What can be expressed via Conic Quadratic and Semidefinite Programming?

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What can be expressed via CQP and SDP?

- **Let us look at three generic families of convex programs:**
  - **Linear Programming:**

$$c^T x \to \min \mid Ax + b \geq 0 \qquad \textbf{(LP)}$$

  - **Conic Quadratic Programming:**

$$c^T x \to \min \mid \| A_i x + b_i \|_2 \leq c_i^T x + d_i, \quad i = 1, ..., m \qquad \textbf{(CQP)}$$

  - **Semidefinite Programming:**

$$c^T x \to \min \mid \sum_{i=1}^{n} x_i A_i + B \succeq 0 \qquad \textbf{(SDP)}$$

- **Geometrically, all these problems are of the form**

$$c^T x \to \min \mid Ax + b \in \mathbf{K}, \qquad \textbf{(CP)}$$

where K is a closed pointed convex cone with a nonempty interior belonging to a *specific for the generic problem in question* **family $\mathcal{K}$ of convex cones.**

$$c^T x \to \min \mid Ax + b \in \mathbf{K} \quad [\mathbf{K} \in \mathcal{K}] \qquad \textbf{(CP)}$$

<u>LP</u>: $\mathcal{K}$ is comprised of direct products of rays $\mathbf{R}_+$

<u>CQP</u>: $\mathcal{K}$ is comprised of direct products of the Lorentz cones

$$\mathbf{L}^n = \{(x,t) \in \mathbf{R}^{n+1} : \| x \|_2 \leq t\}$$

<u>SDP</u>: $\mathcal{K}$ is comprised of direct products of the semidefinite cones

$$\mathbf{S}^n_+ = \{A \in \mathbf{S}^n : A \succeq 0\}$$

<u>Note:</u> *The above families of cones are closed w.r.t.*

(1) *taking (finite) direct products of cones;*
(2) *passing from a cone* $\mathbf{K}$ *to its dual cone*

$$\mathbf{K}_* = \{\eta \mid \eta^T x \geq 0 \quad \forall x \in \mathbf{K}\}.$$

• **Assume that we are given a family $\mathcal{K}$ of finite-dimensional convex cones (closed, pointed and with a nonempty interior) and know how to solve problems of the form**

$$c^T y \to \min \mid Ay + b \in \mathbf{K} \qquad (*)$$

with all possible data (c, A, b) **and all** $\mathbf{K} \in \mathcal{K}$**.**
Question: *What is the family of problems we can actually solve? When an optimization problem*

$$e^T x \to \min \mid x \in X \subset \mathbf{R}^n \qquad \textbf{(P)}$$

can be equivalently reformulated in the form of $(*)$*?*

An answer: This is the case when X **is a** $\mathcal{K}$**-representable set (**$\mathcal{K}$**-r.s.), i.e., there exists a** $\mathcal{K}$*-representation* **(**$\mathcal{K}$**-r.) of** X

$$X = \left\{ x \in \mathbf{R}^n \mid \exists u \in \mathbf{R}^k : \ Px + Qu + b \in \mathbf{K} \subset \mathcal{K}. \right\} \qquad \textbf{(R)}$$

Indeed, given the data $P, Q, b, \mathbf{K}$ **of (R), we can rewrite (P) equivalently in the form of** $(*)$**:**

$$d^T x \to \min \mid x \in X$$

$$\Updownarrow$$

$$c^T y \equiv d^T x \to \min \mid Ay + b \equiv Px + Qu + b \in \mathbf{K}$$
$$\left[y = \begin{pmatrix} x \\ u \end{pmatrix} \right]$$

- **Thus, a natural interpretation of the question**

  Given a possibility to solve problems

$$c^T x \to \min \mid\ Ax + b \in \mathbf{K} \in \mathcal{K}$$

  what can we actually solve?

is

- *What are $\mathcal{K}$-representable sets?*
- *How to recognize $\mathcal{K}$-representability?*

• **Claim:** **Consider a family $\mathcal{K}$ of finite-dimensional closed pointed cones with a nonempty interior, and let this family be closed w.r.t.**

- **taking direct products**
- **passing from a cone to its dual.**

There exists a simple and powerful "calculus" of $\mathcal{K}$-representable sets: essentially,

Every standard convexity-preserving operation as applied to (finitely many) $\mathcal{K}$-representable sets $X_1, ..., X_k$, yields a $\mathcal{K}$-representable result X.

Moreover, a $\mathcal{K}$-representation of X is "readily given" by $\mathcal{K}$-representations of $X_1, ..., X_k$.

• **Applying the "calculus machinery" to (specific for a family $\mathcal{K}$) collection of "raw materials" – simple $\mathcal{K}$-representable sets – we get a possibility to recognize complicated $\mathcal{K}$-representable sets and thus to pose various optimization problems in the "$\mathcal{K}$-form".**

Calculus of $\mathcal{K}$-representable sets

I. The intersection of finitely many $\mathcal{K}$-r.s.'s is a $\mathcal{K}$-r.s.:

$$X_i = \{x \mid \exists u_i : P_i x + Q_i u_i + b_i \in \mathbf{K}_i\}, \; i = 1, ..., k$$

$$\Downarrow$$

$$\bigcap_{i=1}^{k} X = \left\{u \mid \exists u = \begin{pmatrix} u_1 \\ ... \\ u_k \end{pmatrix} : \begin{pmatrix} P_1 x + Q_i u_1 + b_1 \\ ... \\ P_k x + Q_k u_k + b_k \end{pmatrix} \in \mathbf{K}_1 \times ... \times \mathbf{K}_k \right\}$$

II. The direct product of finitely many $\mathcal{K}$-r.s.'s is a $\mathcal{K}$-r.s.:

$$X_i = \{x_i \mid \exists u_i : P_i x_i + Q_i u_i + b_i \in \mathbf{K}_i\}, \; i = 1, ..., k$$

$$\Downarrow$$

$$X_1 \times ... \times X_k = \left\{x = \begin{pmatrix} x_1 \\ ... \\ x_k \end{pmatrix} \mid \exists u = \begin{pmatrix} u_1 \\ ... \\ u_k \end{pmatrix} : \begin{pmatrix} P_1 x_1 + Q_1 u_1 + b_1 \\ ... \\ P_k x_k + Q_k u_k + b_k \end{pmatrix} \in \mathbf{K}_1 \times ... \times \mathbf{K}_k \right\}$$

Calculus of $\mathcal{K}$-representable sets (cont.)

III. The image of a $\mathcal{K}$-r.s. under an affine mapping is a $\mathcal{K}$-r.s.:

$$X = \{x \mid \exists u : Px + Qu + b \in \mathbf{K}\}$$

$$\Downarrow$$

$$AX + a = \{z \mid \exists v \in \mathrm{Ker}\, A \;\; \exists u : PBz + Pv + Qu + [b - PBa] \in \mathbf{K}\}$$

$$[A \text{ is an onto mapping, } ABz = z \quad \forall z]$$

IV. The inverse image of a $\mathcal{K}$-r.s. under an affine mapping is a $\mathcal{K}$-r.s.:

$$X = \{x \mid \exists u : Px + Qu + b \in \mathbf{K}\}, Z = \{z \mid Az + a \in X\}$$

$$\Downarrow$$

$$Z = \{z \mid \exists u : PAz + Qu + [b + Pa] \in \mathbf{K}\}$$

V. The arithmetic sum of finitely many $\mathcal{K}$-r.s.'s

$$X = X_1 + ... + X_k = \{x \mid \exists x_i \in X_i : x = \sum_i x_i\}$$

is a $\mathcal{K}$-r.s.
Indeed, the arithmetic sum of k sets is the image of their direct product under the linear mapping

$$\begin{pmatrix} x_1 \\ ... \\ x_k \end{pmatrix} \mapsto x_1 + ... + x_k.$$

VI. A polyhedral set

$$X = \{x \mid a_i^T x + b_i \geq 0,\ i = 1, ..., k\}$$

is a $\mathcal{K}$-r.s.

Indeed, let $\mathbf{K} \in \mathcal{K}$ and $0 \neq e \in \mathbf{K}$. Then

$$X = \left\{ x \mid \begin{pmatrix} [a_1^T x + b_1]e \\ ... \\ [a_k^T x + b_k]e \end{pmatrix} \in \mathbf{K} \times ... \times \mathbf{K} \right\}.$$

- **Intersection of infinitely many half-spaces not necessarily is $\mathcal{K}$-representable. It, however, is so when the half-spaces are "well-organized".**

VIa. Assume that the data (α, β) defining a half-space

$$X_{\alpha,\beta} = \{x \mid \alpha^T x + \beta \geq 0\}$$

vary in a $\mathcal{K}$-r.s. $\mathcal{U}$:

$$\mathcal{U} = \{(\alpha, \beta) \mid \exists u : P\alpha + \beta p + Qu + b \in \mathbf{K}\}$$

and let the representation be *strictly feasible***:**

$$\exists \bar{\alpha}, \bar{\beta}, \bar{u} : P\bar{\alpha} + \bar{\beta} p + Q\bar{u} + b \in \mathrm{int}\, \mathbf{K}.$$

Then the set

$$X = \bigcap_{(\alpha,\beta)\in\mathcal{U}} X_{\alpha,\beta} = \left\{x \mid \alpha^T x + \beta \geq 0 \quad \forall (\alpha, \beta) \in \mathcal{U}\right\}$$

is a $\mathcal{K}$-r.s.

Indeed,

$$X = \left\{x \mid 0 \leq \inf_{\alpha,\beta,u} \left\{\alpha^T x + \beta : P\alpha + \beta p + Qu + b \in \mathbf{K}\right\}\right\}$$

$\Downarrow$ **[Conic Duality Theorem]**

$$X = \left\{x \mid 0 \leq \max_{\eta} \left\{-b^T \eta : \begin{array}{c} \eta \in \mathbf{K}_* \\ P^T \eta = x \\ p^T \eta = 1 \\ Q^T \eta = 0 \end{array}\right\}\right\}$$

$$\Updownarrow$$

$$X = \left\{x \mid \exists \eta : \begin{array}{c} \eta \in \mathbf{K}_* \\ P^T \eta = x \\ p^T \eta = 1 \\ Q^T \eta = 0 \\ b^T \eta \leq 0 \end{array}\right\}$$

Thus, X is the projection of the intersection of the $\mathcal{K}$-r.s. $\mathbf{K}_*$ and a polyhedral set and thus is a $\mathcal{K}$-r.s.

Calculus of $\mathcal{K}$-representable sets (cont.)

Several operations with sets "nearly preserve" $\mathcal{K}$-representability.

Definition. A set $X \subset \mathbf{R}$ is called *nearly $\mathcal{K}$-representable*, **if there exists a $\mathcal{K}$-r.s. X' such that**

$$X \subset X' \subset \mathrm{cl}\, X. \qquad (*)$$

A *nearly $\mathcal{K}$-representation* **of X is a $\mathcal{K}$-representation of a set X' satisfying $(*)$.**

Note: **Nearly $\mathcal{K}$-representable** *closed* **set X is $\mathcal{K}$-representable, and every nearly $\mathcal{K}$-representation of such a set is its $\mathcal{K}$-representation.**

VII. The conic hull

$$\mathrm{Cone}(X) = \{0\} \bigcup \left\{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid t > 0, t^{-1}x \in X\right\}$$

of a nonempty $\mathcal{K}$-r.s. X is a nearly $\mathcal{K}$-r.s.:

$$X = \{x \mid \exists u : Px + Qu + b \in \mathbf{K}\}$$

$$\Downarrow$$

$$\mathrm{Cone}(X) \subset X' \subset \overline{\mathrm{Cone}(X)},$$

$$X' = \left\{(x,t) \mid \exists u : \begin{pmatrix} Px + Qu + tb \\ te \end{pmatrix} \in \mathbf{K} \times \mathbf{K}\right\}$$
$$[0 \neq e \in \mathbf{K}]$$

VIII. [Yu. Nesterov] Convex hull of finitely many $\mathcal{K}$-r.s.'s is a nearly $\mathcal{K}$-r.s.:

$$X_i = \{x \mid \exists u_i : P_i x + Q_i u_i + b_i \in \mathbf{K}_i\}, \; i = 1, ..., k$$

$$\Downarrow$$

$$\mathrm{Conv}\left(\bigcup_{i=1}^{k} X_i\right) \subset X \subset \overline{\mathrm{Conv}}\left(\bigcup_{i=1}^{k} X_i\right),$$

$$\begin{array}{rl} X = & \Big\{x \mid \exists (x_1, ..., x_k, u_1, ..., u_k, t_1, ..., t_k) : \\ & \begin{pmatrix} P_1 x_1 + Q_1 u_1 + t_1 b_1 \\ ... \\ P_k x_k + Q_k u_k + t_k b_k \end{pmatrix} \in \mathbf{K}_1 \times ... \times \mathbf{K}_k \\ & x = x_1 + ... + x_k \\ & t_1 + ... + t_k = 1 \\ & t_i \geq 0, \, i = 1, ..., k\Big\} \end{array}$$

$\mathcal{K}$-representability of X is readily given by the fact that X is the projection on the x-space of the intersection of the polyhedral (and thus – $\mathcal{K}$-representable) set

$$\left\{ (x_1, ..., x_k, u_1, ..., u_k, t_1, ..., t_k) \mid \begin{array}{c} x = x_1 + ... + x_k \\ t_1 + ... + t_k = 1 \\ t_i > 0, \;\; i = 1, ..., k \end{array} \right\}$$

and the clearly $\mathcal{K}$-r.s.

$$\Big\{ (x_1, ..., x_k, u_1, ..., u_k, t_1, ..., t_k) \mid \begin{pmatrix} P_1 x_1 + Q_1 u_1 + t_1 b_1 \\ \cdots \\ P_k x_k + Q_k u_k + t_k b_k \end{pmatrix} \in \mathbf{K}_1 \times ... \times \mathbf{K}_k \Big\}$$

IX. Let

$$X = \{x \mid \exists u : Px + Qu + b \in \mathbf{K}\} \qquad (*)$$

be a $\mathcal{K}$-r.s., and let the representation $(*)$ be *strictly feasible*:

$$\exists (\bar{x}, \bar{u}) : P\bar{x} + Q\bar{u} + b \in \operatorname{int} \mathbf{K}.$$

Then the *polar cone* **of X – the set**

$$X_* = \{(\eta, s) \mid \eta^T x \geq s \quad \forall x \in X\}$$

– is a $\mathcal{K}$-r.s.:

$$X^* = \left\{ (\eta, s) \mid \exists \zeta \in \mathbf{K}_* : \begin{array}{r} P^T \zeta = \eta \\ Q^T \zeta = 0 \\ s + b^T \zeta \leq 0 \end{array} \right\}$$

$\mathcal{K}$-representable functions

- **In many applications, sets are represented by (systems of) inequalities**

$$f(x) \leq 0 \quad [f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}] \qquad (*)$$

Definition: f is called a $\mathcal{K}$-representable function ($\mathcal{K}$-r.f.), if the epigraph of f

$$\mathrm{Epi}(f) = \{(x,t) \mid t \geq f(x)\}$$

is a $\mathcal{K}$-r.s.

A $\mathcal{K}$-representation of $\mathrm{Epi}(f)$ is called a $\mathcal{K}$-representation ($\mathcal{K}$-r.) of f.

Observation: If f is $\mathcal{K}$-representable, then so are all level sets

$$\{x \mid f(x) \leq a\} \quad [a \in \mathbf{R}]$$

of f.

Indeed,

$$f(x) \leq t \Leftrightarrow \exists u : Px + tp + Qu + b \in \mathbf{K}$$

$$\Downarrow$$

$$\{x \mid f(x) \leq a\} = \{x \mid \exists u : Px + Qu + [ap + b] \in \mathbf{K}\}$$

$\mathcal{K}$-representable functions (cont.)

• Calculus of $\mathcal{K}$-representable sets can be straightforwardly converted into a "calculus of $\mathcal{K}$-representable functions". The latter, essentially, can be summarized in the following statements:

(i). [Conic combinations and taking maximum] A linear combination, with nonnegative coefficients, and the maximum of finitely many $\mathcal{K}$-r.f.'s are themselves $\mathcal{K}$-r.f.'s.

(ii). [Adding affine form] The sum of a $\mathcal{K}$-representable and an affine function is $\mathcal{K}$-representable

(iii). [Affine substitution of argument] The superposition of a $\mathcal{K}$-r.f. and an affine mapping is a $\mathcal{K}$-r.f.

$\mathcal{K}$-representable functions (cont.)

(iv). [Taking superpositions] Let functions

$$f_1, ..., f_k : \mathbf{R}^n \to \mathbf{R} \bigcup \{+\infty\}$$

and a function

$$g : \mathbf{R}^k \to \mathbf{R} \bigcup \{+\infty\}$$

be $\mathcal{K}$-representable:

$$\begin{array}{c} f_i(x) \le t \Leftrightarrow \exists u_i : P_i x + t p_i + Q_i u_i + b_i \in \mathbf{K}_i, \;\; i = 1, ..., k \\ g(z) \le t \Leftrightarrow \exists v : Pz + Qv + b \in \mathbf{K}. \end{array}$$

Let also g be monotone on its domain:

$$g(z) < \infty, z \le z' \Rightarrow g(z) \le g(z').$$

Then the superposition

$$h(x) = g(f_1(x), ..., f_k(x))$$

is a $\mathcal{K}$-r.f.:

$$h(x) \le t$$

$$\Updownarrow$$

$$\begin{array}{l} \exists \tau = (\tau_1, ..., \tau_k)^T, u_1, ..., u_k, v : \\ \underbrace{P_i x + \tau_i p_i + Q_i u_i + b_i \in \mathbf{K}_i, \;\; i = 1, ..., k}_{\text{says that } f_i(x) \le \tau_i} \\ \qquad\qquad \underbrace{P\tau + Qv + b \in \mathbf{K}}_{\text{says that } g(\tau) \le t} \end{array}$$

(v). [Legendre transformation] The Legendre transformation

$$f_*(\eta) = \sup_x [\eta^T x - f(x)]$$

of a $\mathcal{K}$-r.f. f *with strictly feasible $\mathcal{K}$-r.* **is itself a $\mathcal{K}$-r.f.:**

$$\begin{array}{c} f(x) \leq t \Leftrightarrow \exists u : Px + tp + Qu + b \in \mathbf{K} \\ [\exists \bar{x}, \bar{t}, \bar{u} : P\bar{x} + \bar{t}p + Q\bar{u} + b \in \mathrm{int}\, \mathbf{K}] \end{array}$$

$$\Downarrow$$

$$f_*(\eta) \leq \tau \Leftrightarrow \exists \zeta : \begin{array}{c} \zeta \in \mathbf{K}_* \\ P^T\zeta + \eta = 0 \\ Q^T\zeta = 0 \\ p^T\zeta = 1 \\ \tau - b^T\zeta \geq 0 \end{array}$$

$\mathcal{K}$-representable functions (cont.)

(vi). [Conic transformation] Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a $\mathcal{K}$-r.f:

$$f(x) \le t \Leftrightarrow \exists u : Px + tp + Qu + b \in \mathbf{K}$$

Then the *conic transformation* **of f – the function**

$$\mathrm{Cone}\,[f](x,s) = \begin{cases} sf(x/s), & s > 0 \\ +\infty. & s \le 0 \end{cases}$$

is a nearly $\mathcal{K}$-r.f.:

$$\mathrm{Epi}(\mathrm{Cone}\,[f]) \subset X \subset \overline{\mathrm{Epi}(\mathrm{Cone}\,[f])},$$

$$X = \{(x,s,t) \mid \exists u : Px + tp + Qu + sb \in \mathbf{K}\}$$

(vii). [Partial minimization] Let $f(x,y) : \mathbf{R}^m_x \times \mathbf{R}^n_y \to \mathbf{R} \cup \{+\infty\}$ **be a** $\mathcal{K}$**-r.f.:**

$$f(x,y) \leq t \Leftrightarrow \exists u : Px + Ry + tp + Qu + b \in \mathbf{K}.$$

Let

$$\phi(x) = \inf_y f(x,y),$$

and let inf **be attained for every** x **where it is not** $+\infty$**. Then** ϕ **is a** $\mathcal{K}$**-r.f.:**

$$\phi(x) \leq t \Leftrightarrow \exists y, u : Px + tp + [Ry + Qu] + b \in \mathbf{K}.$$

(viii). [inf**-convolution] Let** $f_i(x)$, $i = 1, ..., k$, **be** $\mathcal{K}$**-r.f.'s:**

$$f_i(x) \leq t_i \Leftrightarrow \exists u_i : P_i x + t_i p_i + Q_i u_i + b_i \in \mathbf{K}_i, \quad i = 1, ..., k.$$

Let

$$\phi(x) = \inf_{x_1,...,x_k : \sum_i x_i = x} \sum_i f_i(x_i),$$

and let the infinum be attained at every x **where it is not** $+\infty$**. Then** ϕ **is a** $\mathcal{K}$**-r.f.:**

$$\phi(x) \leq t \Leftrightarrow \exists \{x_i, u_i, t_i\} : \left\{ \begin{array}{c} P_i x_i + t_i p_i + Q_i u_i + b_i \in \mathbf{K}_i, \ i = 1, ..., k \\ \sum_i x_i = x \\ \sum_i t_i \leq t \end{array} \right.$$

Note: All the rules of the Calculus of $\mathcal{K}$-representable functions/sets are "algorithmic and efficient": a $\mathcal{K}$-r. of the result of an operation is readily given by $\mathcal{K}$-r.'s of the operands. E.g.,

$$\boxed{\begin{array}{c} f_i(x) \leq t \Leftrightarrow \exists u_i : P_i x + t p_i + Q_i u_i + b_i \in \mathbf{K}_i,\ \ i = 1, ..., k \\ \alpha_i \geq 0,\ \ i = 1, ..., k \end{array}}$$

$$\Downarrow$$

$$\sum_i \alpha_i f_i(x) \leq t$$

$$\Updownarrow$$

$\exists t_1, ..., t_k, u_1, ..., u_k :$

$$\underbrace{P_i x + t_i p_i + Q_i u_i + b_i \in \mathbf{K}_i}_{\textbf{says that } f_i(x) \leq t_i},\ \ i = 1, ..., k$$

$$\sum_i \alpha_i t_i \leq t$$

What can be expressed via LP?

$$\boxed{\mathcal{K} = \mathcal{LP} \equiv \{\mathbf{R}^n_+\}_{n=1}^{\infty}}$$

- **Of course,**
  - **$\mathcal{LP}$-representable sets are exactly the polyhedral sets,**
  - **$\mathcal{LP}$-representable functions are exactly convex piecewise linear functions.**

E.g., the function

$$f(x_1, ..., x_n) = \prod_{i=1}^{n} x_i^{-\frac{i}{n}} : \text{int } \mathbf{R}^n_+ \to \mathbf{R}$$

is not $\mathcal{LP}$-representable. But...

What can be expressed via CQP?

$$\boxed{\begin{array}{c}\mathcal{K} = \mathcal{CQ} \equiv \{\mathbf{L}^n\}_{n=0}^{\infty} \\ \mathbf{L}^n = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid \| x \|_2 \leq t\}\end{array}}$$

Elementary $\mathcal{CQ}$-representable functions/sets

- **The following functions/sets are $\mathcal{CQ}$-representable with explicit $\mathcal{CQ}$-r.'s:**

1. An affine function $f(x) = a^T x + b$

2. The Euclidean norm $\| x \|_2$: $\mathbf{R}^n \to \mathbf{R}$

3. The set

$$\{(x,s,t) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} : s,t \geq 0, x^T x \leq st\}. \qquad (*)$$

Indeed, this set is the inverse image of $\mathbf{L}^{n+1}$ under the linear transformation

$$\begin{pmatrix} x \\ t \\ s \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{t-s}{2} \\ \frac{t+s}{2} \end{pmatrix}$$

4. The squared Euclidean norm $f(x) = x^T x : \mathbf{R}^n \to \mathbf{R}$

Indeed, the epigraph of $x^T x$ is the intersection of the set $(*)$ and the hyperplane $s = 1$.

Note: by Calculus it follows from 4. that every convex quadratic form is $\mathcal{CQ}$-representable.

5. A simple fractional-quadratic function

$$f(x,s) = \frac{x^2}{s} : \{(x,s) \in \mathbf{R}^2 \mid s \geq 0\} \to \mathbf{R} \bigcup \{+\infty\}$$

is $\mathcal{CQ}$-representable.

Indeed, the epigraph of f is the $\mathcal{CQ}$-r.s.

$$\{(x,t,s) \in \mathbf{R}^3 \mid \ s,t \geq 0, x^2 \leq st\}$$

Application example: The *Multi-Load Truss Topology Design problem*

$$\tau \to \min \mid \begin{cases} \begin{pmatrix} \tau & f_\ell^T \\ f_\ell & \sum\limits_{i=1}^n t_i B_i B_i^T \end{pmatrix} \succeq 0, \ \ell = 1, ..., k \\ t \geq 0, \ Pt + p \geq 0 \end{cases}$$
$$\left[\sum_i B_i B_i^T \succ 0, \exists \bar{t} > 0 : P\bar{t} + p > 0\right]$$

is equivalent to the Conic Quadratic problem

$$\begin{array}{c} \tau \to \min \\ \sum\limits_{i=1}^n s_{i,\ell} \leq \tau, \ \ell = 1, ..., k \\ \left\| \begin{pmatrix} y_{i,\ell} \\ \frac{t_i - s_{i,\ell}}{2} \end{pmatrix} \right\|_2 \leq \frac{t_i + s_{i,\ell}}{2}, \ i = 1, ..., n, \ell = 1, ..., k \\ \sum\limits_{i=1}^n B_i y_{i,\ell} = f_\ell, \ \ell = 1, ..., k \\ Pt + p \geq 0 \\ t \geq 0 \end{array}$$

6. Let $\pi_1, ..., \pi_k$ be positive rationals with $\sum\limits_{i=1}^{k} \pi_i \leq 1$. Then the function

$$f(x) = -\prod_{i=1}^{k} x_i^{\pi_i} : \mathbf{R}_+^k \to \mathbf{R}$$

admits an explicit $\mathcal{CQ}$-r. The "size" of the representation is proportional to the magnitude of the common denominator of the fractions $\pi_1, ..., \pi_k$.

Indeed,

A. The set

$$\{(t, u_1, u_2) \in \mathbf{R}_+^3 : t^2 \leq u_1 u_2\} \tag{1}$$

is $\mathcal{CQ}$-representable.

B. It follows that the set

$$X_r = \{(t, u_1, ..., u^{2^r}) \in \mathbf{R}^{2^r+1} : t \leq (u_1, ..., u_{2^r})^{\frac{1}{2^r}}\}$$

also is $\mathcal{CQ}$-representable.
Indeed, X_r is the projection onto the (t, u)-space of the set given by the inequalities

$$\begin{array}{c} u_1, ..., u_{2^r} \geq 0 \\ 0 \leq v_{1,j} \leq \sqrt{u_{2j-1} u_{2j}}, \quad 1 \leq j \leq 2^{r-1} \\ 0 \leq v_{2,j} \leq \sqrt{v_{1,2j-1} v_{1,2j}}, \quad 1 \leq j \leq 2^{r-2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \leq v_{r-1,1} \leq \sqrt{v_{r-2,1} v_{r-2,2}}, 0 \leq v_{r-1,2} \leq \sqrt{v_{r-2,3} v_{r-2,4}} \\ 0 \leq t \leq \sqrt{v_{r-1,1} v_{r-1,2}} \end{array}$$

and the latter set is $\mathcal{CQ}$-representable as the intersection of $2^r - 1$ $\mathcal{CQ}$-representable sets (direct products of linear spaces and sets of type (1)).

C. Further, the set

$$\{(x_1,...,x_k,t)\in \mathbf{R}_+^{k+1} \mid t\le x_1^{\frac{p_1}{q}}...x_k^{\frac{p_k}{q}}\} \qquad (2)$$

is the inverse image of X_r with $2^r\ge q$ under the affine mapping

$$(x_1,...,x_k,t)\mapsto (t,\underbrace{x_1,...,x_1}_{p_1},...,\underbrace{x_2,...,x_k}_{p_k}\underbrace{t,...,t}_{2^r-q},\ \underbrace{1,...,1}_{q-p_1-...-p_k}\ )$$

D. Since the set (2) is $\mathcal{CQ}$-representable, so is the set

$$\{(x_1,...,x_k,t,s)\mid x_1,...,x_k\ge 0,\ 0\le s-t\le x_1^{\pi_1}...x_k^{\pi_k}\}$$

(it is the inverse image of (2) under an affine mapping) and thus – the set

$$\{(x_1,...,x_k,t,s)\mid s,x_1,...,x_k\ge 0,\ 0\le s-t\le x_1^{\pi_1}...x_k^{\pi_k}\} \quad (3)$$

E. Finally, the epigraph

$$\{(x_1,...,x_k,t)\mid x_1,...,x_k\ge 0,\ t\ge -x_1^{\pi_1}...x_k^{\pi_k}\}$$

of the function $f(x)=-\prod_{i=1}^{k} x_i^{\pi_i}$ of $x\ge 0$ is the projection on the (x,t)-space of the $\mathcal{CQ}$-r.s. (3).

7. Let $\pi_1, ..., \pi_k$ be positive rationals. Then the function

$$f(x) = \prod_{i=1}^{k} x_i^{-\pi_i} : \mathrm{int}\, \mathbf{R}^k_+ \to \mathbf{R}$$

admits an explicit $\mathcal{CQ}$-r.

8. Let $\pi \geq 1$ be a rational. Then the function

$$f_+(x) = (\max[x, 0])^{\pi}$$

admits an explicit $\mathcal{CQ}$-r.

So does the function

$$g(x) = |x|^{\pi} \quad [\equiv f(x) + f(-x)]$$

Indeed,

$$\{(t, x) \mid t \geq (\max[x, 0])^{\frac{p}{q}}\} = \{(t, x) \mid -x \geq -t^{\frac{q}{p}},\ t \geq 0\}.$$

9. For rational $\pi \geq 1$, the π-norm $f(x) = \| x \|_{\pi}$: $\mathbf{R}^n \to \mathbf{R}$ is $\mathcal{CQ}$-representable:

$$f(x) \leq t \Leftrightarrow \exists u_1, ..., u_n \geq 0 : \begin{cases} |x_i| \leq t^{\frac{\pi-1}{\pi}} u_i^{\frac{1}{\pi}},\ i = 1, ..., n \\ \sum_{i=1}^{n} u_i \leq t \end{cases}$$

Examples:

- **A $\mathcal{CQ}$-r. of the set**

$$t \geq -x_1^{\frac{1}{7}} x_2^{\frac{2}{7}} x_3^{\frac{3}{7}},\ \ x \geq 0$$

is

$$\begin{array}{c}
\exists s, u_1, u_2, u_3, u_4 : \\
\begin{pmatrix} u_1 \\ \frac{x_1 - x_3}{2} \\ \frac{x_1 + x_3}{2} \end{pmatrix} \in \mathbf{L}^2 \qquad \begin{pmatrix} u_2 \\ \frac{s - t - 1}{2} \\ \frac{s - t + 1}{2} \end{pmatrix} \in \mathbf{L}^2 \\
\begin{pmatrix} u_3 \\ \frac{u_1 - u_2}{2} \\ \frac{u_1 + u_2}{2} \end{pmatrix} \in \mathbf{L}^2 \qquad \begin{pmatrix} u_4 \\ \frac{x_2 - x_3}{2} \\ \frac{x_2 + x_3}{2} \end{pmatrix} \in \mathbf{L}^2 \\
\begin{pmatrix} s - t \\ \frac{u_3 - u_4}{2} \\ \frac{u_3 + u_4}{2} \end{pmatrix} \in \mathbf{L}^2 \\
s \geq 0
\end{array}$$

- **A $\mathcal{CQ}$-r. of the set**

$$t \geq x_1^{-\frac{1}{7}} x_2^{-\frac{2}{7}} x_3^{-\frac{3}{7}}$$

is

$$\begin{array}{c}
\exists u_1, u_2, u_3, u_4, u_5, u_6 : \\
\begin{pmatrix} u_1 \\ \frac{x_2 - x_3}{2} \\ \frac{x_2 + x_3}{2} \end{pmatrix} \in \mathbf{L}^2 \qquad \begin{pmatrix} u_2 \\ \frac{t - 1}{2} \\ \frac{t + 1}{2} \end{pmatrix} \in \mathbf{L}^2 \qquad \begin{pmatrix} u_3 \\ \frac{x_2 - x_3}{2} \\ \frac{x_2 + x_3}{2} \end{pmatrix} \in \mathbf{L}^2 \\
\begin{pmatrix} u_4 \\ \frac{u_1 - u_2}{2} \\ \frac{u_1 + u_2}{2} \end{pmatrix} \in \mathbf{L}^2 \qquad \begin{pmatrix} u_5 \\ \frac{u_3 - u_4}{2} \\ \frac{u_3 + u_4}{2} \end{pmatrix} \in \mathbf{L}^2 \qquad \begin{pmatrix} u_6 \\ \frac{u_2 - t}{2} \\ \frac{u_2 + t}{2} \end{pmatrix} \in \mathbf{L}^2 \\
\begin{pmatrix} 1 \\ \frac{u_3 - u_6}{2} \\ \frac{u_3 + u_6}{2} \end{pmatrix} \in \mathbf{L}^2
\end{array}$$

• **There are, of course, nice convex functions which are** *not* $\mathcal{CQ}$**-representable, e.g., the exponent**

$$f(x) = \exp\{x\}.$$

<u>But</u>: for every $p \geq 1$**,**

$$\exp\{x\} = \lim_{r\to\infty} \Big(\underbrace{1 + \frac{x}{2^r} + \frac{1}{2}\left(\frac{x}{2^r}\right)^2 + ... + \frac{1}{p!}\left(\frac{x}{2^r}\right)^p}_{h_p\left(\frac{x}{2^r}\right)}\Big)^{(2^r)}$$

It follows that if p **is such that** $h_p(\cdot)$ **is** $\mathcal{CQ}$**-representable and nonnegative, then** $\exp\{x\}$ **can be** *approximated* **by** $\mathcal{CQ}$**-r.f.**

$$g_{p,r}(x) = h_p^{(2^r)}\left(\frac{x}{2^r}\right).$$

For p **fixed and for a given** *bounded* **range** $|x| \leq T$ **of values of** x**, the quality of this approximation grows rapidly with** r**:**

$$r \geq O\left(\ln\left(\tfrac{1}{\varepsilon}\right) + T\right), \ |x| \leq T$$

$$\Downarrow$$

$$(1-\varepsilon)\exp\{x\} \leq g_{p,r}(x) \leq (1+\varepsilon)\exp\{x\}.$$

E.g.,

- $g_{4,18}(x)$ **approximates** $\exp\{x\}$ **in the segment**

$$\Delta = \{x \mid |x| \le 512\} = \{x \mid \texttt{4.38e-223} \le \exp\{x\} \le \texttt{2.28e222}\}$$

within relative accuracy `1.e-10`

- **At the same time,** $g_{4,18}\big|_\Delta$ **admits the following simple** $\mathcal{CQ}$**-r.:**

$$\boxed{|x| \le 512, g_{4,18}(x) \le t}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{c}
\exists u_1, ..., u_{21} : \\
\begin{pmatrix} 1 + \frac{x}{2^{18}} \\ \frac{1-u_1}{2} \\ \frac{1+u_1}{2} \end{pmatrix} \in \mathbf{L}^2 \quad
\begin{pmatrix} \frac{5}{6} + \frac{x}{2^{19}} \\ \frac{1-u_2}{2} \\ \frac{1+u_2}{2} \end{pmatrix} \in \mathbf{L}^2 \quad
\begin{pmatrix} u_1 \\ \frac{1-u_3}{2} \\ \frac{1+u_3}{2} \end{pmatrix} \in \mathbf{L}^2 \\
\frac{19}{72} + u_2 + \frac{1}{24} u_3 \le u_4 \\
\begin{pmatrix} u_{\ell-1} \\ \frac{1-u_\ell}{2} \\ \frac{u_\ell+1}{2} \end{pmatrix} \in \mathbf{L}^2, \ \ell = 5, ..., 21 \\
\begin{pmatrix} u_{21} \\ \frac{1-t}{2} \\ \frac{1+t}{2} \end{pmatrix} \in \mathbf{L}^2 \\
-512 \le x \le 512
\end{array}}$$

Thus, for any practical purpose $\exp\{x\}$ **is** $\mathcal{CQ}$**-representable!**

“Whether Conic Quadratic Programming does exist?”

• **Surprisingly, Lorentz cones (and thus – all $\mathcal{CQ}$-representable sets!) are “nearly polyhedrally representable”.**

Theorem [**BT-N, ’98**] *For every $n > 0$ and every $\varepsilon \in (0, 1/2)$, there exist (and can be explicitly written down) matrices $P_{n,\varepsilon}$, $Q_{n,\varepsilon}$ and vector $b_{n,\varepsilon}$ such that*

(i) The row and the column sizes of $P_{n,\varepsilon}, Q_{n,\varepsilon}, b_{n,\varepsilon}$ do not exceed $O\left(n \ln \frac{1}{\varepsilon}\right)$

(ii) If

$$(x,t) \in \mathbf{L}^n = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid \| x \|_2 \leq t\},$$

then there exists a vector u such that

$$P_{n,\varepsilon}x + Q_{n,\varepsilon}u + b_{n,\varepsilon} \geq 0, \qquad (*)$$

and “nearly vice versa”: if (x,t) can be extended by some u to a solution of $(*)$, then

$$(x,t) \in \mathbf{L}^n_\varepsilon = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid \| x \|_2 \leq (1+\varepsilon)t\}.$$

Thus, $\mathbf{L}^n$ “can be approximated within accuracy ε” by a polyhedral cone $\mathbf{P}^n_\varepsilon$:

$$\mathbf{L}^n \subset \mathbf{P}^n_\varepsilon \subset \mathbf{L}^n_\varepsilon$$

admitting a “short” – of the size $O\left(n \ln \frac{1}{\varepsilon}\right)$ – $\mathcal{LP}$-r.:

$$\mathbf{P}^n_\varepsilon = \{(x,t) \mid \exists u : P_{n,\varepsilon}x + Q_{n,\varepsilon}u + b_{n,\varepsilon} \geq 0\}.$$

Corollary. **Consider a conic quadratic program**

$$c^T x \to \min\ Ax + b \geq 0,\ \ Px + q \in \mathbf{K} = \mathbf{L}^{n_1} \times ... \times \mathbf{L}^{n_k} \quad \text{(CQP)}$$

and let the problem be both

r-strictly feasible for some $r > 0$:

$$\exists \bar{x} : A\bar{x} + b \geq 0\ \&\ \forall(\xi : \| \xi \|_2 \leq r) : P\bar{x} + q + \xi \in \mathbf{K}$$

R-semibounded:

$$Ax + b \geq 0, Px + q \in \mathbf{K} \Rightarrow \| Px + q \|_2 \leq R.$$

For every $\varepsilon \in (0,1)$, there exists an LP program

$$c^T x \to \min \mid\ Bx + Cv + d \geq 0 \qquad \text{(LP)}$$

which is an ε-relaxation of (CQP) of polynomial complexity:

- **If x is feasible for (CQP), then x can be extended to a feasible solution (x, v) of (LP);**
- **If (x, v) is feasible for (LP), then the vector $(1-\varepsilon)x + \varepsilon\bar{x}$ is feasible for (CQP).**
- **The size** $\dim x + \dim v + \dim d$ **of** (LP) **does not exceed**

$$\dim x + \dim b + O\left([\dim q] \ln \frac{2R}{\varepsilon r}\right)$$

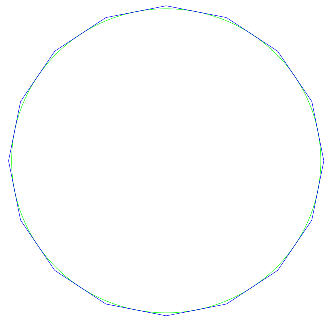
and the data (c, B, C, d) of (LP) is readily given by the data (c, A, b, P, q) of (CQP).

• **"Fast polyhedral approximations" of the Lorentz cone are based on the fact that if $\mathbf{P}^\nu$ is the polytope defined as**

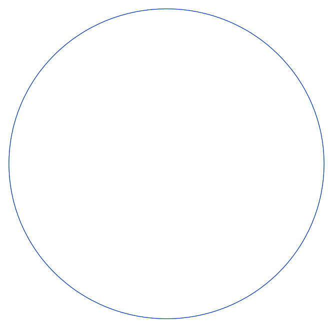
$$\left\{(x,u)\in\mathbf{R}^2\times\mathbf{R}^{2\nu+4}:\left\{\begin{array}{rcl} u_0 &\geq& |x_1|;\\ u_1 &\geq& |x_2|;\\ u_{2j} &=& \cos\left(\frac{\pi}{2^{j+1}}\right)u_{2j-2}+\sin\left(\frac{\pi}{2^{j+1}}\right)u_{2j-1},\\ && j=1,\dots,\nu;\\ u_{2j+1} &\geq& \left|-\sin\left(\frac{\pi}{2^{j+1}}\right)u_{2j-2}+\cos\left(\frac{\pi}{2^{j+1}}\right)u_{2j-1}\right|,\\ && j=1,\dots,\nu;\\ u_{2\nu+3} &\leq& 1;\\ u_{2\nu+4} &\leq& \tan\left(\frac{\pi}{2^{\nu+1}}\right)u_{2\nu+3} \end{array}\right.\right\}$$

then the projection $\mathbf{P}^\nu_x$ of the polytope on the x-plane approximates the unit 2D disk $\mathbf{D}_2$ within accuracy

$$\frac{1}{\cos\left(\frac{\pi}{2^{\nu+1}}\right)}-1.$$



- $\mathbf{P}^3$ "lives" in $\mathbf{R}^9$ and is given by 12 linear inequalities.
  $\mathbf{P}^3_x$ approximates $\mathbf{D}_2$ within accuracy 5.e-3
  (as good as the 16-side perfect polygon)



- $\mathbf{P}^6$ "lives" in $\mathbf{R}^{12}$ and is given by 18 linear inequalities.
  $\mathbf{P}^6_x$ approximates $\mathbf{D}_2$ within accuracy 3.e-4
  (as good as the 127-side perfect polygon)

- $\mathbf{P}^{12}$ "lives" in $\mathbf{R}^{18}$ and is given by 30 linear inequalities.
  $\mathbf{P}^{12}_x$ approximates $\mathbf{D}_2$ within accuracy 7.e-8
  (as good as the 8,192-side perfect polygon)

- $\mathbf{P}^{24}$ "lives" in $\mathbf{R}^{30}$ and is given by 54 linear inequalities.
  $\mathbf{P}^{24}_x$ approximates $\mathbf{D}_2$ within accuracy 4.e-15
  (as good as the 34,200,933-side perfect polygon)

What can be expressed via SDP?

$$\mathcal{K} = \mathcal{SD} \equiv \left\{ \begin{array}{l} \textbf{Direct products of the cones of} \\ \textbf{positive semidefinite matrices} \end{array} \right\}$$

Examples of $\mathcal{SD}$-representable functions/sets

1-9. Every $\mathcal{CQ}$-r. function/set is $\mathcal{SD}$-r. as well.

Indeed, the Lorentz cone $\mathbf{L}^n$ is $\mathcal{SD}$-r.:

$$(x,t) \in \mathbf{L}^n$$

$$\Updownarrow$$

$$\underbrace{\begin{pmatrix} t & x_1 & x_2 & \dots & x_n \\ x_1 & t & & & \\ x_2 & & t & & \\ \vdots & & & \ddots & \\ x_n & & & & t \end{pmatrix}}_{\Theta_n(x)} \succeq 0$$

Consequently, a $\mathcal{CQ}$-r. of a set X:

$$X = \{x \mid \exists u_1, ..., u_k : P_i x + Q_i u + b_i \in \mathbf{L}^{n_i},\ i = 1, ..., k\}$$

can be straightforwardly converted into an $\mathcal{SD}$-r. of X:

$$X = \{x \mid \exists u_1, ..., u_k : \Theta_{n_i}(P_i x + Q_i u_i + b_i) \succeq 0,\ i = 1, ..., k\}.$$

Functions of eigenvalues of symmetric matrices, I

10. Maximum eigenvalue $\lambda_{\max}(X)$ of a symmetric matrix X is $\mathcal{SD}$-representable:

$$t \geq \lambda_{\max}(X) \Leftrightarrow tI - X \succeq 0.$$

11. The sum

$$S_k(X) = \lambda_1(X) + ... + \lambda_k(X)$$

of k largest eigenvalues of a symmetric matrix X is $\mathcal{SD}$-representable:

$$S_k(X) \leq t \Leftrightarrow \exists s, Z : \begin{cases} t - ks - \mathrm{Tr}(Z) \geq 0 \\ Z \succeq 0 \\ Z - X + sI \succeq 0 \end{cases}$$

Functions of eigenvalues of symmetric matrices, II

12. Let a function f on $\mathbf{R}^n$ be symmetric (i.e., invariant w.r.t. permutations of coordinates) and $\mathcal{SD}$-representable:

$$f(x) \leq t \Leftrightarrow \exists u : Px + tp + Qu + b \succeq 0.$$

Then the function

$$\phi(X) = f(\lambda(X))$$

is $\mathcal{SD}$-r. with the representation

$$t \geq F(X) \Leftrightarrow \exists u, x_1, ..., x_n : \begin{cases} Px + tp + Qu + b \succeq 0 \\ x_1 \geq x_2 \geq ... \geq x_n \\ x_1 + ... + x_k \geq S_k(X), \\ k = 1, ..., n-1 \\ x_1 + ... + x_n = \mathrm{Tr}(X) \end{cases}$$

E.g., the following functions of a symmetric $n \times n$ matrix X are $\mathcal{SD}$-r.f.'s with explicit $\mathcal{SD}$-r.'s:

- $-(\mathrm{Det}\ X)^{\pi}$, $X \succeq 0$, $0 < \pi \leq \frac{1}{n}$ is rational
  $[f(x) = -(\prod_{i=1}^{n} x_i)^{\pi},\ x \geq 0]$
- $\| X \|_{\pi} \equiv \| \lambda(X) \|_{\pi}$, $\pi \geq 1$ is rational
  $[f(x) = \| x \|_{\pi}]$
- $\| X_+ \|_{\pi} \equiv \| \{\max[\lambda_i(X), 0]\}_{i=1}^{n} \|_{\pi}$, $\pi \geq 1$ is rational
  $[f(x) = \| \{\max[x_i, 0]\}_{i=1}^{n} \|_{\pi}]$

13. Let a function f on $\mathbf{R}^m_+$ be symmetric, monotone and $\mathcal{SD}$-representable:

$$f(x) \le t \Leftrightarrow \exists u : Px + tp + Qu + b \succeq 0,$$

and let

$$\sigma(X) = \lambda\left(\sqrt{XX^T}\right)$$

be the vector of singular values of a rectangular $m \times n$ matrix ($m \le n$). Then the function

$$\phi(X) = f(\sigma(X)) : \mathbf{R}^{m\times n} \to \mathbf{R} \cup \{+\infty\}$$

is $\mathcal{SD}$-r.:

$$\phi(X) \le t \Leftrightarrow \exists u, x_1, ..., x_m : \left\{\begin{array}{c} Px + tp + Qu + b \succeq 0 \\ x_1 \ge ... \ge x_m \\ x_1 + ... + x_k \ge S_k\left(\left(\begin{array}{cc} & X \\ X^T & \end{array}\right)\right), \\ \hfill k = 1, ..., m. \end{array}\right.$$

E.g., the following functions of rectangular $m \times n$ matrix X are $\mathcal{SD}$-r.f.'s with explicit $\mathcal{SD}$-r.:

- $\| X \| = \sigma_{\max}(X)$ $\left[\| X \| \le t \Leftrightarrow \left(\begin{array}{cc} tI_m & X \\ X^T & tI_n \end{array}\right) \succeq 0\right]$
- $\Sigma_k(X) = \sum\limits_{i=1}^{k} \sigma_k(X)$, $1 \le k \le m$
- $\| X \|_\pi = \| \sigma(X) \|_\pi$, $\pi \ge 1$ is rational

Convex hulls of some sets of matrices

14. Let $S \subset \{x \in \mathbf{R}^n \mid x_1 \geq x_2 \geq ... \geq x_n\}$ **be an** $\mathcal{SD}$**-r.s.:**

$$S = \{x \in \mathbf{R}^n \mid \exists u : Px + Qu + b \succeq 0\}.$$

14a. The set

$$\mathcal{CS}[S] = \mathrm{Conv}\{X \in \mathbf{S}^n \mid \lambda(X) \in S\}.$$

is $\mathcal{SD}$**-r. with** $\mathcal{SD}$**-r. induced by the relation**

$$\mathcal{CS}[S] = \left\{ X \in \mathbf{S}^n \mid \exists u, x : \begin{array}{c} Px + Qu + b \succeq 0 \\ S_k(X) \leq x_1 + ... + x_k, \; k < n \\ \mathrm{Tr}(X) = x_1 + ... + x_n \end{array} \right\}$$

14b. Let, in addition, S **be monotone:**

$$(y \geq 0, y_1 \geq y_2 \geq ... \geq y_n, \exists x \in S : y \leq x) \Rightarrow y \in S.$$

The set

$$\mathcal{CR}[S] = \mathrm{Conv}\{X \in \mathbf{R}^{m \times n} \mid \sigma(x) \in S\} \quad [n \geq m].$$

is $\mathcal{SD}$**-representable with** $\mathcal{SD}$**-r. induced by the relation**

$$\mathcal{CR}[S] = \left\{ X \in \mathbf{R}^{m \times n} \mid \exists u, x : \begin{array}{c} Px + Qu + b \succeq 0 \\ \Sigma_k(X) \leq x_1 + ... + x_k, \; k \leq m \end{array} \right\}$$

"$\succeq$-epigraphs" of matrix-valued functions, I

15. ["General $\succeq$-convex quadratic matrix inequality"] The set

$$\{\ (X,Y)\in \mathbf{R}^{k\times\ell}\times\mathbf{R}^m\mid F(X)\equiv(AXB)(AXB)^T+CXD+(CXD)^T+E\preceq Y\}$$

is $\mathcal{SD}$-representable:

$$F(X)\preceq Y \Leftrightarrow \begin{pmatrix} I & (AXB)^T \\ AXB & Y-E-CXD-(CXD)^T\end{pmatrix}\succeq 0$$

16. ["Fractional-quadratic matrix inequality"] The set

$$\mathcal{F}=\mathrm{cl}\{\ (X,V,Y)\in\mathbf{R}^{p\times q}\times\mathbf{S}^q\times\mathbf{S}^p\mid V\succ 0,\ F(X,V)=XV^{-1}X^T\preceq Y\}$$

is $\mathcal{SD}$-representable:

$$\mathcal{F}=\left\{(X,V,Y)\mid \begin{pmatrix} Y & X\\ X^T & V\end{pmatrix}\succeq 0\right\}$$

17. The set

$$\mathcal{X}=\mathrm{cl}\{(X,Y)\in\mathbf{S}^n\times\mathbf{S}^m\mid X\succ 0,\ Y\preceq(A^TX^{-1}A)^{-1}\}$$
$$\left[A^TA\succ 0\right]$$

is $\mathcal{SD}$-representable:

$$\mathcal{X}=\left\{(X,Y)\mid \exists Z\in\mathbf{S}^m:\ \begin{array}{c} Z\succeq 0\\ Y\leq Z\\ X\succeq AZA^T\end{array}\right\}$$

Elementary $\mathcal{SD}$-representable functions/sets (cont.)

$\succeq$-epigraphs of matrix-valued functions, II

18. [Square root of positive semidefinite matrix] The set

$$\mathcal{X} = \{(X,Y) \in \mathbf{S}^n \times \mathbf{S}^n \mid X \succeq 0,\ Y \preceq X^{1/2}\}$$

is $\mathcal{SD}$-representable:

$$\mathcal{X} = \left\{ (X,Y) \mid \exists Z \in \mathbf{S}^n : \begin{array}{c} \begin{pmatrix} X & Z \\ Z & I \end{pmatrix} \succeq 0 \\ Z \succeq 0 \\ Y \preceq Z \end{array} \right\}$$

Nonnegative univariate polynomials
[Yu. Nesterov '97]

19. Let $\Delta \subset \mathbf{R}$ be a segment (perhaps infinite), and let

$$\mathcal{P}_k^+(\Delta) = \left\{ (x_0, ..., x_k)^T \in \mathbf{R}^{k+1} : x(\tau) \equiv \sum_{p=0}^{k} x_p \tau^p \geq 0 \;\; \forall \tau \in \Delta \right\}$$

be the set of coefficients of nonnegative on Δ polynomials of degree $\leq k$. It turns out that $\mathcal{P}_k^+(\Delta)$ admits an explicit $\mathcal{SD}$-r.

In particular, the function

$$f(x) = \sup_{\tau \in \Delta} x(\tau)$$

admits explicit $\mathcal{SD}$-r. induced by the relation

$$f(x) \leq t \Leftrightarrow (t - x_0, -x_1, ..., -x_k) \in \mathcal{P}_k^+(\Delta).$$

$\mathcal{SD}$**-r.'s of the sets** $\mathcal{P}_k^+(\Delta)$ **are as follows:**

- **$\underline{\Delta = \mathbf{R}}$:**

$$x(\tau) \equiv \sum_{i=0}^{2m} x_i \tau^i \geq 0 \quad \forall \tau \in \mathbf{R}$$

$$\Updownarrow$$

$$\begin{array}{l} \exists X \in \mathbf{S}^m : \\ \qquad X \succeq 0 \\ \qquad x(\tau) \equiv \underbrace{\sum_{i,j=1}^{m} X_{ij} \tau^{i+j-2}}_{X(\tau)} \end{array}$$

Indeed,

$$X \succeq 0$$
$$\Downarrow$$
$$X = \sum_i \xi_i \xi_i^T$$
$$\Downarrow$$
$$X(\tau) = \sum_i \left(\sum_p \xi_{i,p} \tau^p \right)^2 \geq 0$$

Vice versa, a nonnegative polynomial is the sum of squares of two polynomials, whence

$$x(\cdot) \geq 0$$
$$\Downarrow$$
$$\exists \xi_1, \xi_2 : x(\tau) = \sum_{i=1}^{2} \left(\sum_p \xi_{i,p} \tau^p \right)^2$$
$$\Downarrow$$
$$x(\cdot) = X(\cdot), \;\; X = \xi_1 \xi_1^T + \xi_2 \xi_2^T \succeq 0$$

• $\underline{\Delta = \mathbf{R}_+}$:
The set

$$\mathcal{P}_k^+(\mathbf{R}_+) = \left\{ \begin{array}{l} \textbf{coefficients of nonnegative on } \mathbf{R}_+ \\ \textbf{polynomials of degree} \leq k \end{array} \right\}$$

is the inverse image of the set

$$\mathcal{P}_{2k}^+(\mathbf{R}) = \left\{ \begin{array}{l} \textbf{coefficients of nonnegative on } \mathbf{R} \\ \textbf{polynomials of degree} \leq 2k \end{array} \right\}$$

under the linear mapping of the polynomial coefficients induced by the transformation

$$x(\tau) \mapsto (Ax)(\tau) \equiv x(\tau^2).$$

• $\underline{\Delta = [0,1]}$:
The set

$$\mathcal{P}_k^+([0,1]) = \left\{ \begin{array}{l} \textbf{coefficients of nonnegative on } [0,1] \\ \textbf{polynomials of degree} \leq k \end{array} \right\}$$

is the inverse image of $\mathcal{P}_{2k}^+(\mathbf{R})$ under the linear mapping of the polynomial coefficients induced by the transformation

$$x(\tau) \mapsto [Bx](\tau) = (1+\tau^2)^k x\left(\frac{\tau^2}{1+\tau^2}\right).$$

Thus, $\mathcal{SD}$-r.'s of $\mathcal{P}_+^k(\mathbf{R}^+)$ and $\mathcal{P}_k^+([0,1])$ are readily given by the above $\mathcal{SD}$-r. of $\mathcal{P}_{2k}^+(\mathbf{R})$.

Nonnegative trigonometric polynomials [Yu. Nesterov '97]

20. Let $\Delta \subset [-\pi, \pi]$ be a segment, and let

$$\mathcal{T}_k^+(\Delta) = \Big\{(\xi_0, \xi_1, ..., \xi_{2k})^T \in \mathbf{R}^{2k+1} : \\ \xi(\phi) \equiv \xi_0 + \sum_{p=1}^{k} (\xi_{2p-1}\cos(p\phi) + \xi_{2p}\sin(p\phi)) \geq 0 \ \ \forall \phi \in \Delta\Big\}$$

be the set of coefficients of the nonnegative on Δ trigonometric polynomials of degree $\leq k$. The set $\mathcal{T}_k^+(\Delta)$ admits an explicit $\mathcal{SD}$-r.

In particular, the function

$$f(\xi) = \max_{\phi \in \Delta} \xi(\phi)$$

admits an explicit $\mathcal{SD}$-r.

Indeed, $\mathcal{T}_k^+(\Delta)$ is the inverse image of $\mathcal{P}_{2k}^+(\Gamma)$ under the linear mapping of coefficients induced by the transformation

$$\xi(\phi) \mapsto x(\tau) = (1 + \tau^2)^k \xi(2\,\mathrm{atan}(\tau));$$

here

$$\Gamma = \{\tau : 2\,\mathrm{atan}(\tau) \in \Delta\}.$$

Convex hulls of epigraphs of univariate polynomials

21. For a segment $\Delta \subset \mathbf{R}$ and a continuous function $p(\cdot)$ on Δ, let

$$\mathrm{ConvEpi}(\Delta, p) = \mathrm{Conv}\left\{(\tau, t) \mid \tau \in \Delta, t \geq p(\tau)\right\}$$

be the convex hull of the epigraph of Δ.

For every one of the following pairs (Δ, p), the set $\mathrm{ConvEpi}(\Delta, p)$ admits an explicit $\mathcal{SD}$-r.:

21.a. $\Delta = \mathbf{R}$, $p(\tau) = \sum\limits_{\ell=0}^{2k} p_\ell \tau^\ell$ **with** $p_{2k} > 0$

21.b. $\Delta = \mathbf{R}_+$, $p(\tau) = \sum\limits_{\ell=0}^{k} p_\ell \tau^\ell$ **with** $p_k > 0$

21.c. $\Delta = [0, 1]$, $p(\tau) = \sum\limits_{\ell=0}^{k} p_\ell \tau^\ell$

E.g. in the case of 21.a an $\mathcal{SD}$-r. of $\mathrm{ConvEpi}(\Delta, p)$ is

$$\left\{ (\tau, t) \mid \exists \tau_2, \tau_3, ..., \tau_{2k} : \begin{array}{l} \begin{pmatrix} 1 & \tau & \tau_2 & \tau_3 & \cdots & \tau_k \\ \tau & \tau_2 & \tau_3 & \tau_4 & \cdots & \tau_{k+1} \\ \tau_2 & \tau_3 & \tau_4 & \tau_5 & \cdots & \tau_{k+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau_k & \tau_{k+1} & \tau_{k+2} & \tau_{k+3} & \cdots & \tau_{2k} \end{pmatrix} \succeq 0 \\ p_0 + p_1\tau + p_2\tau_2 + p_3\tau_3 + ... + p_{2k}\tau_{2k} \leq t \end{array} \right\}$$

• $\mathcal{SD}$**-r. of the above sets** $\mathrm{ConvEpi}(\Delta, p)$ **are given by the following construction.**

1. Let

$$p(\tau) = p_0 + p_1\tau + \sum_{\ell=2}^{n} p_\ell f_\ell(\tau),$$
$$\phi(\tau) = (1, \tau, f_2(\tau), ..., f_n(\tau))^T : \Delta \to \mathbf{R}^{n+1}$$

Assume that

A. The set $Y = \overline{\mathrm{Conv}}\{\phi(\tau) \mid \tau \in \Delta\}$ **is** $\mathcal{SD}$**-r. with a known** $\mathcal{SD}$**-r.**

<u>**Remark**</u> Y **is the intersection of the cone** *dual* **to the "cone of nonnegative polynomials"**

$$\mathcal{P}^+ = \{(p_0, p_1, ..., p_n) \mid p(\tau) \geq 0 \quad \forall \tau \in \Delta\}$$

and the hyperplane $\{(1, \pi_1, \pi_2, ..., \pi_n)^T \in \mathbf{R}^{n+1}\}$**.**

B. The set $\mathrm{ConvEpi}(\Delta, p)$ **is closed.**

Under these assumptions $\mathrm{ConvEpi}(\Delta, p)$ **admits the representation**

$$\left\{(\tau, t) \mid \exists \tau_1, \tau_2, ..., \tau_n : \begin{array}{c} (\tau, \tau_1, \tau_2, ..., \tau_n)^T \in Y \\ \sum_{i=1}^{n} p_i \tau_i \leq t \end{array}\right\}$$

Thus, $\mathrm{ConvEpi}(\Delta, p)$ **is a projection of a** $\mathcal{SD}$**-r.s. (the intersection of** Y **and a half-space). By Calculus, an** $\mathcal{SD}$**-r. of** $\mathrm{ConvEpi}(\Delta, p)$ **is readily given by an** $\mathcal{SD}$**-r. of** Y**.**

2. In the situations 21.a – 21.c

- **Assumption A is satisfied by the following reasons:**

  - **The corresponding "cone of nonnegative polynomials" is $\mathcal{P}_k^+(\Delta)$, for which we know $\mathcal{SD}$-r. which in fact is strictly feasible.**
  - **By Calculus, a strictly feasible $\mathcal{SD}$-r. of the cone $\mathcal{P}_k^+(\Delta)$ explicitly induces an $\mathcal{SD}$-r. of the cone dual to $\mathcal{P}_k^+(\Delta)$ and thus, by Remark – for Y.**

- **Assumption B holds true (a straightforward verification).**

Elementary $\mathcal{SD}$-representable functions/sets (cont.)

Families of ellipsoids

• **Let**

$$V(X,x) = \{u \in \mathbf{R}^n \mid u = Xv + x, v^Tv \le 1\} \qquad [X \in \mathbf{R}^{n\times n}]$$

$$W(Y,y) = \{u \in \mathbf{R}^n \mid u^TY^TYu - 2u^TY^Ty + y^Ty \le 1\} \quad [Y \in \mathbf{R}^{n\times n}]$$

• $V(\cdot,\cdot)$ **is a natural parameterization of ellipsoids in** $\mathbf{R}^n$**, including "flat" ones (***an ellipsoid is the image of the unit Euclidean ball under affine mapping***)**

• $W(\cdot,\cdot)$ **is a natural parameterization of ellipsoids in** $\mathbf{R}^n$**, including "elliptic cylinders" (***an ellipsoid is a level set of a below bounded convex quadratic function***)**

Proposition [Boyd et al?]

$$V(X,x) \subset W(Y,y)$$

$$\Updownarrow$$

$$\exists \lambda : \left(\begin{array}{ccc} I & Yx-y & YX \\ x^TY^T - y^T & 1-\lambda & \\ X^TY^T & & \lambda I \end{array}\right) \succeq 0 \qquad (1)$$

<u>Note</u>: When one of the ellipsoids $V(X,x)$ or $W(Y,y)$ is fixed, (*) becomes a Linear Matrix Inequality in the parameters of the other ellipsoid and λ.
Thus, both the sets

$$\{(X,x) \in \mathbf{S}^N_+ \mid V(X,x) \subset W(A,a)\}$$
$$\{(Y,y) \mid V(A,a) \subset W(Y,y)\}$$

admit explicit $\mathcal{SD}$-r.'s.

As a result, both the problems

Find the smallest volume ellipsoid containing the union of a given finite family of ellipsoids

and

Find the largest volume ellipsoid contained in the intersection of a given finite family of ellipsoids

can be posed as semidefinite programs.

• **The problem**

Find the largest volume ellipsoid $V(X,x)$ contained in every one of given ellipsoids $W(Y_\ell, y_\ell)$, $\ell = 1, ..., k$

is equivalent to

$$\begin{array}{c} \tau \to \max \\ (\mathrm{Det}\ X)^{\frac{1}{n}} \geq \tau \\ \begin{pmatrix} I & Y_\ell x - y_\ell & Y_\ell X \\ x^T Y_\ell - y_\ell^T & 1 - \lambda_\ell & \\ X Y_\ell & & \lambda_\ell I \end{pmatrix} \succeq 0, \ \ell = 1, ..., k \\ X \succeq 0 \end{array} \qquad \textbf{(I)}$$

with design variables $x, X, \lambda_1, ..., \lambda_k$**.**

• **The problem**

Find the smallest volume ellipsoid $W(Y,y)$ containing every one of given ellipsoids $V(X_\ell, y_\ell)$, $\ell = 1, ..., k$

is equivalent to

$$\begin{array}{c} \tau \to \max \\ (\mathrm{Det}\ Y)^{\frac{1}{n}} \geq \tau \\ \begin{pmatrix} I & Y x_\ell - y & Y X_\ell \\ x_\ell^T Y & 1 - \lambda_\ell & \\ X_\ell Y & & \lambda_\ell I \end{pmatrix} \succeq 0, \ \ \ell = 1, ..., k \\ Y \succeq 0 \end{array} \qquad \textbf{(O)}$$

with design variables $y, Y, \lambda_1, ..., \lambda_k$**.**

Since the feasible sets of every one of the constraints in (I) and (O) admit explicit $\mathcal{SD}$-r.'s, both problems can be straightforwardly converted to semidefinite programs.