What can be expressed via Conic Quadratic and Semidefinite Programming?

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What can be expressed via CQP and SDP?

- Let us look at three generic families of convex programs:
  - Linear Programming:
    \[ c^T x \rightarrow \min | \ A x + b \geq 0 \]  
    (LP)
  - Conic Quadratic Programming:
    \[ c^T x \rightarrow \min | \ || A_i x + b_i ||_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \]  
    (CQP)
  - Semidefinite Programming:
    \[ c^T x \rightarrow \min | \ \sum_{i=1}^{n} x_i A_i + B \succeq 0 \]  
    (SDP)
- Geometrically, all these problems are of the form
  \[ c^T x \rightarrow \min | \ A x + b \in K, \]  
  (CP)
  where K is a closed pointed convex cone with a nonempty interior belonging to a specific family of convex cones.


\[ c^T x \rightarrow \min \mid Ax + b \in K \quad [K \in \mathcal{K}] \quad \text{(CP)} \]

**LP:** \( \mathcal{K} \) is comprised of direct products of rays \( \mathbb{R}_+ \)

**CQP:** \( \mathcal{K} \) is comprised of direct products of the Lorentz cones

\[ L^n = \{(x, t) \in \mathbb{R}^{n+1} : \|x\|_2 \leq t\} \]

**SDP:** \( \mathcal{K} \) is comprised of direct products of the semidefinite cones

\[ S^n_+ = \{A \in S^n : A \succeq 0\} \]

**Note:** The above families of cones are closed w.r.t.

(1) taking (finite) direct products of cones;
(2) passing from a cone \( K \) to its dual cone

\[ K^* = \{\eta \mid \eta^T x \geq 0 \quad \forall x \in K\}. \]
Assume that we are given a family $\mathcal{K}$ of finite-dimensional convex cones (closed, pointed and with a nonempty interior) and know how to solve problems of the form
\[
c^Ty \to \min | Ay + b \in K
\] (*)
with all possible data $(c, A, b)$ and all $K \in \mathcal{K}$.

**Question:** What is the family of problems we can actually solve? When an optimization problem
\[
e^Tx \to \min | x \in X \subset \mathbb{R}^n
\] (P)
can be equivalently reformulated in the form of (*)?

**An answer:** This is the case when $X$ is a $\mathcal{K}$-representable set ($\mathcal{K}$-r.s.), i.e., there exists a $\mathcal{K}$-representation ($\mathcal{K}$-r.) of $X$
\[
X = \{ x \in \mathbb{R}^n | \exists u \in \mathbb{R}^k : Px + Qu + b \in K \subset \mathcal{K} \} \tag{R}
\]

Indeed, given the data $P, Q, b, K$ of (R), we can rewrite (P) equivalently in the form of (*):
\[
d^Tx \to \min | x \in X
\]
\[
\uparrow
\]
\[
c^Ty \equiv d^Tx \to \min | Ay + b \equiv Px + Qu + b \in K
\]
\[
\begin{bmatrix}
y = \\
\begin{pmatrix}
x \\
u
\end{pmatrix}
\end{bmatrix}
\]
Thus, a natural interpretation of the question

*Given a possibility to solve problems*

\[ c^T x \rightarrow \min \mid Ax + b \in K \in \mathcal{K} \]

*what can we actually solve?*

is

- **What are \( \mathcal{K} \)-representable sets?**

- **How to recognize \( \mathcal{K} \)-representability?**
• **Claim:** Consider a family \( \mathcal{K} \) of finite-dimensional closed pointed cones with a nonempty interior, and let this family be closed w.r.t.
  • taking direct products
  • passing from a cone to its dual.

There exists a simple and powerful “calculus” of \( \mathcal{K} \)-representable sets: essentially,

*Every standard convexity-preserving operation as applied to (finitely many) \( \mathcal{K} \)-representable sets \( X_1, \ldots, X_k \), yields a \( \mathcal{K} \)-representable result \( X \).*

Moreover, a \( \mathcal{K} \)-representation of \( X \) is “readily given” by \( \mathcal{K} \)-representations of \( X_1, \ldots, X_k \).

• Applying the “calculus machinery” to (specific for a family \( \mathcal{K} \)) collection of “raw materials” — simple \( \mathcal{K} \)-representable sets — we get a possibility to recognize complicated \( \mathcal{K} \)-representable sets and thus to pose various optimization problems in the “\( \mathcal{K} \)-form”.


I. The intersection of finitely many $\mathcal{K}$-r.s.’s is a $\mathcal{K}$-r.s.:

\[ X_i = \{ x \mid \exists u_i : P_i x + Q_i u_i + b_i \in K_i \} , \ i = 1, \ldots, k \]

\[
\bigcap_{i=1}^{k} X = \{ u \mid \exists u = \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} : \\
\begin{pmatrix} P_1 x + Q_1 u_1 + b_1 \\ \vdots \\ P_k x + Q_k u_k + b_k \end{pmatrix} \in K_1 \times \ldots \times K_k \} \]

II. The direct product of finitely many $\mathcal{K}$-r.s.’s is a $\mathcal{K}$-r.s.:

\[ X_i = \{ x_i \mid \exists u_i : P_i x_i + Q_i u_i + b_i \in K_i \} , \ i = 1, \ldots, k \]

\[
X_1 \times \ldots \times X_k = \{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \mid \exists u = \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} : \\
\begin{pmatrix} P_1 x_1 + Q_1 u_1 + b_1 \\ \vdots \\ P_k x_k + Q_k u_k + b_k \end{pmatrix} \in K_1 \times \ldots \times K_k \} \]
III. The image of a $\mathcal{K}$-r.s. under an affine mapping is a $\mathcal{K}$-r.s.:

$$X = \{x \mid \exists u : Px + Qu + b \in \mathbf{K}\}$$

$$\downarrow$$

$$AX + a = \{z \mid \exists v \in \text{Ker } A \; \exists u : PBz + Pv + Qu + [b - PBa] \in \mathbf{K}\}$$

[A is an onto mapping, $ABz = z \quad \forall z$]

IV. The inverse image of a $\mathcal{K}$-r.s. under an affine mapping is a $\mathcal{K}$-r.s.:

$$X = \{x \mid \exists u : Px + Qu + b \in \mathbf{K}\}, \; Z = \{z \mid Az + a \in X\}$$

$$\downarrow$$

$$Z = \{z \mid \exists u : PAz + Qu + [b + Pa] \in \mathbf{K}\}$$

V. The arithmetic sum of finitely many $\mathcal{K}$-r.s.’s

$$X = X_1 + \ldots + X_k = \{x \mid \exists x_i \in X_i : x = \sum_i x_i\}$$

is a $\mathcal{K}$-r.s.

Indeed, the arithmetic sum of $k$ sets is the image of their direct product under the linear mapping

$$\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \mapsto x_1 + \ldots + x_k.$$
VI. A polyhedral set

\[ X = \{ x \mid a_i^T x + b_i \geq 0, \ i = 1, \ldots, k \} \]

is a \( \mathcal{K} \)-r.s.

Indeed, let \( K \in \mathcal{K} \) and \( 0 \neq e \in K \). Then

\[
X = \left\{ x \mid \begin{pmatrix} [a_1^T x + b_1]e \\ \vdots \\ [a_k^T x + b_k]e \end{pmatrix} \in K \times \cdots \times K \right\}.
\]

- Intersection of infinitely many half-spaces not necessarily is \( \mathcal{K} \)-representable. It, however, is so when the half-spaces are “well-organized”.
VIa. Assume that the data \((\alpha, \beta)\) defining a half-space
\[X_{\alpha, \beta} = \{x \mid \alpha^T x + \beta \geq 0\}\]
vary in a \(\mathcal{K}\)-r.s. \(\mathcal{U}\):
\[\mathcal{U} = \{(\alpha, \beta) \mid \exists u : P\alpha + \beta p + Qu + b \in \mathbf{K}\}\]
and let the representation be strictly feasible:
\[\exists \bar{\alpha}, \bar{\beta}, \bar{u} : P\bar{\alpha} + \bar{\beta} p + Q\bar{u} + b \in \text{int } \mathbf{K}\]
Then the set
\[X = \bigcap_{(\alpha, \beta) \in \mathcal{U}} X_{\alpha, \beta} = \{x \mid \alpha^T x + \beta \geq 0 \quad \forall (\alpha, \beta) \in \mathcal{U}\}\]
is a \(\mathcal{K}\)-r.s.

Indeed,
\[X = \left\{ x \mid 0 \leq \inf_{\alpha, \beta, u} \{\alpha^T x + \beta : P\alpha + \beta p + Qu + b \in \mathbf{K}\} \right\}\]
\[\Downarrow \quad \text{[Conic Duality Theorem]}\]
\[X = \left\{ x \mid 0 \leq \max_{\eta} \left\{-b^T \eta : \begin{array}{l} \eta \in \mathbf{K}_* \\
\quad \quad P^T \eta = x \\
\quad \quad p^T \eta = 1 \\
\quad \quad Q^T \eta = 0 \end{array} \right\} \right\}\]
\[\Uparrow\]
\[X = \left\{ x \mid \exists \eta : \begin{array}{l} \eta \in \mathbf{K}_* \\
\quad \quad P^T \eta = x \\
\quad \quad p^T \eta = 1 \\
\quad \quad Q^T \eta = 0 \\
\quad \quad b^T \eta \leq 0 \end{array} \right\}\]

Thus, \(X\) is the projection of the intersection of the \(\mathcal{K}\)-r.s. \(\mathbf{K}_*\) and a polyhedral set and thus is a \(\mathcal{K}\)-r.s.
Several operations with sets “nearly preserve” \( \mathcal{K} \)-representability.

**Definition.** A set \( X \subset \mathbb{R} \) is called nearly \( \mathcal{K} \)-representable, if there exists a \( \mathcal{K} \)-r.s. \( X' \) such that

\[
X \subset X' \subset \text{cl} X. \tag{*}
\]

A nearly \( \mathcal{K} \)-representation of \( X \) is a \( \mathcal{K} \)-representation of a set \( X' \) satisfying (*)..

**Note:** Nearly \( \mathcal{K} \)-representable closed set \( X \) is \( \mathcal{K} \)-representable, and every nearly \( \mathcal{K} \)-representation of such a set is its \( \mathcal{K} \)-representation.

**VII. The conic hull**

\[
\text{Cone}(X) = \{0\} \cup \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid t > 0, t^{-1}x \in X\}
\]

of a nonempty \( \mathcal{K} \)-r.s. \( X \) is a nearly \( \mathcal{K} \)-r.s.:

\[
X = \{x \mid \exists u : Px + Qu + b \in \mathbb{K}\}
\]

\[
\downarrow
\]

\[
\text{Cone}(X) \subset X' \subset \overline{\text{Cone}(X)},
\]

\[
X' = \{(x, t) \mid \exists u : \left(\begin{array}{c} Px + Qu + tb \\ te \end{array}\right) \in \mathbb{K} \times \mathbb{K}\}
\]

\[
[0 \neq e \in \mathbb{K}]
\]
VIII. [Yu. Nesterov] Convex hull of finitely many $\mathcal{K}$-r.s.’s is a nearly $\mathcal{K}$-r.s.:

$$X_i = \{x \mid \exists u_i : P_i x + Q_i u_i + b_i \in K_i \}, \ i = 1, \ldots, k$$

$$\downarrow$$

$$\text{Conv} \left( \bigcup_{i=1}^{k} X_i \right) \subset X \subset \overline{\text{Conv} \left( \bigcup_{i=1}^{k} X_i \right)},$$

$$X = \{x \mid \exists (x_1, \ldots, x_k, u_1, \ldots, u_k, t_1, \ldots, t_k) :$$

$$\left( \begin{array}{c}
P_1 x_1 + Q_1 u_1 + t_1 b_1 \\
\vdots \\
P_k x_k + Q_k u_k + t_k b_k 
\end{array} \right) \in K_1 \times \ldots \times K_k$$

$$x = x_1 + \ldots + x_k$$

$$t_1 + \ldots + t_k = 1$$

$$t_i \geq 0, \ i = 1, \ldots, k \}$$
\( \mathcal{K} \)-representability of \( X \) is readily given by the fact that \( X \) is the projection on the \( x \)-space of the intersection of the polyhedral (and thus \( \mathcal{K} \)-representable) set

\[
\begin{align*}
(x_1, \ldots, x_k, u_1, \ldots, u_k, t_1, \ldots, t_k) & : x = x_1 + \ldots + x_k \quad t_1 + \ldots + t_k = 1 \\
& \quad t_i > 0, \ i = 1, \ldots, k
\end{align*}
\]

and the clearly \( \mathcal{K} \)-r.s.

\[
\begin{align*}
(x_1, \ldots, x_k, u_1, \ldots, u_k, t_1, \ldots, t_k) & : P_1 x_1 + Q_1 u_1 + t_1 b_1 \\
& \quad \quad \ldots \\
& \quad P_k x_k + Q_k u_k + t_k b_k 
\end{align*}
\in \mathbf{K}_1 \times \ldots \times \mathbf{K}_k
\]
Calculus of $\mathcal{K}$-representable sets (cont.)

IX. Let

$$X = \{ x \mid \exists u : Px + Qu + b \in K \}$$

be a $\mathcal{K}$-r.s., and let the representation (*) be strictly feasible:

$$\exists (\bar{x}, \bar{u}) : P\bar{x} + Q\bar{u} + b \in \text{int } K.$$ 

Then the polar cone of $X$ – the set

$$X_* = \{ (\eta, s) \mid \eta^T x \geq s \quad \forall x \in X \}$$

– is a $\mathcal{K}$-r.s.:

$$X^* = \left\{ (\eta, s) \mid \exists \zeta \in K_* : \begin{array}{l}
            P^T \zeta = \eta \\
            Q^T \zeta = 0 \\
            s + b^T \zeta \leq 0
          \end{array} \right\}$$
\( f(x) \leq 0 \)
\[ f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \]

\textbf{Definition:} \( f \) is called a \( K \)-representable function (\( K \)-r.f.), if the epigraph of \( f \)
\[ \text{Epi}(f) = \{(x, t) \mid t \geq f(x)\} \]
is a \( K \)-r.s.

A \( K \)-representation of \( \text{Epi}(f) \) is called a \( K \)-representation (\( K \)-r.) of \( f \).

\textbf{Observation:} If \( f \) is \( K \)-representable, then so are all level sets
\[ \{x \mid f(x) \leq a\} \quad [a \in \mathbb{R}] \]
of \( f \).

Indeed,
\[ f(x) \leq t \Leftrightarrow \exists u : Px + tp + Qu + b \in K \]
\[ \downarrow \]
\[ \{x \mid f(x) \leq a\} = \{x \mid \exists u : Px + Qu + [ap + b] \in K\} \]
$\mathcal{K}$-representable functions (cont.)

- Calculus of $\mathcal{K}$-representable sets can be straightforwardly converted into a “calculus of $\mathcal{K}$-representable functions”. The latter, essentially, can be summarized in the following statements:

(i). [Conic combinations and taking maximum] A linear combination, with nonnegative coefficients, and the maximum of finitely many $\mathcal{K}$-r.f.’s are themselves $\mathcal{K}$-r.f.’s.

(ii). [Adding affine form] The sum of a $\mathcal{K}$-representable and an affine function is $\mathcal{K}$-representable.

(iii). [Affine substitution of argument] The superposition of a $\mathcal{K}$-r.f. and an affine mapping is a $\mathcal{K}$-r.f.
(iv). [Taking superpositions] Let functions

\[ f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \]

and a function

\[ g : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\} \]

be \( \mathcal{K} \)-representable:

\[ f_i(x) \leq t \iff \exists u_i : P_i x + t p_i + Q_i u_i + b_i \in K_i, \ i = 1, \ldots, k \]

\[ g(z) \leq t \iff \exists v : P z + Q v + b \in K. \]

Let also \( g \) be monotone on its domain:

\[ g(z) < \infty, z \leq z' \Rightarrow g(z) \leq g(z'). \]

Then the superposition

\[ h(x) = g(f_1(x), \ldots, f_k(x)) \]

is a \( \mathcal{K} \)-r.f.:

\[ h(x) \leq t \]

\[ \Downarrow \]

\[ \exists \tau = (\tau_1, \ldots, \tau_k)^T, u_1, \ldots, u_k, v : \]

\[ P_i x + \tau_i p_i + Q_i u_i + b_i \in K_i, \ i = 1, \ldots, k \]

says that \( f_i(x) \leq \tau_i \)

\[ P \tau + Q v + b \in K \]

says that \( g(\tau) \leq t \)
\( (v). \) [Legendre transformation] The Legendre transformation

\[ f^*(\eta) = \sup_x [\eta^T x - f(x)] \]

of a \( \mathcal{K}\text{-r.f.} \) \( f \) with strictly feasible \( \mathcal{K}\text{-r.} \) is itself a \( \mathcal{K}\text{-r.f.:} \)

\[
f(x) \leq t \Leftrightarrow \exists u : Px + tp + Qu + b \in \mathcal{K} \\
[\exists \bar{x}, t, \bar{u} : P\bar{x} + t\bar{p} + Q\bar{u} + b \in \text{int } \mathcal{K}] \\
\downarrow \\
\]

\[
f^*(\eta) \leq \tau \Leftrightarrow \exists \zeta : \begin{align*}
\zeta & \in \mathcal{K}^* \\
P^T\zeta + \eta & = 0 \\
Q^T\zeta & = 0 \\
p^T\zeta & = 1 \\
\tau - b^T\zeta & \geq 0
\end{align*}
\]
(vi). [Conic transformation] Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{ +\infty \} \) be a \( \mathcal{K} \)-r.f:

\[
f(x) \leq t \iff \exists u : Px + tp + Qu + b \in \mathcal{K}
\]

Then the conic transformation of \( f \) — the function

\[
\text{Cone}[f](x, s) = \begin{cases} sf(x/s), & s > 0 \\ +\infty, & s \leq 0 \end{cases}
\]

is a nearly \( \mathcal{K} \)-r.f.:

\[
\text{Epi}(\text{Cone}[f]) \subset X \subset \overline{\text{Epi}(\text{Cone}[f])},
\]

\[
X = \{(x, s, t) \mid \exists u : Px + tp + Qu + sb \in \mathcal{K}\}
\]
(vii). [Partial minimization] Let \( f(x, y) : \mathbb{R}_x^m \times \mathbb{R}_y^n \to \mathbb{R} \cup \{+\infty\} \) be a \( \mathcal{K} \)-r.f.:

\[
f(x, y) \leq t \iff \exists u : Px + Ry + tp + Qu + b \in \mathcal{K}.
\]

Let

\[
\phi(x) = \inf_y f(x, y),
\]

and let \( \inf \) be attained for every \( x \) where it is not \( +\infty \). Then \( \phi \) is a \( \mathcal{K} \)-r.f.:

\[
\phi(x) \leq t \iff \exists y, u : Px + tp + [Ry + Qu] + b \in \mathcal{K}.
\]

(viii). [inf-convolution] Let \( f_i(x), i = 1, \ldots, k, \) be \( \mathcal{K} \)-r.f.’s:

\[
f_i(x) \leq t_i \iff \exists u_i : P_i x + t_i p_i + Q_i u_i + b_i \in \mathcal{K}_i, \quad i = 1, \ldots, k.
\]

Let

\[
\phi(x) = \inf_{x_1, \ldots, x_k, \sum x_i = x} \sum f_i(x_i),
\]

and let the infimum be attained at every \( x \) where it is not \( +\infty \). Then \( \phi \) is a \( \mathcal{K} \)-r.f.:

\[
\phi(x) \leq t \iff \exists \{x_i, u_i, t_i\} : \begin{cases}
P_i x_i + t_i p_i + Q_i u_i + b_i \in \mathcal{K}_i, \ i = 1, \ldots, k \\
\sum x_i = x \\
\sum t_i \leq t
\end{cases}
\]
Note: All the rules of the Calculus of $\mathcal{K}$-representable functions/sets are “algorithmic and efficient”: a $\mathcal{K}$-r. of the result of an operation is readily given by $\mathcal{K}$-r.’s of the operands. E.g.,

\[
\begin{align*}
    f_i(x) \leq t & \iff \exists u_i : P_i x + t p_i + Q_i u_i + b_i \in K_i, \quad i = 1, \ldots, k \\
    \alpha_i & \geq 0, \quad i = 1, \ldots, k \\

\end{align*}
\]

\[
\begin{align*}
    \sum_i \alpha_i f_i(x) & \leq t \\
    \exists t_1, \ldots, t_k, u_1, \ldots, u_k : \\
    P_i x + t_i p_i + Q_i u_i + b_i & \in K_i, \quad i = 1, \ldots, k \\
\text{says that} \quad f_i(x) & \leq t_i \\
    \sum_i \alpha_i t_i & \leq t
\end{align*}
\]
What can be expressed via LP?

\[ \mathcal{K} = \mathcal{LP} \equiv \{ \mathbb{R}^n_+ \}_{n=1}^\infty \]

- Of course,
  - \( \mathcal{LP} \)-representable sets are exactly the polyhedral sets,
  - \( \mathcal{LP} \)-representable functions are exactly convex piecewise linear functions.

E.g., the function

\[ f(x_1, \ldots, x_n) = \prod_{i=1}^n x_i^{-\frac{1}{n}} : \text{int } \mathbb{R}^n_+ \to \mathbb{R} \]

is not \( \mathcal{LP} \)-representable. But...
What can be expressed via CQP?

\[
\mathcal{K} = \mathcal{CQ} \equiv \left\{ \mathbf{L}^n \right\}_{n=0}^{\infty} \\
\mathbf{L}^n = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \| x \|_2 \leq t \right\}
\]

Elementary \( \mathcal{CQ} \)-representable functions/sets

- The following functions/sets are \( \mathcal{CQ} \)-representable with explicit \( \mathcal{CQ} \)-r.’s:
  1. An affine function \( f(x) = a^T x + b \)
  2. The Euclidean norm \( \| x \|_2: \mathbb{R}^n \to \mathbb{R} \)
  3. The set

\[
\{(x, s, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : s, t \geq 0, x^T x \leq st \}. \quad (*)
\]

Indeed, this set is the inverse image of \( \mathbf{L}^{n+1} \) under the linear transformation

\[
\begin{pmatrix} x \\ t \\ s \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{t-s}{2} \\ \frac{t+s}{2} \end{pmatrix}
\]

4. The squared Euclidean norm \( f(x) = x^T x : \mathbb{R}^n \to \mathbb{R} \)

Indeed, the epigraph of \( x^T x \) is the intersection of the set \((*)\) and the hyperplane \( s = 1 \).

**Note:** by Calculus it follows from 4. that every convex quadratic form is \( \mathcal{CQ} \)-representable.
5. A simple fractional-quadratic function

\[ f(x, s) = \frac{x^2}{s} : \{ (x, s) \in \mathbb{R}^2 \mid s \geq 0 \} \to \mathbb{R} \cup \{ +\infty \} \]

is \( CQ \)-representable.

Indeed, the epigraph of \( f \) is the \( CQ \)-r.s.

\[ \{ (x, t, s) \in \mathbb{R}^3 \mid s, t \geq 0, x^2 \leq st \} \]

**Application example:** The *Multi-Load Truss Topology Design problem*

\[ \tau \to \min \mid \left\{ \left( \begin{array}{c} \tau \\ f_{\ell} \\ \sum_{i=1}^{n} t_i B_i B_i^T \end{array} \right) \geq 0, \ \ell = 1, \ldots, k \\
\sum_{i} t_i \geq 0, \ Pt + p \geq 0 \\
\sum_i B_i B_i^T \succeq 0, \ \exists \bar{t} > 0 : P\bar{t} + p > 0 \right\} \]

is equivalent to the Conic Quadratic problem

\[ \tau \to \min \]

\[ \sum_{i=1}^{n} s_{i, \ell} \leq \tau, \ \ell = 1, \ldots, k \]

\[ \left\| \left( \begin{array}{c} y_{i, \ell} \\ t_i - s_{i, \ell} \end{array} \right) \right\|_2 \leq \frac{t_i + s_{i, \ell}}{2}, \ i = 1, \ldots, n, \ell = 1, \ldots, k \]

\[ \sum_{i=1}^{n} B_i y_{i, \ell} = f_{\ell}, \ \ell = 1, \ldots, k \]

\[ Pt + p \geq 0 \]

\[ t \geq 0 \]
6. Let \( \pi_1, \ldots, \pi_k \) be positive rationals with \( \sum_{i=1}^{k} \pi_i \leq 1 \). Then the function

\[
f(x) = - \prod_{i=1}^{k} x_{i}^{\pi_i} : \mathbb{R}_{+}^{k} \to \mathbb{R}
\]

admits an explicit \( CQ \)-r. The “size” of the representation is proportional to the magnitude of the common denominator of the fractions \( \pi_1, \ldots, \pi_k \).

Indeed,

A. The set

\[
\{(t, u_1, u_2) \in \mathbb{R}_{+}^{3} : t^2 \leq u_1 u_2\}
\]

is \( CQ \)-representable.

B. It follows that the set

\[
X_r = \{(t, u_1, \ldots, u_{2^r}) \in \mathbb{R}_{+}^{2^r+1} : t \leq (u_1, \ldots, u_{2^r})^{\frac{1}{2^r}}\}
\]

also is \( CQ \)-representable.

Indeed, \( X_r \) is the projection onto the \((t, u)\)-space of the set given by the inequalities

\[
\begin{align*}
0 \leq u_1, \ldots, u_{2^r} & \geq 0 \\
0 \leq v_{1,j} & \leq \sqrt{u_{2^r-j}u_{2^r}}, \quad 1 \leq j \leq 2^{r-1} \\
0 \leq v_{2,j} & \leq \sqrt{v_{1,2^r-j}v_{1,2^r}}, \quad 1 \leq j \leq 2^{r-2} \\
& \quad \ldots \ldots \ldots \ldots \\
0 \leq v_{r-1,1} & \leq \sqrt{v_{r-2,1}v_{r-2,2}}, 0 \leq v_{r-1,2} \leq \sqrt{v_{r-2,3}v_{r-2,4}} \\
0 \leq t & \leq \sqrt{v_{r-1,1}v_{r-1,2}}
\end{align*}
\]

and the latter set is \( CQ \)-representable as the intersection of \( 2^r - 1 \) \( CQ \)-representable sets (direct products of linear spaces and sets of type \((1)\)).
C. Further, the set
\[ \{(x_1, \ldots, x_k, t) \in \mathbb{R}^{k+1}_+ \mid t \leq x_1^{q_{p_1}} \cdots x_k^{q_{p_k}} \} \]  
(2)
is the inverse image of \( X_r \) with \( 2^r \geq q \) under the affine mapping
\[
(x_1, \ldots, x_k, t) \mapsto (t, x_1, \ldots, x_{1/p_1}, \ldots, x_{2/p_k}, t, \ldots, t, 1, \ldots, 1)
\]
\[ p_1 \]
\[ p_k \]
\[ 2^r - q \]
\[ q - p_1 - \ldots - p_k \]

D. Since the set (2) is \( \mathcal{CQ} \)-representable, so is the set
\[
\{(x_1, \ldots, x_k, t, s) \mid x_1, \ldots, x_k \geq 0, 0 \leq s - t \leq x_1^{\frac{q_{p_1}}{p_1}} \cdots x_k^{\frac{q_{p_k}}{p_k}} \}
\]
(it is the inverse image of (2) under an affine mapping) and thus – the set
\[
\{(x_1, \ldots, x_k, t, s) \mid s, x_1, \ldots, x_k \geq 0, 0 \leq s - t \leq x_1^{\frac{q_{p_1}}{p_1}} \cdots x_k^{\frac{q_{p_k}}{p_k}} \}
\]  
(3)

E. Finally, the epigraph
\[
\{(x_1, \ldots, x_k, t) \mid x_1, \ldots, x_k \geq 0, t \geq -x_1^{\frac{q_{p_1}}{p_1}} \cdots x_k^{\frac{q_{p_k}}{p_k}} \}
\]
of the function \( f(x) = -\prod_{i=1}^k x_i^{q_{p_i}} \) of \( x \geq 0 \) is the projection on the \((x, t)\)-space of the \( \mathcal{CQ} \)-r.s. (3).
7. Let \( \pi_1, \ldots, \pi_k \) be positive rationals. Then the function
\[
f(x) = \prod_{i=1}^k x_i^{-\pi_i} : \text{int } \mathbb{R}_+^k \to \mathbb{R}
\]
admits an explicit \( \mathcal{CQ} \)-r.

8. Let \( \pi \geq 1 \) be a rational. Then the function
\[
f_+(x) = (\max[x, 0])^\pi
\]
admits an explicit \( \mathcal{CQ} \)-r.

So does the function
\[
g(x) = |x|^\pi \quad [\equiv f(x) + f(-x)]
\]
Indeed,
\[
\{(t, x) \mid t \geq (\max[x, 0])^{\frac{p}{q}}\} = \{(t, x) \mid -x \geq -t^{\frac{q}{p}}, \, t \geq 0\}.
\]

9. For rational \( \pi \geq 1 \), the \( \pi \)-norm \( f(x) = \|x\|_\pi : \mathbb{R}^n \to \mathbb{R} \) is \( \mathcal{CQ} \)-representable:
\[
f(x) \leq t \iff \exists u_1, \ldots, u_n \geq 0 : \begin{cases} |x_i| \leq t^{\frac{\pi-1}{\pi}} u_{i^{\frac{1}{\pi}}}, \; i = 1, \ldots, n \\ \sum_{i=1}^n u_i \leq t \end{cases}
\]
Examples:

- A $C_Q$-r. of the set

\[ t \geq -x_1^\frac{1}{2} x_2^\frac{2}{3} x_3^\frac{3}{4}, \quad x \geq 0 \]

is

\[ \exists s, u_1, u_2, u_3, u_4 : \]

\[
\begin{pmatrix}
    u_1 \\
    \frac{x_1-x_3}{x_1+x_3} \\
    \frac{x_1-x_3}{2}
\end{pmatrix} \in L^2 \quad \begin{pmatrix}
    u_2 \\
    \frac{s-t-1}{s-t+1} \\
    \frac{2}{2}
\end{pmatrix} \in L^2 \\
\begin{pmatrix}
    u_3 \\
    \frac{u_1-u_2}{u_1+u_2} \\
    \frac{2}{2}
\end{pmatrix} \in L^2 \quad \begin{pmatrix}
    u_4 \\
    \frac{x_3-x_2}{x_2+x_3} \\
    \frac{2}{2}
\end{pmatrix} \in L^2 \\
\begin{pmatrix}
    s-t \\
    \frac{u_3-u_4}{u_3+u_4} \\
    \frac{2}{s} 
\end{pmatrix} \in L^2 \\
\begin{pmatrix}
    u_5 \\
    \frac{u_3-u_4}{u_3+u_4} \\
    \frac{2}{2}
\end{pmatrix} \in L^2 \quad \begin{pmatrix}
    u_6 \\
    \frac{u_3-u_6}{u_3+u_6} \\
    \frac{2}{2}
\end{pmatrix} \in L^2
\]

- A $C_Q$-r. of the set

\[ t \geq x_1^\frac{1}{t} x_2^\frac{2}{t} x_3^\frac{3}{t} \]

is

\[ \exists u_1, u_2, u_3, u_4, u_5, u_6 : \]

\[
\begin{pmatrix}
    u_1 \\
    \frac{x_2-x_3}{x_2+x_3} \\
    \frac{x_2-x_3}{2}
\end{pmatrix} \in L^2 \quad \begin{pmatrix}
    u_2 \\
    \frac{t+1}{t+1} \\
    \frac{2}{2}
\end{pmatrix} \in L^2 \\
\begin{pmatrix}
    u_3 \\
    \frac{x_2-x_3}{x_2+x_3} \\
    \frac{x_2-x_3}{2}
\end{pmatrix} \in L^2 \\
\begin{pmatrix}
    u_4 \\
    \frac{u_3-u_2}{u_3+u_2} \\
    \frac{2}{2}
\end{pmatrix} \in L^2 \quad \begin{pmatrix}
    u_5 \\
    \frac{u_3-u_4}{u_3+u_4} \\
    \frac{2}{2}
\end{pmatrix} \in L^2 \\
\begin{pmatrix}
    u_6 \\
    \frac{u_3-u_6}{u_3+u_6} \\
    \frac{2}{2}
\end{pmatrix} \in L^2 \\
\begin{pmatrix}
    \frac{1}{2} \\
    \frac{u_3-u_6}{u_3+u_6} \\
    \frac{2}{2}
\end{pmatrix} \in L^2
\]
• There are, of course, nice convex functions which are not \( CQ \)-representable, e.g., the exponent
\[
f(x) = \exp\{x\}.
\]

**But:** for every \( p \geq 1 \),
\[
\exp\{x\} = \lim_{r \to \infty} \left( 1 + \frac{x}{2r} + \frac{1}{2} \left( \frac{x}{2r} \right)^2 + \ldots + \frac{1}{p!} \left( \frac{x}{2r} \right)^p \right)^{(2r)}_{h_p\left(\frac{x}{2r}\right)}
\]

It follows that if \( p \) is such that \( h_p(\cdot) \) is \( CQ \)-representable and nonnegative, then \( \exp\{x\} \) can be approximated by \( CQ \)-r.f.
\[
g_{p,r}(x) = h_p^{(2r)} \left( \frac{x}{2r} \right).
\]

For \( p \) fixed and for a given bounded range \(|x| \leq T\) of values of \( x \), the quality of this approximation grows rapidly with \( r \):
\[
r \geq O \left( \ln \left( \frac{1}{\varepsilon} \right) + T \right), \quad |x| \leq T
\]
\[
\Downarrow
\]
\[
(1 - \varepsilon) \exp\{x\} \leq g_{p,r}(x) \leq (1 + \varepsilon) \exp\{x\}.
\]
E.g.,
- \( g_{4,18}(x) \) approximates \( \exp\{x\} \) in the segment
  \[ \Delta = \{ x \mid |x| \leq 512 \} = \{ x \mid 4.38 \times 10^{-233} \leq \exp\{x\} \leq 2.28 \times 10^{222} \} \]
within relative accuracy \( 1 \times 10^{-10} \)
- At the same time, \( g_{4,18}\big|_\Delta \) admits the following simple \( CQ \)-r.:

\[
|x| \leq 512, g_{4,18}(x) \leq t
\]

\[
\exists u_1, \ldots, u_{21}:
\]

\[
\left( \frac{1 + \frac{x}{218}}{1 - \frac{u_1}{2}} \right) \in \mathbb{L}^2, \quad \left( \frac{\frac{5}{6} + \frac{x}{219}}{1 - \frac{u_2}{2}} \right) \in \mathbb{L}^2, \quad \left( \frac{\frac{u_1}{2}}{1 + \frac{u_3}{2}} \right) \in \mathbb{L}^2
\]

\[
\frac{19}{72} + u_2 + \frac{1}{24} u_3 \leq u_4
\]

\[
\left( \frac{u_{\ell - 1}}{\frac{1 - u_\ell}{2}} \right) \in \mathbb{L}^2, \quad \ell = 5, \ldots, 21
\]

\[
\left( \frac{u_{21}}{\frac{1 - t}{2}} \right) \in \mathbb{L}^2, \quad \ell = 5, \ldots, 21
\]

\[-512 \leq x \leq 512
\]

Thus, for any practical purpose \( \exp\{x\} \) is \( CQ \)-representable!
“Whether Conic Quadratic Programming does exist?”

• Surprisingly, Lorentz cones (and thus all \(\mathcal{CQ}\)-representable sets!) are “nearly polyhedrally representable”.

**Theorem** [BT-N, ’98] For every \(n > 0\) and every \(\varepsilon \in (0, 1/2)\), there exist (and can be explicitly written down) matrices \(P_{n,\varepsilon}\), \(Q_{n,\varepsilon}\) and vector \(b_{n,\varepsilon}\) such that

(i) The row and the column sizes of \(P_{n,\varepsilon}, Q_{n,\varepsilon}, b_{n,\varepsilon}\) do not exceed \(O(n \ln \frac{1}{\varepsilon})\)

(ii) If

\[
(x, t) \in L^n = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 \leq t\},
\]

then there exists a vector \(u\) such that

\[
P_{n,\varepsilon}x + Q_{n,\varepsilon}u + b_{n,\varepsilon} \geq 0,
\]

and “nearly vice versa”: if \((x, t)\) can be extended by some \(u\) to a solution of \((*)\), then

\[
(x, t) \in L^n_\varepsilon = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 \leq (1 + \varepsilon)t\}.
\]

Thus, \(L^n\) “can be approximated within accuracy \(\varepsilon\)” by a polyhedral cone \(P^n_\varepsilon\):

\[
L^n \subset P^n_\varepsilon \subset L^n_\varepsilon
\]

admitting a “short” – of the size \(O(n \ln \frac{1}{\varepsilon}) - \mathcal{LP}\)-r.:

\[
P^n_\varepsilon = \{(x, t) \mid \exists u : P_{n,\varepsilon}x + Q_{n,\varepsilon}u + b_{n,\varepsilon} \geq 0\}.
\]
Corollary. Consider a conic quadratic program
\[ c^T x \rightarrow \min \ Ax + b \geq 0, \ Px + q \in K = L^{n_1} \times \ldots \times L^{n_k} \] (CQP)
and let the problem be both
\[ r \text{-strictly feasible for some } r > 0:\]
\[ \exists \bar{x} : A\bar{x} + b \geq 0 \land \forall (\xi : \|\xi\|_2 \leq r) : P\bar{x} + q + \xi \in K \]
\[ R \text{-semibounded:} \]
\[ Ax + b \geq 0, \ Px + q \in K \Rightarrow \|Px + q\|_2 \leq R. \]
For every \( \varepsilon \in (0, 1) \), there exists an LP program
\[ c^T x \rightarrow \min | Bx + Cv + d \geq 0 \] (LP)
which is an \( \varepsilon \)-relaxation of (CQP) of polynomial complexity:

- If \( x \) is feasible for (CQP), then \( x \) can be extended to a feasible solution \((x, v)\) of (LP);
- If \((x, v)\) is feasible for (LP), then the vector \((1 - \varepsilon)x + \varepsilon \bar{x}\) is feasible for (CQP).
- The size \( \dim x + \dim v + \dim d \) of (LP) does not exceed
\[ \dim x + \dim b + O \left( [\dim q] \ln \frac{2R}{\varepsilon r} \right) \]
and the data \((c, B, C, d)\) of (LP) is readily given by the data \((c, A, b, P, q)\) of (CQP).
“Fast polyhedral approximations” of the Lorentz cone are based on the fact that if $P^\nu$ is the polytope defined as

$$(x, u) \in \mathbb{R}^2 \times \mathbb{R}^{2\nu+4} : \begin{cases} 
  u_0 & \geq |x_1|; \\
  u_1 & \geq |x_2|; \\
  u_{2j} &= \cos \left( \frac{\pi}{2j+1} \right) u_{2j-2} + \sin \left( \frac{\pi}{2j+1} \right) u_{2j-1}, \\
  j &= 1, \ldots, \nu; \\
  u_{2j+1} &\geq \left| - \sin \left( \frac{\pi}{2j+1} \right) u_{2j-2} + \cos \left( \frac{\pi}{2j+1} \right) u_{2j-1} \right|, \\
  j &= 1, \ldots, \nu; \\
  u_{2\nu+3} &\leq 1; \\
  u_{2\nu+4} &\leq \tan \left( \frac{\pi}{2\nu+1} \right) u_{2\nu+3} 
\end{cases}$$

then the projection $P_x^\nu$ of the polytope on the $x$-plane approximates the unit 2D disk $D_2$ within accuracy

$$\frac{1}{\cos \left( \frac{\pi}{2\nu+1} \right)} - 1.$$
• $P^3$ “lives” in $R^9$ and is given by 12 linear inequalities. $P^3_x$ approximates $D_2$ within accuracy $5.e-3$
  (as good as the 16-side perfect polygon)

• $P^6$ “lives” in $R^{12}$ and is given by 18 linear inequalities. $P^6_x$ approximates $D_2$ within accuracy $3.e-4$
  (as good as the 127-side perfect polygon)

• $P^{12}$ “lives” in $R^{18}$ and is given by 30 linear inequalities. $P^{12}_x$ approximates $D_2$ within accuracy $7.e-8$
  (as good as the 8,192-side perfect polygon)

• $P^{24}$ “lives” in $R^{30}$ and is given by 54 linear inequalities. $P^{24}_x$ approximates $D_2$ within accuracy $4.e-15$
  (as good as the 34,200,933-side perfect polygon)
What can be expressed via SDP?

\[ \mathcal{K} = S\mathcal{D} \equiv \{ \text{Direct products of the cones of positive semidefinite matrices} \} \]

Examples of \( S\mathcal{D} \)-representable functions/sets

1-9. Every \( \mathcal{CQ} \)-r. function/set is \( S\mathcal{D} \)-r. as well.

Indeed, the Lorentz cone \( \mathbb{L}^n \) is \( S\mathcal{D} \)-r.:

\[(x, t) \in \mathbb{L}^n \]

\[\uparrow\]

\[
\begin{pmatrix}
  t & x_1 & x_2 & \ldots & x_n \\
x_1 & t & & & \\
x_2 & & t & & \\
& \ddots & \ddots & \ddots & \\
x_n & & & t &
\end{pmatrix}
\geq 0

\Theta_n(x)
\]

Consequently, a \( \mathcal{CQ} \)-r. of a set \( X \):

\[ X = \{ x \mid \exists u_1, \ldots, u_k : P_i x + Q_i u + b_i \in \mathbb{L}^{n_i}, \ i = 1, \ldots, k \} \]

can be straightforwardly converted into an \( S\mathcal{D} \)-r. of \( X \):

\[ X = \{ x \mid \exists u_1, \ldots, u_k : \Theta_{n_i} (P_i x + Q_i u_i + b_i) \geq 0, \ i = 1, \ldots, k \} \].
10. Maximum eigenvalue $\lambda_{\text{max}}(X)$ of a symmetric matrix $X$ is $SD$-representable:

$$t \geq \lambda_{\text{max}}(X) \Leftrightarrow tI - X \succeq 0.$$ 

11. The sum

$$S_k(X) = \lambda_1(X) + \ldots + \lambda_k(X)$$

of $k$ largest eigenvalues of a symmetric matrix $X$ is $SD$-representable:

$$S_k(X) \leq t \Leftrightarrow \exists s, Z : \begin{cases} t - ks - \text{Tr}(Z) \geq 0 \\ Z \succeq 0 \\ Z - X + sI \succeq 0 \end{cases}$$
12. Let a function \( f \) on \( \mathbb{R}^n \) be symmetric (i.e., invariant w.r.t. permutations of coordinates) and \( SD \)-representable:

\[
f(x) \leq t \iff \exists u : Px + tp + Qu + b \geq 0.
\]

Then the function

\[
\phi(X) = f(\lambda(X))
\]

is \( SD \)-r. with the representation

\[
t \geq F(X) \iff \exists u, x_1, \ldots, x_n : \begin{cases} Px + tp + Qu + b \geq 0 \\ x_1 \geq x_2 \geq \ldots \geq x_n \\ x_1 + \ldots + x_k \geq S_k(X), \\ k = 1, \ldots, n - 1 \\ x_1 + \ldots + x_n = \text{Tr}(X) \end{cases}
\]

E.g., the following functions of a symmetric \( n \times n \) matrix \( X \) are \( SD \)-r.f.’s with explicit \( SD \)-r.’s:

- \( -(\text{Det } X)^\pi \), \( X \succeq 0, \ 0 < \pi \leq \frac{1}{n} \) is rational
  \[ f(x) = -(\prod_{i=1}^n x_i)^\pi, \ x \geq 0 \]

- \( \| X \|_\pi \equiv \| \lambda(X) \|_\pi, \ \pi \geq 1 \) is rational
  \[ f(x) = \| x \|_\pi \]

- \( \| X⁺ \|_\pi \equiv \| \{\max[\lambda_i(X), 0]\}_{i=1}^n \|_\pi, \ \pi \geq 1 \) is rational
  \[ f(x) = \| \{\max[x_i, 0]\}_{i=1}^n \|_\pi \]
13. Let a function $f$ on $\mathbb{R}_+^m$ be symmetric, monotone and $S\mathcal{D}$-representable:

$$f(x) \leq t \iff \exists u : Px + tp + Qu + b \geq 0,$$

and let

$$\sigma(X) = \lambda \left( \sqrt{XX^T} \right)$$

be the vector of singular values of a rectangular $m \times n$ matrix $(m \leq n)$. Then the function

$$\phi(X) = f(\sigma(X)) : \mathbb{R}^{m \times n} \to \mathbb{R} \cup \{+\infty\}$$

is $S\mathcal{D}$-r.:

$$\phi(X) \leq t \iff \exists u, x_1, \ldots, x_m : \begin{cases} Px + tp + Qu + b \geq 0 \\ x_1 \geq \ldots \geq x_m \\ x_1 + \ldots + x_k \geq S_k \left( \left( \begin{array}{c} X \\ X^T \end{array} \right) \right), \\ k = 1, \ldots, m. \end{cases}$$

E.g., the following functions of rectangular $m \times n$ matrix $X$ are $S\mathcal{D}$-r.f.’s with explicit $S\mathcal{D}$-r.:

- $\|X\| = \sigma_{\max}(X) \left[ \|X\| \leq t \iff \left( tI_m \begin{array}{c} X \\ X^T \end{array} \right) \geq 0 \right]$

- $\Sigma_k(X) = \sum_{i=1}^{k} \sigma_i(X), \ 1 \leq k \leq m$

- $\|X\|_\pi = \|\sigma(X)\|_\pi, \ \pi \geq 1$ is rational

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Elementary $S\mathcal{D}$-representable functions/sets (cont.)

Convex hulls of some sets of matrices

14. Let $S \subset \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \ldots \geq x_n\}$ be an $S\mathcal{D}$-r.s.:

$$S = \{x \in \mathbb{R}^n \mid \exists u : Px + Qu + b \geq 0\}.$$ 

14a. The set

$$\mathcal{C}S[S] = \text{Conv}\{X \in \mathbb{S}^n \mid \lambda(X) \in S\}.$$ 

is $S\mathcal{D}$-r. with $S\mathcal{D}$-r. induced by the relation

$$\mathcal{C}S[S] = \left\{\begin{array}{l} Px + Qu + b \geq 0 \\ X \in \mathbb{S}^n \mid \exists u, x : S_k(X) \leq x_1 + \ldots + x_k, \ k < n \\ \text{Tr}(X) = x_1 + \ldots + x_n \end{array}\right\}$$

14b. Let, in addition, $S$ be monotone:

$$(y \geq 0, y_1 \geq y_2 \geq \ldots \geq y_n, \exists x \in S : y \leq x) \Rightarrow y \in S.$$ 

The set

$$\mathcal{C}R[S] = \text{Conv}\{X \in \mathbb{R}^{m \times n} \mid \sigma(x) \in S\} \quad [n \geq m].$$

is $S\mathcal{D}$-representable with $S\mathcal{D}$-r. induced by the relation

$$\mathcal{C}R[S] = \left\{X \in \mathbb{R}^{m \times n} \mid \exists u, x : Px + Qu + b \geq 0 \\ \Sigma_k(X) \leq x_1 + \ldots + x_k, \ k \leq m \right\}$$
15. [“General $\succeq$-convex quadratic matrix inequality”] 
The set
\[
\{ (X, Y) \in \mathbb{R}^{k \times \ell} \times \mathbb{R}^m \mid 
F(X) \equiv (AXB)(AXB)^T + CXD + (CXD)^T + E \succeq Y \}
\]
is $\mathcal{S}\mathcal{D}$-representable:
\[
F(X) \succeq Y \iff \begin{pmatrix} I \\ AXB \end{pmatrix} \begin{pmatrix} (AXB)^T \\ Y - E - CXD - (CXD)^T \end{pmatrix} \succeq 0
\]

16. [“Fractional-quadratic matrix inequality”] The set
\[
\mathcal{F} = \text{cl}\{(X, V, Y) \in \mathbb{R}^{p \times q} \times \mathbb{S}^q \times \mathbb{S}^p \mid V \succ 0,
F(X, V) = XV^{-1}X^T \preceq Y\}
\]
is $\mathcal{S}\mathcal{D}$-representable:
\[
\mathcal{F} = \left\{ (X, V, Y) \mid \begin{pmatrix} Y \\ X^T \\ V \end{pmatrix} \succeq 0 \right\}
\]

17. The set
\[
\mathcal{X} = \text{cl}\{(X, Y) \in \mathbb{S}^n \times \mathbb{S}^m \mid X \succ 0, \ Y \preceq (A^TX^{-1}A)^{-1} \}
\]
is $\mathcal{S}\mathcal{D}$-representable:
\[
\mathcal{X} = \left\{ (X, Y) \mid \exists Z \in \mathbb{S}^m : \begin{pmatrix} Z \\ Y \preceq Z \\ X \succeq AZAT \end{pmatrix} \right\}
\]
18. [Square root of positive semidefinite matrix] The set

\[ \mathcal{X} = \{(X, Y) \in S^n \times S^n \mid X \succeq 0, \ Y \preceq X^{1/2}\} \]

is \( S^D \)-representable:

\[
\mathcal{X} = \left\{ (X, Y) \mid \exists Z \in S^n : \begin{pmatrix} X & Z \\ Z & I \end{pmatrix} \succeq 0 \right\}
\]

\[
\begin{align*}
&X \\
&Z \\
&I
\end{align*}
\]
19. Let $\Delta \subset \mathbb{R}$ be a segment (perhaps infinite), and let
\[
P_k^+(\Delta) = \left\{ (x_0, \ldots, x_k)^T \in \mathbb{R}^{k+1} : x(\tau) \equiv \sum_{p=0}^{k} x_p \tau^p \geq 0 \ \forall \tau \in \Delta \right\}
\]
be the set of coefficients of nonnegative on $\Delta$ polynomials of degree $\leq k$. It turns out that $P_k^+(\Delta)$ admits an explicit $SD$-r.

In particular, the function
\[
f(x) = \sup_{\tau \in \Delta} x(\tau)
\]
adopts explicit $SD$-r. induced by the relation
\[
f(x) \leq t \iff (t - x_0, -x_1, \ldots, -x_k) \in P_k^+(\Delta).
\]
$SD$-r.'s of the sets $\mathcal{P}_k^+(\Delta)$ are as follows:

- $\Delta = \mathbb{R}$:

  \[
  x(\tau) \equiv \sum_{i=0}^{2m} x_i \tau^i \geq 0 \quad \forall \tau \in \mathbb{R}
  \]

  \[
  \Downarrow
  \]

  \[
  \exists X \in \mathbb{S}^m : \quad X \succeq 0
  \]

  \[
  x(\tau) \equiv \sum_{i,j=1}^{m} X_{ij} \tau^{i-j-2}
  \]

  \[
  X(\tau) = \bigg( \sum_{i} \xi_i \xi_i^T \bigg)^2 \quad \Downarrow \quad X(\tau) = \sum_{i} \left( \sum_{p} \xi_{i,p} \tau^p \right)^2 \geq 0
  \]

  Indeed,

  \[
  X \succeq 0 \Downarrow \quad X = \sum_{i} \xi_i \xi_i^T \Downarrow \quad X(\tau) = \sum_{i} \left( \sum_{p} \xi_{i,p} \tau^p \right)^2 \geq 0
  \]

  Vice versa, a nonnegative polynomial is the sum of squares of two polynomials, whence

  \[
  x(\cdot) \geq 0 \Downarrow \quad \exists \xi_1, \xi_2 : \quad x(\tau) = \sum_{i=1}^{2} \left( \sum_{p} \xi_{i,p} \tau^p \right)^2 \Downarrow \quad x(\cdot) = X(\cdot), \quad X = \xi_1 \xi_1^T + \xi_2 \xi_2^T \geq 0
  \]
\( \Delta = \mathbb{R}_+ \):

The set
\[
\mathcal{P}_k^+(\mathbb{R}_+) = \left\{ \begin{array}{l}
\text{coefficients of nonnegative on } \mathbb{R}_+ \\
\text{polynomials of degree } \leq k
\end{array} \right\}
\]
is the inverse image of the set
\[
\mathcal{P}_{2k}^+(\mathbb{R}) = \left\{ \begin{array}{l}
\text{coefficients of nonnegative on } \mathbb{R} \\
\text{polynomials of degree } \leq 2k
\end{array} \right\}
\]
under the linear mapping of the polynomial coefficients induced by the transformation
\[
x(\tau) \mapsto (Ax)(\tau) \equiv x(\tau^2).
\]

\( \Delta = [0, 1] \):

The set
\[
\mathcal{P}_k^+([0, 1]) = \left\{ \begin{array}{l}
\text{coefficients of nonnegative on } [0, 1] \\
\text{polynomials of degree } \leq k
\end{array} \right\}
\]
is the inverse image of \( \mathcal{P}_{2k}^+(\mathbb{R}) \) under the linear mapping of the polynomial coefficients induced by the transformation
\[
x(\tau) \mapsto [Bx](\tau) = (1 + \tau^2)^k x \left( \frac{\tau^2}{1 + \tau^2} \right).
\]

Thus, \( S\mathcal{D}\)-r.'s of \( \mathcal{P}_k^+(\mathbb{R}_+) \) and \( \mathcal{P}_k^+([0, 1]) \) are readily given by the above \( S\mathcal{D}\)-r. of \( \mathcal{P}_{2k}^+(\mathbb{R}) \).
20. Let $\Delta \subset [-\pi, \pi]$ be a segment, and let

$$\mathcal{T}^+_k(\Delta) = \left\{ (\xi_0, \xi_1, \ldots, \xi_{2k})^T \in \mathbb{R}^{2k+1} : \right.$$ 

$$\xi(\phi) \equiv \xi_0 + \sum_{p=1}^k (\xi_{2p-1} \cos(p\phi) + \xi_{2p} \sin(p\phi)) \geq 0 \quad \forall \phi \in \Delta \right\}$$

be the set of coefficients of the nonnegative on $\Delta$ trigonometric polynomials of degree $\leq k$. The set $\mathcal{T}^+_k(\Delta)$ admits an explicit $S\mathcal{D}$-r.

In particular, the function

$$f(\xi) = \max_{\phi \in \Delta} \xi(\phi)$$

admits an explicit $S\mathcal{D}$-r.

Indeed, $\mathcal{T}^+_k(\Delta)$ is the inverse image of $\mathcal{P}^+_k(\Gamma)$ under the linear mapping of coefficients induced by the transformation

$$\xi(\phi) \mapsto x(\tau) = (1 + \tau^2)^k \xi(2\tan(\tau));$$

here

$$\Gamma = \{ \tau : 2\tan(\tau) \in \Delta \}.$$
21. For a segment $\Delta \subset \mathbb{R}$ and a continuous function $p(\cdot)$ on $\Delta$, let

$$\text{ConvEpi}(\Delta, p) = \text{Conv} \{ (\tau, t) \mid \tau \in \Delta, t \geq p(\tau) \}$$

be the convex hull of the epigraph of $\Delta$.

For every one of the following pairs $(\Delta, p)$, the set $\text{ConvEpi}(\Delta, p)$ admits an explicit $\mathcal{SD}$-r.:

21.a. $\Delta = \mathbb{R}$, $p(\tau) = \sum_{\ell=0}^{2k} p_{\ell} \tau^\ell$ with $p_{2k} > 0$

21.b. $\Delta = \mathbb{R}_+$, $p(\tau) = \sum_{\ell=0}^{k} p_{\ell} \tau^\ell$ with $p_k > 0$

21.c. $\Delta = [0, 1]$, $p(\tau) = \sum_{\ell=0}^{k} p_{\ell} \tau^\ell$

E.g. in the case of 21.a an $\mathcal{SD}$-r. of $\text{ConvEpi}(\Delta, p)$ is

$$\left\{ (\tau, t) \mid \exists \tau_2, \tau_3, \ldots, \tau_{2k} : \begin{pmatrix} 1 & \tau & \tau_2 & \tau_3 & \cdots & \tau_k \\ \tau & \tau_2 & \tau_3 & \tau_4 & \cdots & \tau_{k+1} \\ \tau_2 & \tau_3 & \tau_4 & \tau_5 & \cdots & \tau_{k+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau_k & \tau_{k+1} & \tau_{k+2} & \tau_{k+3} & \cdots & \tau_{2k} \end{pmatrix} \succeq 0 \right\}$$

$$p_0 + p_1 \tau + p_2 \tau^2 + p_3 \tau^3 + \cdots + p_{2k} \tau^{2k} \leq t$$
• \( S^D \)-r. of the above sets \( \text{ConvEpi}(\Delta, p) \) are given by the following construction.

1. Let

\[
p(\tau) = p_0 + p_1 \tau + \sum_{\ell=2}^{n} p_\ell f_\ell(\tau),
\]

\[
\phi(\tau) = (1, \tau, f_2(\tau), \ldots, f_n(\tau))^T : \Delta \rightarrow \mathbb{R}^{n+1}
\]

Assume that

A. The set \( Y = \overline{\text{Conv}} \{ \phi(\tau) \mid \tau \in \Delta \} \) is \( S^D \)-r. with a known \( S^D \)-r.

Remark \( Y \) is the intersection of the cone dual to the “cone of nonnegative polynomials”

\[
\mathcal{P}^+ = \{(p_0, p_1, \ldots, p_n) \mid p(\tau) \geq 0 \quad \forall \tau \in \Delta \}
\]

and the hyperplane \( \{(1, \pi_1, \pi_2, \ldots, \pi_n)^T \in \mathbb{R}^{n+1} \} \).

B. The set \( \text{ConvEpi}(\Delta, p) \) is closed.

Under these assumptions \( \text{ConvEpi}(\Delta, p) \) admits the representation

\[
\left\{ (\tau, t) \mid \exists \tau_1, \tau_2, \ldots, \tau_n : (\tau, \tau_1, \tau_2, \ldots, \tau_n)^T \in Y \right\} \\

\sum_{i=1}^{n} p_i \tau_i \leq t
\]

Thus, \( \text{ConvEpi}(\Delta, p) \) is a projection of a \( S^D \)-r.s. (the intersection of \( Y \) and a half-space). By Calculus, an \( S^D \)-r. of \( \text{ConvEpi}(\Delta, p) \) is readily given by an \( S^D \)-r. of \( Y \).
2. In the situations 21.a – 21.c

- Assumption A is satisfied by the following reasons:
  - The corresponding “cone of nonnegative polynomials” is $\mathcal{P}_k^+(\Delta)$, for which we know $SD$-r. which in fact is strictly feasible.
  - By Calculus, a strictly feasible $SD$-r. of the cone $\mathcal{P}_k^+(\Delta)$ explicitly induces an $SD$-r. of the cone dual to $\mathcal{P}_k^+(\Delta)$ and thus, by Remark – for $Y$.

- Assumption B holds true (a straightforward verification).
Elementary SD-representable functions/sets (cont.)

Families of ellipsoids

• Let

\[ V(X, x) = \{ u \in \mathbb{R}^n \mid u = Xv + x, v^Tv \leq 1 \} \quad [X \in \mathbb{R}^{n \times n}] \]

\[ W(Y, y) = \{ u \in \mathbb{R}^n \mid u^TY^TYu - 2u^TLYT + y^Ty \leq 1 \} \quad [Y \in \mathbb{R}^{n \times n}] \]

• \( V(\cdot, \cdot) \) is a natural parameterization of ellipsoids in \( \mathbb{R}^n \), including “flat” ones (an ellipsoid is the image of the unit Euclidean ball under affine mapping)

• \( W(\cdot, \cdot) \) is a natural parameterization of ellipsoids in \( \mathbb{R}^n \), including “elliptic cylinders” (an ellipsoid is a level set of a below bounded convex quadratic function)

**Proposition** [Boyd et al?]

\[
V(X, x) \subset W(Y, y) \downarrow
\]

\[
\exists \lambda : \\
\begin{pmatrix}
I & Yx - y & YX \\
x^TY^T - y^T & 1 - \lambda & \lambda I \\
X^TY^T & & \lambda I
\end{pmatrix} \succeq 0 \quad (1)
\]
Note: When one of the ellipsoids $V(X, x)$ or $W(Y, y)$ is fixed, (*) becomes a Linear Matrix Inequality in the parameters of the other ellipsoid and $\lambda$. Thus, both the sets

$$
\{(X, x) \in S_+^N \mid V(X, x) \subset W(A, a)\}
$$

$$
\{(Y, y) \mid V(A, a) \subset W(Y, y)\}
$$

admit explicit $\mathcal{S}\mathcal{D}$-r.’s.

As a result, both the problems

Find the smallest volume ellipsoid containing the union of a given finite family of ellipsoids

and

Find the largest volume ellipsoid contained in the intersection of a given finite family of ellipsoids

can be posed as semidefinite programs.
- The problem

Find the largest volume ellipsoid \( V(X, x) \) contained in every one of given ellipsoids \( W(Y_\ell, y_\ell), \ell = 1, \ldots, k \)

is equivalent to

\[
\tau \to \max \left\{ \left( \frac{\text{Det } X}{\pi} \right)^\frac{1}{n} \geq \tau \right. 
\left. \begin{pmatrix}
I & Y_\ell x - y_\ell & Y_\ell X \\
x^T Y_\ell - y_\ell^T & 1 - \lambda_\ell & \lambda_\ell I \\
Y_\ell X^T & \lambda_\ell I & X \geq 0
\end{pmatrix} \right\} \geq 0, \ \ell = 1, \ldots, k \tag{I}
\]

with design variables \( x, X, \lambda_1, \ldots, \lambda_k \).

- The problem

Find the smallest volume ellipsoid \( W(Y, y) \) containing every one of given ellipsoids \( V(X_\ell, y_\ell), \ell = 1, \ldots, k \)

is equivalent to

\[
\tau \to \max \left\{ \left( \frac{\text{Det } Y}{\pi} \right)^\frac{1}{n} \geq \tau \right. 
\left. \begin{pmatrix}
I & Y x_\ell - y & Y X_\ell \\
x^T Y_\ell & 1 - \lambda_\ell & \lambda_\ell I \\
X_\ell Y & \lambda_\ell I & Y \geq 0
\end{pmatrix} \right\} \geq 0, \ \ell = 1, \ldots, k \tag{O}
\]

with design variables \( y, Y, \lambda_1, \ldots, \lambda_k \).

Since the feasible sets of every one of the constraints in (I) and (O) admit explicit \( S\mathcal{D}\)-r.’s, both problems can be straightforwardly converted to semidefinite programs.