_n, u\rangle \leq 1,\ X \succeq 0\} \\ &= \min_{S,\lambda}\{\lambda : \ \lambda 1_n = \mathrm{diag}(S),\ S \succeq Q,\ S \succeq 0\}. \end{aligned} \tag{13.2.49}$$

Note that for any x, $\| x \|_1 \leq 1$, the pair $(X = xx^T, u = \mathrm{abs}[x])$ is feasible for the primal form of the relaxation (13.2.49). Therefore we can guarantee that $\psi^* \geq \phi^*$. However, the relative accuracy of such a bound is not known.

13.3 QUADRATIC CONSTRAINTS

Yinyu Ye

Consider the quadratic programming (QP) problem with diagonally quadratic equality and inequality constraints

$$(QP) \qquad \begin{array}{rll} \bar{q}(Q) := & \text{Maximize} & q(x) := x^T Q x \\ & \text{Subject to} & \sum_{j=1}^n a_{ij} x_j^2 = b_i,\ i = 1, \ldots, m, \\ & & \sum_{j=1}^n c_{ij} x_j^2 \leq d_i,\ i = 1, \ldots, p \end{array}$$

where the symmetric matrix $Q \in \mathcal{S}^n$, $A = \{a_{ij}\} \in \mathcal{M}_{m,n}$, $C = \{c_{ij}\} \in \mathcal{M}_{p,n}$, $b \in \Re^m$, and $d \in \Re^p$ are given. We assume that the QP problem is feasible and its feasible set is bounded (this can be checked by a linear program considering x_j^2 as nonnegative variables). Let $\bar{x}(Q)$ be a maximizer of the problem.

The (QP) problem has applications in combinatorial and global optimization problems, see, e.g., Gibbons et al. [470]. Note that this quadratic problem