

On Dualization of Hypergraphs with Bounded Edge-Intersections and Other Related Classes of Hypergraphs ^{*}

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Abstract. Given a finite set V , and integers $k \geq 1$ and $r \geq 0$, denote by $\mathbb{A}(k, r)$ the class of hypergraphs $\mathcal{A} \subseteq 2^V$ with (k, r) -bounded intersections, i.e. in which the intersection of any k distinct hyperedges has size at most r . We consider the problem $MIS(\mathcal{A}, \mathcal{I})$: given a hypergraph \mathcal{A} and a subfamily $\mathcal{I} \subseteq \mathcal{I}(\mathcal{A})$, of its maximal independent sets (MIS) $\mathcal{I}(\mathcal{A})$, either extend this subfamily by constructing a new MIS $I \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{I}$ or prove that there are no more MIS, that is $\mathcal{I} = \mathcal{I}(\mathcal{A})$. It is known that, for hypergraphs of bounded dimension $\mathbb{A}(1, c)$, as well as for hypergraphs of bounded degree $\mathbb{A}(c, 0)$, problem $MIS(\mathcal{A}, \mathcal{I})$ can be solved in incremental polynomial time. In this paper we extend this result to any integers k, r such that $k + r = c$ is a constant. More precisely, we show that for hypergraphs $\mathcal{A} \in \mathbb{A}(k, r)$ with $k + r \leq \text{const}$, problem $MIS(\mathcal{A}, \mathcal{I})$ is NC-reducible to problem $MIS(\mathcal{A}', \emptyset)$ of generating a single MIS for a partial subhypergraph \mathcal{A}' of \mathcal{A} . In particular, this implies that $MIS(\mathcal{A}, \mathcal{I})$ is polynomial and we get an incremental polynomial algorithm for generating all MIS. Furthermore, combining this result with the currently known algorithms for finding a single maximal independent set of a hypergraph, we obtain efficient parallel algorithms for incrementally generating all MIS for hypergraphs in the classes $\mathbb{A}(1, c)$, $\mathbb{A}(c, 0)$, and $\mathbb{A}(2, 1)$, where c is a constant. We also show that, for $\mathcal{A} \in \mathbb{A}(k, r)$, where $k + r \leq \text{const}$, the problem of generating all MIS of \mathcal{A} can be solved in incremental polynomial-time and with space polynomial only in the size of \mathcal{A} .

Key Words: Bounded degree, bounded dimension, conformal hypergraph, dualization, incremental generating, polynomial space, hypergraph, maximal independent set, minimal transversal

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1 Introduction

Let $\mathcal{A} \subseteq 2^V$ be a hypergraph (set family) on a finite vertex set V . A vertex set $I \subseteq V$ is called *independent* if I contains no hyperedge of \mathcal{A} . Let $\mathcal{I}(\mathcal{A}) \subseteq 2^V$ denote the family of all maximal independent sets (MIS) of \mathcal{A} . We assume that \mathcal{A} is given by the list of its hyperedges and consider problem $MIS(\mathcal{A})$ of incrementally generating all sets in $\mathcal{I}(\mathcal{A})$. Clearly, this problem can be solved by performing $|\mathcal{I}(\mathcal{A})| + 1$ calls to the following problem:

MIS(\mathcal{A}, \mathcal{I}): Given a hypergraph \mathcal{A} and a collection $\mathcal{I} \subseteq \mathcal{I}(\mathcal{A})$ of its maximal independent sets, either find a new maximal independent set $I \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{I}$, or prove that the given collection is complete, $\mathcal{I} = \mathcal{I}(\mathcal{A})$.

Note that if $I \in \mathcal{I}(\mathcal{A})$ is an independent set, the complement $B = V \setminus I$ is a *transversal* to \mathcal{A} , that is $B \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, and vice versa. Hence $\{B \mid B = V \setminus I, I \in \mathcal{I}(\mathcal{A})\} = \mathcal{A}^d$, where

$$\mathcal{A}^d \stackrel{\text{def}}{=} \{B \mid B \text{ is a minimal transversal to } \mathcal{A}\}$$

is the *transversal* or *dual* hypergraph of \mathcal{A} . For this reason, $MIS(\mathcal{A}, \mathcal{I})$ can be equivalently stated as the *hypergraph dualization problem*:

DUAL(\mathcal{A}, \mathcal{B}): Given a hypergraph \mathcal{A} and a collection $\mathcal{B} \subseteq \mathcal{A}^d$ of minimal transversals to \mathcal{A} , either find a new minimal transversal $B \in \mathcal{A}^d \setminus \mathcal{B}$ or show that $\mathcal{B} = \mathcal{A}^d$.

This problem has applications in combinatorics, graph theory, artificial intelligence, reliability theory, database theory, integer programming, and learning theory (see, e.g. [5, 11]). It is an open question whether problem $DUAL(\mathcal{A}, \mathcal{B})$, or equivalently $MIS(\mathcal{A}, \mathcal{I})$, can be solved in polynomial time for arbitrary hypergraphs. The fastest currently known algorithm [14] for $DUAL(\mathcal{A}, \mathcal{B})$ is quasi-polynomial and runs in time $O(nm) + m^{o(\log m)}$, where $n = |V|$ and $m = |\mathcal{A}| + |\mathcal{B}|$. The fastest known randomized parallel algorithm [20], for problem $MIS(\mathcal{A}, \emptyset)$ of computing a single MIS of a hypergraph \mathcal{A} on n vertices, runs in time $O(\sqrt{n})$ on $n^{3/2}$ processors.

It was shown in [6, 11] that in the case of hypergraphs of bounded dimension,

$$\dim(\mathcal{A}) \stackrel{\text{def}}{=} \max_{A \in \mathcal{A}} |A| \leq \text{const.} \quad (1)$$

problem $MIS(\mathcal{A}, \mathcal{I})$ can be solved in polynomial time. Moreover, [4] shows that the problem can be efficiently solved in parallel, $MIS(\mathcal{A}, \mathcal{I}) \in NC$ for $\dim(\mathcal{A}) \leq 3$ and $MIS(\mathcal{A}, \mathcal{I}) \in RNC$ for $\dim(\mathcal{A}) = 4, 5, \dots$. Let us also mention that for graphs, $\dim(\mathcal{A}) \leq 2$, all MIS can be generated with polynomial delay, see [18] and also [25].

In [10], a total polynomial time generation algorithm was obtained for the hypergraphs of bounded degree,

$$\deg(\mathcal{A}) \stackrel{\text{def}}{=} \max_{v \in V} |\{A : v \in A \in \mathcal{A}\}| \leq \text{const.} \quad (2)$$

This result was recently strengthened in [12], where a polynomial delay algorithm was obtained for a wider class of hypergraphs.

In this paper we consider the class $\mathbb{A}(k, r)$ of hypergraphs with (k, r) -bounded intersections: $\mathcal{A} \in \mathbb{A}(k, r)$ if the intersection of each (at least) k distinct hyperedges of \mathcal{A} is of cardinality at most r . We will always assume that $k \geq 1$ and $r \geq 0$ are fixed integers whose sum is bounded, $k + r \leq c = \text{const}$. Note that

$$\dim(\mathcal{A}) \leq r \quad \text{iff} \quad \mathcal{A} \in \mathbb{A}(1, r) \quad \text{and} \quad \deg(\mathcal{A}) < k \quad \text{iff} \quad \mathcal{A} \in \mathbb{A}(k, 0),$$

and hence, the class $\mathbb{A}(k, r)$ contains both the bounded-dimension and bounded-degree hypergraphs as subclasses.

It will be shown that problem $\text{MIS}(\mathcal{A}, \mathcal{I})$ can be solved in polynomial time for hypergraphs with (k, r) -bounded intersections. It is not difficult to see that for any hypergraph $\mathcal{A} \in \mathbb{A}(k, r)$ the following property holds for every vertex-set $X \subseteq V$: X is contained in a hyperedge of \mathcal{A} whenever each subset of X of cardinality at most $c = k + r$ is contained in a hyperedge of \mathcal{A} . Hypergraphs $\mathcal{A} \subseteq 2^V$ with this property were introduced by Berge [3] under the name of *c-conformal hypergraphs*, and clearly define a wider class of hypergraphs than $\mathbb{A}(k, r)$ with $k + r = c$. In fact, we will prove our result for this wider class of *c-conformal* hypergraphs.

Theorem 1. *For the c-conformal hypergraphs, $c \leq \text{const}$, and in particular for $\mathcal{A} \in \mathbb{A}(k, r)$, $k + r \leq c = \text{const}$, problem $\text{MIS}(\mathcal{A}, \mathcal{I})$ is polynomial and hence $\mathcal{I}(\mathcal{A})$, the set all MIS of \mathcal{A} , can be generated in incremental polynomial time.*

Theorem 1 is a corollary of the following stronger theorem which will be proved in Section 2.

Theorem 2. *For any c-conformal hypergraph \mathcal{A} , where c is a constant, problem $\text{MIS}(\mathcal{A}, \mathcal{I})$ is NC-reducible to $\text{MIS}(\mathcal{A}', \emptyset)$, where \mathcal{A}' is a partial sub-hypergraph of \mathcal{A} .*

In Section 2, we also derive some further consequences of Theorem 2, related to the parallel complexity of problem $\text{MIS}(\mathcal{A}, \mathcal{I})$ for certain classes of hypergraphs.

Let us note that our algorithm of generating $\mathcal{I}(\mathcal{A})$ based on Theorem 1 is incremental, since it requires solving problem $\text{MIS}(\mathcal{A}, \mathcal{I})$ iteratively $|\mathcal{I}(\mathcal{A})| + 1$ times. Thus, this algorithm may require space exponential in the size of the input hypergraph $N = N(\mathcal{A}) = \sum_{A \in \mathcal{A}} |A|$. A generation algorithm for $\mathcal{I}(\mathcal{A})$ is said to work in *polynomial space* if the total space required by the algorithm to output all the elements of $\mathcal{I}(\mathcal{A})$ is polynomial in N . In Section 3, we prove the following.

Theorem 3. *For the hypergraphs of bounded intersections, $\mathcal{A} \in \mathbb{A}(k, r)$, where $k + r \leq \text{const}$, all MIS of \mathcal{A} can be enumerated in incremental polynomial time and with polynomial space.*

Finally, we conclude in Section 4, with a third algorithm for generating all maximal independent sets of a hypergraph $\mathcal{A} \in \mathbb{A}(k, r)$, $k + r \leq \text{const}$.

2 NC-reduction for c -Conformal Hypergraphs

The results of [4] show that, for hypergraphs of bounded dimension $\mathcal{A}(1, c)$, there is an NC-reduction from $\text{MIS}(\mathcal{A}, \mathcal{I})$ to $\text{MIS}(\mathcal{A}', \emptyset)$, where \mathcal{A}' is a partial sub-hypergraph of \mathcal{A} . In other words, the problem of extending in parallel a given list of MIS of \mathcal{A} can be reduced to the problem of generating in parallel a single MIS for a partial sub-hypergraph of \mathcal{A} . In this section we extend this reduction to the class of c -conformal hypergraphs, when c is a constant.

2.1 c -Conformal Hypergraphs

Given a hypergraph $\mathcal{A} \subseteq 2^V$, we say that \mathcal{A} is *Sperner* if no hyperedge of \mathcal{A} contains another hyperedge. By definition, for every hypergraph \mathcal{A} , its MIS hypergraph $\mathcal{I}(\mathcal{A})$ is Sperner. Let us inverse the operator \mathcal{I} . Given a Sperner hypergraph $\mathcal{B} \subseteq 2^V$, introduce the hypergraph $\mathcal{A} = \mathcal{I}^{-1}(\mathcal{B}) \subseteq 2^V$ whose hyperedges are all minimal subsets $A \subseteq V$ which are not contained in any hyperedge of \mathcal{B} , that is $A \subseteq B$ for no $A \in \mathcal{A}, B \in \mathcal{B}$ and $A' \subseteq B$ for some $B \in \mathcal{B}$ for each proper subset $A' \subset A \in \mathcal{A}$. The hypergraph $\mathcal{A} = \mathcal{I}^{-1}(\mathcal{B})$ is Sperner by definition, too. It is also easy to see that \mathcal{B} is the MIS hypergraph of \mathcal{A} . In other words, for Sperner hypergraphs $\mathcal{B} = \mathcal{I}(\mathcal{A})$ if and only if $\mathcal{A} = \mathcal{I}^{-1}(\mathcal{B})$. In [3], Berge introduced the class of c -conformal hypergraphs and characterized them in several equivalent ways as follows.

Proposition 1 ([3]). *For each hypergraph $\mathcal{A} \subseteq 2^V$ the following statements are equivalent:*

- (i) \mathcal{A} is c -conformal;
- (ii) The transposed hypergraph \mathcal{A}^T (whose incidence matrix is the transposed incidence matrix of \mathcal{A}) satisfies the $(c-1)$ -dimensional Helly property: a subset of hyperedges from \mathcal{A}^T has a common vertex whenever every at most c hyperedges of this subset have one;
- (iii) For each partial hypergraph $\mathcal{A}' \subseteq \mathcal{A}$ having $c+1$ edges, the set $\{x \in V \mid d_{\mathcal{A}'}(x) \geq c\}$ of vertices of degree at least c in \mathcal{A}' , is contained in an edge of \mathcal{A} .

It is not difficult to see that we can add to the above list the following equivalent characterization:

- (iv) $\dim(\mathcal{I}^{-1}(\mathcal{A})) \leq c$.

Note also that (iii) gives a polynomial-time membership test for c -conformal hypergraphs, for a fixed constant c . The following lemma states that the hypergraphs with (k, r) -bounded intersections are $(k+r)$ -conformal.

Lemma 1. *If $\mathcal{A} \in \mathbb{A}(k, r)$ then \mathcal{A} is $k+r$ -conformal. In particular, $\dim(\mathcal{I}^{-1}(\mathcal{A})) \leq k+r$.*

Proof. Suppose that $\dim(\mathcal{I}^{-1}(\mathcal{A})) > k + r$. This exactly means that there is a subset $B \subseteq V$ such that (i) $|B| > k + r$, (ii) $B \subseteq A$ for no $A \in \mathcal{A}$, and (iii) $B' \subseteq A$ for some $A \in \mathcal{A}$ for each proper subset $B' \subset B$. Let us fix arbitrary k distinct elements $e_1, \dots, e_k \in B$ and consider k proper subsets $B_i = B \setminus \{e_i\}$ for $i = 1, \dots, k$. According to (iii), each B_i is contained in a hyperedge of \mathcal{A} . Further, the corresponding k hyperedges are pairwise distinct, since otherwise B would be contained in a hyperedge of \mathcal{A} in contradiction to (ii). Finally, according to (i), the cardinality of the intersection of these k distinct hyperedges of \mathcal{A} is greater than r which contradicts to $\mathcal{A} \in \mathbb{A}(k, r)$. \square

It is not difficult to see that in the above Lemma the inverse implication does not hold. Hence the Lemma shows that bounding the dimension,

$$\dim(\mathcal{I}^{-1}(\mathcal{A})) \leq c = \text{const}, \quad (3)$$

we define a wider class of hypergraphs than $\mathbb{A}(k, r)$ with $k + r = c$. Thus even though, given a hypergraph \mathcal{A} , the precise computation of $\dim(\mathcal{I}^{-1}(\mathcal{A}))$ is an NP-complete problem (it can be reduced from *stability number for graphs*), verifying condition (iv) is polynomial for every fixed c by Proposition 1.

Given a hypergraph $\mathcal{A} \subseteq 2^V$, let us introduce the *complementary* hypergraph $\mathcal{A}^c = \{V \setminus A \mid A \in \mathcal{A}\}$ whose hyperedges are complementary to the hyperedges of \mathcal{A} . It is easy to see that $\mathcal{A}^{dd} = \mathcal{A}$, $\mathcal{A}^{cc} = \mathcal{A}$ for each Sperner hypergraph \mathcal{A} . In other words, both operations, duality and complementation, are involutions. It is also clear that

$$\mathcal{A}^{dc} = \mathcal{I}(\mathcal{A}), \quad \mathcal{A}^{cd} = \mathcal{I}^{-1}(\mathcal{A}).$$

A vertex set S is called a *sub-transversal* of \mathcal{A} if $S \subseteq B$ for some minimal transversal $B \in \mathcal{A}^d$. Our proof of Theorem 2 makes use of a characterization of sub-transversals suggested in [6]. Even though it is NP-hard in general to test whether a given set $S \subseteq V$ is a sub-transversal of \mathcal{A} , for $|S| \leq \text{const}$ the sub-transversal criterion of [6] is polynomial (moreover, it is in *NC*). This turns out to be sufficient for the proof of Theorem 2.

2.2 Characterization of Sub-transversals to a Hypergraph

Given a hypergraph $\mathcal{A} \subseteq 2^V$, a subset $S \subseteq V$, and a vertex $v \in S$, let $\mathcal{A}_v(S) = \{A \in \mathcal{A} \mid A \cap S = \{v\}\}$ denote the family of all hyperedges of \mathcal{A} whose intersection with S is exactly v . Let further $\mathcal{A}_0(S) = \{A \in \mathcal{A} \mid A \cap S = \emptyset\}$ denote the partial hypergraph consisting of the hyperedges of \mathcal{A} disjoint from S . A selection of $|S|$ hyperedges $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ is called *covering* if there exists a hyperedge $A \in \mathcal{A}_0(S)$ such that $A \subseteq \bigcup_{v \in S} A_v$. Proposition 2 below states that a non-empty set S is a sub-transversal of \mathcal{A} if and only if there exists a non-covering selection for S .

Proposition 2 (cf. [6]). *Let $S \subseteq V$ be a non-empty vertex set in a hypergraph $\mathcal{A} \in 2^V$.*

- i) If S is a sub-transversal for \mathcal{A} then there exists a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S .
- ii) Given a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S , we can extend S to a minimal transversal of \mathcal{A} by solving problem $MIS(\mathcal{A}', \emptyset)$ for the induced partial hypergraph

$$\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}_0(S)\} \subseteq 2^U, \quad (4)$$

where

$$U = V \setminus \bigcup_{v \in S} A_v. \quad (5)$$

Proof. Let us start with the following observations:

- (a) If $S \subseteq B \subseteq V$ then $\mathcal{A}_v(B) \subseteq \mathcal{A}_v(S)$ holds for all $v \in S$.
- (b) If B is a transversal to \mathcal{A} then B is minimal iff $\mathcal{A}_v(B) \neq \emptyset$ for all $v \in B$.

Observation (a) follows directly from the definitions of $\mathcal{A}_v(S)$ and $\mathcal{A}_v(B)$. To see (b) note that if $\mathcal{A}_v(B) = \emptyset$ for some $v \in B$ then $B \setminus \{v\}$ is still a transversal to \mathcal{A} .

Proof of (i) Suppose that $\emptyset \neq S \subseteq B$, where $B \in \mathcal{A}^d$ is a minimal transversal. By observations (a) and (b) we have $\emptyset \neq \mathcal{A}_v(B) \subseteq \mathcal{A}_v(S)$ for each $v \in S$. Consider then a selection of the form $\{A_v \in \mathcal{A}_v(B) \mid v \in S\}$. If it covers a hyperedge $A \in \mathcal{A}_0(S)$ then A would be disjoint from B , contradicting the fact that $B \in \mathcal{A}^d$.

Proof of (ii) Suppose we are given a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$. If $\mathcal{A}_0(S) = \emptyset$ then S is obviously a transversal to \mathcal{A} . Hence by (b), S itself is a minimal transversal to \mathcal{A} . Let us assume now that $\mathcal{A}_0(S) \neq \emptyset$ and consider the hypergraph \mathcal{A}' , as defined in (4)-(5). Since the given selection is non-covering and $\mathcal{A}_0(S) \neq \emptyset$, we conclude that the vertex and edge sets of \mathcal{A}' are not empty and \mathcal{A}' contains no empty edges. Let T be a minimal transversal to \mathcal{A}' . (Such a transversal can be computed by letting $T = U \setminus I$, where $I = \text{output}(MIS(\mathcal{A}', \emptyset))$.) It is easy to see that $S \cup T$ is a transversal to \mathcal{A} . Moreover, $S \cup T$ is minimal, since if we delete a vertex $v \in S$ then $A_v \cap [(S \setminus \{v\}) \cup T] = \emptyset$, while deleting a vertex $v \in T$ results in an empty intersection with some $A \in \mathcal{A}_0(S)$. \square

Unfortunately, finding a non-covering selection for S (or equivalently, testing if S is a sub-transversal) is NP-hard if the cardinality of S is not bounded. In fact, this is so even for $\dim(\mathcal{A}) = 2$, that is for graphs (see [4]). However, if the size of S is bounded by a constant then there are only polynomially many selections $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S . All of these selections, including the non-covering ones, can be easily enumerated in polynomial time (moreover, it can be done in parallel).

Corollary 1. *For any fixed c there is an NC algorithm which, given a hypergraph $\mathcal{A} \subseteq 2^V$ and a set S of at most c vertices, determines whether S is a sub-transversal to \mathcal{A} and if so finds a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$.*

Note that this Corollary holds for hypergraphs of arbitrary dimension.

2.3 Proof of Theorem 2

We prove the theorem for the equivalent problem $\text{DUAL}(\mathcal{A}, \mathcal{B})$. We may assume without loss of generality that \mathcal{A} is Sperner. Our reduction consists of the following steps:

Step 1. By definition, each set $B \in \mathcal{B}$ is a minimal transversal to \mathcal{A} . This implies that each set $A \in \mathcal{A}$ is transversal to \mathcal{B} . Check whether each $A \in \mathcal{A}$ is a *minimal* transversal to \mathcal{B} . Suppose that some $A^o \in \mathcal{A}$ is not minimal, i.e. there is a vertex $u \in A^o$ such that $A^* = A^o \setminus \{u\}$ is still transversal to \mathcal{B} . Then we can proceed as follows.

- Let $\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}\}$, where $U = V \setminus A^*$.
- Since \mathcal{A} is Sperner, we have $A \cap U \neq \emptyset$ for each hyperedge $A \in \mathcal{A}$. Hence any minimal transversal T to \mathcal{A}' is also a minimal transversal for \mathcal{A} .
- It is easy to see that $T \not\subseteq \mathcal{B}$. This is because any set $B \in \mathcal{B}$ intersects A^* whereas T is disjoint from A^* . This reduces the computation of a new element in $\mathcal{A}^d \setminus \mathcal{B}$ to problem $\text{MIS}(\mathcal{A}', \emptyset)$.

Thus we assume in the sequel that each set in \mathcal{A} is a minimal transversal to \mathcal{B} :

$$\mathcal{A} \subseteq \mathcal{B}^d. \quad (6)$$

Recall that $\mathcal{A}^{dd} = \mathcal{A}$ for each Sperner hypergraph \mathcal{A} . Therefore, if $B \neq \mathcal{A}^d$ then $\mathcal{A} \neq \mathcal{B}^d$. By (6), we then have $\mathcal{B}^d \setminus \mathcal{A} \neq \emptyset$. Hence we arrive at the following duality criterion: $\mathcal{A}^d \setminus \mathcal{B} \neq \emptyset$ iff there is a sub-transversal S to \mathcal{B} such that

$$S \subseteq A \text{ for no } A \in \mathcal{A}. \quad (7)$$

Hence we can apply the sub-transversal test only to S such that

$$|S| \leq \dim(\mathcal{I}^{-1}(\mathcal{A})). \quad (8)$$

So far, we have not relied on the assumption that $\dim(\mathcal{I}^{-1}(\mathcal{A}))$ is bounded. We need it to guarantee that the next step of our reduction is polynomial (and moreover, is in NC).

Step 2 (Duality test.) For each set S satisfying (7), (8) and the condition that

$$A \not\subseteq S \text{ for all } A \in \mathcal{A}, \quad (9)$$

check whether or not

$$S \text{ is a sub-transversal to } \mathcal{B}. \quad (10)$$

Recall that by Proposition 2, S satisfies (10) iff there is a selection

$$\{B_v \in \mathcal{B}_v(S) \mid v \in S\} \quad (11)$$

which covers no set $B \in \mathcal{B}_0(S)$. Here as before, $\mathcal{B}_0(S) = \{B \in \mathcal{B} \mid B \cap S = \emptyset\}$ and $\mathcal{B}_v(S) = \{B \in \mathcal{B} \mid B \cap S = \{v\}\}$ for $v \in S$.

If conditions (7), (8), (9) and (10) cannot be met, we conclude that $\mathcal{B} = \mathcal{A}^d$ and halt.

Step 3. Suppose we have found a non-covering selection (11) for some set S satisfying (7), (8), (9) (and hence (10)). Then it is easy to see that the set

$$Z = S \cup \left[V \setminus \bigcup_{v \in S} B_v \right]$$

is independent in \mathcal{A} . Suppose to the contrary that $A \subseteq Z$ for some $A \in \mathcal{A}$. By (7), there is a vertex $u \in S$ such that $u \notin A$. Then $A \cap B_u = \emptyset$, yielding a contradiction. Furthermore, Z is transversal to \mathcal{B} , because selection (11) is non-covering. Let $\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}\}$, where $U = V \setminus Z$, and let T be a minimal transversal to \mathcal{A}' . (As before, we can let $T = U \setminus \text{output}(\text{MIS}(\mathcal{A}', \emptyset))$.) Since Z is an independent set of \mathcal{A} , we have $T \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, that is T is transversal to \mathcal{A} . Clearly, T is minimal, that is $T \in \mathcal{A}^d$. It remains to argue that T is a *new* minimal transversal to \mathcal{A} , that is $T \notin \mathcal{B}$. This follows from the fact that Z is transversal to \mathcal{B} and disjoint from T . \square

Note that Theorem 2 does not imply that $\text{MIS}(\mathcal{A}, \mathcal{I}) \in \text{NC}$ because the parallel complexity of the resulting problem $\text{MIS}(\mathcal{A}', \emptyset)$ is not known. The question whether it is in NC in general (for arbitrary hypergraphs) was raised in [19]. The affirmative answers were obtained in [1, 8, 16, 17, 21, 23] for the following special cases. For hypergraphs of bounded dimension, $\mathcal{A} \in \mathbb{A}(1, c)$, it is known that $\text{MIS}(\mathcal{A}', \emptyset) \in \text{NC}$ for $c \leq 3$, and $\text{MIS}(\mathcal{A}', \emptyset) \in \text{RNC}$ for $c = 4, 5, \dots$, see [2, 21]. Furthermore, it was shown in [23, 24] that $\text{MIS}(\mathcal{A}', \emptyset) \in \text{NC}$ for the so-called *linear* hyperedges, in which each two hyperedges intersect in at most one vertex, that is for $\mathcal{A}' \in \mathbb{A}(2, 1)$. Finally, it follows from [15] that $\text{MIS}(\mathcal{A}', \emptyset) \in \text{NC}$ for hypergraphs of bounded degree (2), that is for $\mathcal{A}' \in \mathbb{A}(c, 0)$. Combining the above results with Theorem 2, we obtain the following corollary.

Corollary 2. (i) *Problem $\text{MIS}(\mathcal{A}, \mathcal{I})$ is in RNC for or $\mathcal{A} \in \mathbb{A}(1, c)$, where c is a constant (hypergraphs of bounded dimension).*
(ii) *$\text{MIS}(\mathcal{A}, \mathcal{I})$ is in NC for $\mathcal{A} \in \mathbb{A}(1, c)$, $c \leq 3$ (hypergraphs of $\dim \leq 3$),*
(iii) *$\text{MIS}(\mathcal{A}, \mathcal{I})$ is in NC for $\mathcal{A} \in \mathbb{A}(c, 0)$, where c is a constant (hypergraphs of bounded degree), and finally*
(iv) *$\text{MIS}(\mathcal{A}, \mathcal{I})$ is in NC for $\mathcal{A} \in \mathbb{A}(2, 1)$ (linear hypergraphs).*

Yet, for a hypergraph \mathcal{A} satisfying $\dim(\mathcal{I}^{-1}(\mathcal{A})) \leq \text{const}$, or even more specifically for $\mathcal{A} \in \mathbb{A}(k, r)$, $k+r \leq \text{const}$, we only have an NC-reduction of $\text{MIS}(\mathcal{A}, \mathcal{I})$ to $\text{MIS}(\mathcal{A}', \emptyset)$, where the parallel complexity of the latter problem is not known.

Remark 1. Theorem 2 can be generalized to families of vectors in the Cartesian product of n lattices. Specifically, given n lattices $\mathcal{P}_1, \dots, \mathcal{P}_n$ and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, consider the problem of generating the family $\mathcal{I}(\mathcal{A})$ of all maximal elements in $\mathcal{P} \setminus \mathcal{A}^+$, where $\mathcal{A}^+ = \{x \in \mathcal{P} \mid x \succcurlyeq a, \text{ for some } a \in \mathcal{A}\}$ denotes the ideal generated by \mathcal{A} . If $\mathcal{P} = \{0, 1\}^n$ is the product of n chains $\{0, 1\}$, then this problem is equivalent to the generation of the transversal hypergraph

for \mathcal{A} . In general, when \mathcal{A} is a set in $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, we define $\dim(\mathcal{A}) = \max\{|\text{Supp}(a)| : a \in \mathcal{A}\}$, where $\text{Supp}(a)$ is the *support* of $a \in \mathcal{P}$, i.e., the set of all non-minimal components of a . Let $\mathcal{A}^- = \{x \in \mathcal{P} \mid x \preceq a, \text{ for some } a \in \mathcal{A}\}$ and denote by $\mathcal{I}^{-1}(\mathcal{A})$ the family of minimal elements in $\mathcal{P} \setminus \mathcal{A}^-$. Then it follows from a generalization of the sub-transversal criterion in [4] that, for families of vectors $\mathcal{A} \subseteq \mathcal{P}$ satisfying

$$\dim(\mathcal{I}^{-1}(\mathcal{A})) \leq \text{const}, \quad (12)$$

the problem of extending a given list $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$ is *NC*-reducible to computing a single maximal element in $\mathcal{P}' \setminus \mathcal{A}'$, where \mathcal{P}' is a sub-lattice of \mathcal{P} and $\mathcal{A}' \subseteq \mathcal{P}'$ satisfies (12), provided that the number of immediate predecessors of any element in each factor-lattice \mathcal{P}_i is also bounded by a constant. \square

2.4 Powers of independence

Theorem 3 states that for hypergraphs $\mathcal{A} \subseteq 2^V$ satisfying (3), problem $\text{MIS}(\mathcal{A}, \mathcal{I})$ is *NC*-reducible to the computation of a single independent set in an induced partial sub-hypergraph of \mathcal{A} . In this section, we examine other related special classes of hypergraphs for which a similar result can be obtained. In particular, we consider the hypergraphs resulting from repeated applications of the operators $\mathcal{I}(\cdot)$ and $\mathcal{I}^{-1}(\cdot)$ to a given hypergraph. Some interesting properties of the powers of these operators can be found in [7, 9, 13].

Let $\mathcal{A} \subseteq 2^V$ be a hypergraph such that

$$\dim(\mathcal{A}^{dcd}) \leq c, \quad (13)$$

for some constant c . It is not clear how to check membership in the family of hypergraphs satisfying (13). However, as far as the generation of the dual hypergraph \mathcal{A}^d is concerned, such a check is not needed. In fact, we shall present below an algorithm that, for any given constant c , will keep generating in incremental polynomial time (and in fact efficiently in parallel) maximal independent sets of \mathcal{A} , and halt only when either all these independent sets have been generated, or the algorithm discovers that (13) is not satisfied.

Let c be a given constant. The algorithm will proceed in the following two steps:

Step 1. Generate the hypergraph $\mathcal{B} \subseteq 2^V$, whose hyperedges are defined as follows:

$$\mathcal{B} = \{S \subseteq V : |S| \leq c \text{ and } S \text{ is a minimal non sub-transversal of } \mathcal{A}\}. \quad (14)$$

For a constant c , the hypergraph \mathcal{B} can be generated efficiently in parallel by Corollary 1.

Step 2. Note that $\dim(\mathcal{B}) \leq c$. Thus Corollary 2 implies that the dual hypergraph \mathcal{B}^d can be generated in incremental *RNC* time. However, we do not need to

generate always all the hyperedges of \mathcal{B}^d . We stop generation when either an edge $X \in \mathcal{B}^d$ is generated such that X is not a maximal independent set of \mathcal{A} , or when all edges of \mathcal{B}^d have been generated, whichever happens sooner.

To verify that the above procedure indeed generates $\mathcal{I}(\mathcal{A})$ in incremental *RNC* time if (13) is satisfied, notice the equivalences

$$\dim(\mathcal{A}^{dcd}) \leq c \iff \mathcal{B} = \mathcal{A}^{dcd} \iff \mathcal{B}^d \subseteq \mathcal{I}(\mathcal{A}).$$

The first equivalence is clear from the definition of \mathcal{B} since a subset $S \subseteq V$ belongs to \mathcal{A}^{dcd} if and only if S is a minimal non sub-transversal of \mathcal{A} . To see the second equivalence, suppose that $\mathcal{B}^d \subseteq \mathcal{I}(\mathcal{A}) = \mathcal{A}^{dc}$, and take an $X \in \mathcal{A}^{dcd}$. Then $\mathcal{B}^d \subseteq \mathcal{A}^{dc}$ implies that X contains a subset $X' \in \mathcal{B}$, while $\mathcal{B} \subseteq \mathcal{A}^{dcd}$ implies that $X' \in \mathcal{A}^{dcd}$. But since \mathcal{A}^{dcd} is Sperner, we get $X = X' \in \mathcal{B}$. Thus we obtain the following.

Corollary 3. *For any hypergraph \mathcal{A} and any constant c satisfying (13), problem $MIS(\mathcal{A}, \mathcal{I})$ is *NC*-reducible to $MIS(\mathcal{A}', \emptyset)$, where \mathcal{A}' is a partial sub-hypergraph of \mathcal{A}^{dcd} .*

Finally, let us consider some other easy consequences of Theorem 3 and Corollary 3. For a hypergraph $\mathcal{A} \subseteq 2^V$, note that

$$\dim(\mathcal{A}^c) \leq c \iff \dim(\mathcal{A}^d) \leq c + 1. \quad (15)$$

Denoting by $\mathcal{A}^{(dc)^l}$ the application of the operator $(\cdot)^{dc}$, l times, to the hypergraph \mathcal{A} , we note by (15) that

$$\dim(\mathcal{A}^{(dc)^l}) \leq c \iff \dim(\mathcal{A}^{(cd)^l}) \leq c + l + 1,$$

and hence for constant c and l , the dualization of hypergraphs \mathcal{A} satisfying $\dim(\mathcal{A}^{(dc)^l}) \leq c$ is *NC*-reducible to the computation of a single maximal independent set in some *NC*-computable hypergraph. Similarly, if $\dim(\mathcal{A}^{(cd)^l}) \leq c$ then $\dim(\mathcal{A}^{dcd}) \leq c + l + 2$, and hence, for constant c and l , the computation of \mathcal{A}^d is incrementally *NC*-reducible to the computation of a single maximal independent set in some *NC*-computable hypergraph. The next weakest cases, of this form, for which no polynomial time dualization algorithm is known, are when $\dim(\mathcal{A}^{cdcd}) \leq c$ and when $\dim(\mathcal{A}^{dcdcd}) \leq c$, for some constant c .

3 Polynomial Space Algorithm for Generating \mathcal{A}^d

For $i = 1, \dots, n$ denote by $[i : n]$ the set $\{i, i + 1, \dots, n\}$, where $[n + 1 : n]$ is assumed to be the empty set. Given a hypergraph $\mathcal{A} \subseteq 2^{[n]}$, we shall say that $X \subseteq [n]$ is an i -minimal transversal for \mathcal{A} if $X \supseteq [i : n]$, X is a transversal of \mathcal{A} , and $X \setminus \{j\}$ is not a transversal for all $j \in X \cap [1 : i - 1]$. Thus, $n + 1$ -minimal

transversals are just the minimal transversals of \mathcal{A} . For $i = 1, \dots, n$, let \mathcal{A}^{d_i} be the family of i -minimal transversals for \mathcal{A} .

Given $i \in [n]$ and $X \in \mathcal{A}^{d_i}$, let $\mathcal{A}_i(X)$ be the hypergraph

$$\mathcal{A}_i(X) = \{A \setminus \{i\} : A \in \mathcal{A}, A \cap X = \{i\}\}.$$

Proposition 3 (see [12, 22]).

(i) $|\mathcal{A}^{d_i}| \leq |\mathcal{A}^d|$, for $i = 1, \dots, n + 1$.

(ii) $|\mathcal{A}_i(X)^d| \leq |\mathcal{A}^{d_{i+1}}|$, for $i \in [n]$ and $X \in \mathcal{A}^{d_i}$.

Proof. (i) follows from the fact that each set $X \in \mathcal{A}^{d_i}$ can be reduced to a set in \mathcal{A}^d by dropping some elements from $X \cap [i : n]$. (ii) follows from the fact that for each set $Y \in \mathcal{A}_i(X)^d$, the set $X \cup Y \setminus \{i\}$ can be reduced to a set in $\mathcal{A}^{d_{i+1}}$ by deleting some elements from $X \cap [i - 1]$. \square

Now consider the following generalization of an algorithm in [25] for generating maximal independent sets in graphs (see also [18] and [22]). Given $i \in [n]$, and $X \in \mathcal{A}^{d_i}$, we assume in the algorithm that the minimal transversals $\mathcal{A}_i(X)^d$ are computed by calling a process $P(i, X)$ that invokes the same algorithm recursively on the partial hypergraph $\mathcal{A}_i(X)$. We further assume that, once $P(i, X)$ finds an element $Y \in \mathcal{A}_i(X)^d$, it returns control to the calling process $\text{GEN}(\mathcal{A}, i, X)$. When called for the next time, $P(i, X)$ returns the next element of $\mathcal{A}_i(X)^d$ that has not been generated yet, if there is such an element.

Algorithm GEN(\mathcal{A}, i, X):

Input: A hypergraph \mathcal{A} , an index $i \in [n]$, and an i -minimal transversal $X \in \mathcal{A}^{d_i}$.

Output: All minimal transversals of \mathcal{A} .

1. if $i = n + 1$ then
2. output X ;
3. else
4. if $X \setminus \{i\}$ is a transversal of \mathcal{A} then
5. $\text{GEN}(\mathcal{A}, i + 1, X \setminus \{i\})$;
6. else
7. $\text{GEN}(\mathcal{A}, i + 1, X)$;
8. for each minimal transversal $Y \in \mathcal{A}_i(X)^d$ (found recursively) do
9. if $X \cup Y \setminus \{i\} \in \mathcal{A}^{d_{i+1}}$ then
10. Compute the *lexicographically largest* set $Z \subseteq X \cup Y$ such that $Z \in \mathcal{A}^{d_i}$.
11. if $Z = X$ then
12. $\text{GEN}(\mathcal{A}, i + 1, X \cup Y \setminus \{i\})$;
13. end if
14. end if
15. end for
16. end if
17. end if

Lemma 2. *When called with $i = 1$ and $X = [n]$, Algorithm $GEN(\mathcal{A}, i, X)$ outputs all minimal transversals of \mathcal{A} with no repetitions.*

Proof. Consider the recursion tree \mathbf{T} traversed by the algorithm. Label each node of tree by the pair (i, X) which represents the input to the algorithm at this node. Clearly i represents the level of node (i, X) in the tree (where the root of \mathbf{T} is at level 1). By induction on $i = 1, \dots, n + 1$, we can verify the following statement:

$$\mathcal{A}^{d_i} = \{X \subseteq [n] : (i, X) \in \mathbf{T}\}. \quad (16)$$

Indeed, this trivially holds at $i = 1$. Assume now that (16) holds for a specific $i \in [n - 1]$. It is easy to see that any node $(i + 1, X) \in \mathbf{T}$ generated at level $i + 1$ of the tree must have $X \in \mathcal{A}^{d_{i+1}}$. Thus it remains to verify that $\mathcal{A}^{d_{i+1}} \subseteq \{X : (i + 1, X) \in \mathbf{T}\}$. To see this, let X' be an arbitrary element of $\mathcal{A}^{d_{i+1}}$. Note first that if $X' \ni i$ then $X' \setminus \{i\}$ is not a transversal of \mathcal{A} and $X' \in \mathcal{A}^{d_i}$, and therefore by induction we have a node $(i, X') \in \mathbf{T}$. Consequently, we get a node $(i + 1, X') \in \mathbf{T}$ as a child of $(i, X') \in \mathbf{T}$, by Step 7 of the algorithm. Let us therefore assume that $X' \not\ni i$. Note that X' must contain a subset $X \setminus \{i\}$, for some $X \in \mathcal{A}^{d_i}$. This is because $X' \cup \{i\}$ is a transversal and therefore it contains an i -minimal transversal X of \mathcal{A} . Among all the sets X satisfying this property, let Z be the lexicographically largest. Now, if $Z \setminus \{i\}$ is a transversal of \mathcal{A} , then $Z \setminus \{i\} = X'$ and Step 5 will create a node $(i + 1, X') \in \mathbf{T}$ as the only child of $(i, Z) \in \mathbf{T}$. On the other hand, if $Z \setminus \{i\}$ is not a transversal, then X' can be written as $X' = Z \cup Y \setminus \{i\}$, for some $Y \in \mathcal{A}_i(Z)^d$. But then node $(i + 1, X')$ will be generated as a child of $(i, Z) \in \mathbf{T}$ by Step 12 of the Algorithm. This completes the proof of (16). Finally, it follows from Step 10 that each node in the tree is generated as the child of exactly one other node. Consequently each leaf is visited, and hence each set $X \in \mathcal{A}^d$ is output, only once and the lemma follows. \square

The next lemma states that, for hypergraphs \mathcal{A} of (k, r) -bounded intersections, Algorithm GEN is a polynomial-space, *output*-polynomial time algorithm for generating all minimal transversals of \mathcal{A} .

Lemma 3. *The time taken by Algorithm GEN until it outputs the last minimal transversal of a hypergraph $\mathcal{A} \in \mathbb{A}(k, r)$ is $O(n^{k+r-1}|\mathcal{A}^d|^{r+1})$, and the total space required is $O(N^{r+1})$.*

Proof. For a hypergraph $\mathcal{A} \in \mathbb{A}(k, r)$, let $T(\mathcal{A})$ and $M(\mathcal{A})$ be respectively the time and space required by Algorithm GEN to output the last minimal transversal of \mathcal{A} . Note that the algorithm basically performs depth-first search on the tree \mathbf{T} (whose leaves are the elements of \mathcal{A}^d), and only generates nodes of \mathbf{T} as needed during the search. Since each node of the tree \mathbf{T} , which is not a leaf, has at least one child, the time between two successive outputs generated by the algorithm does not exceed the time required to generate the children of nodes along a complete path of the tree \mathbf{T} from the root to a leaf. But, as can be seen from the algorithm, for a given node $v = (i, X)$ in \mathbf{T} , where $i \in [n]$ and $X \in \mathcal{A}^{d_i}$, the time required to generate all the children of v , is bounded by the time to

output all the elements of $\mathcal{A}_i(X)^d$. Since the depth of the tree is $n + 1$, we get the recurrence

$$T(\mathcal{A}) \leq n|\mathcal{A}^d| \max\{T(\mathcal{A}_i(X)) : i \in [n], X \in \mathcal{A}^{d_i}\}. \quad (17)$$

Note that $\mathcal{A}_i(X) \in \mathbb{A}(k, r-1)$. Furthermore, by Proposition 3, we have $|\mathcal{A}_i(X)^d| \leq |\mathcal{A}^d|$, and thus (17) gives $T(\mathcal{A}) \leq (n|\mathcal{A}^d|)^r T(\mathcal{A}')$, for some sub-hypergraph $\mathcal{A}' \in \mathbb{A}(k, 0)$ of \mathcal{A} which satisfies $|(\mathcal{A}')^d| \leq |\mathcal{A}^d|$. Now, we observe that for any $i \in [n]$ and $X \in (\mathcal{A}')^{d_i}$, we have $|\mathcal{A}'_i(X)| \leq k - 1$, and hence it follows that $T(\mathcal{A}') = O(n^{k-1}|(\mathcal{A}')^d|)$. The bound on the running time follows.

Now let us consider the total memory required by the algorithm. Since, for each recursion tree (corresponding to a (sub-)hypergraph that is to be dualized), the algorithm maintains only the path from the root to a leaf of the tree, we get the recurrence

$$M(\mathcal{A}) \leq N \max\{M(\mathcal{A}_i(X)) : i \in [n], X \in \mathcal{A}^{d_i}\}.$$

This recurrence again gives $M(\mathcal{A}) \leq N^r M(\mathcal{A}')$, for some sub-hypergraph $\mathcal{A}' \in \mathbb{A}(k, 0)$ of \mathcal{A} . But $M(\mathcal{A}') = O(N)$ and the bound on the space follows. \square

Now Theorem 3 follows by combining Lemma 3 with the following reduction.

Lemma 4. *Let $\mathcal{A} \subseteq 2^{[n]}$ be a hypergraph. Suppose that there is an algorithm P that generates all minimal transversals of \mathcal{A} in time $p(n, |\mathcal{A}^d|)$ and space $q(N(\mathcal{A}))$, for some polynomials $p(\cdot, \cdot)$ and $q(\cdot)$. Then for any integer k , we can generate at least k minimal transversals of \mathcal{A} in time $2n(p(n, k) + 1)$ and space $q(N(\mathcal{A}))$.*

Proof. We start by running algorithm P for time $p(n, k) + 1$ to check whether the number of minimal transversals of \mathcal{A} exceeds k (but we do not yet output any of them). If the algorithm terminates before this time, we conclude that \mathcal{A}^d has at most k elements, which can be generated by re-running the algorithm for time $p(n, k) + 1$. Otherwise, we proceed as follows: First, we initialize $S \leftarrow \emptyset$ and $R \leftarrow \emptyset$. Then, we repeat the following steps, for $i = 1$ to n :

Step 1. Determine if the number of minimal transversals for the restricted hypergraph

$$\mathcal{A}_Y = \{A \cap ([n] \setminus Y) \mid A \in \mathcal{A}\}, \quad (18)$$

with $Y = R \cup \{i\}$, exceeds k . This can be done by running algorithm P for time $p(n, k) + 1$ on the hypergraph \mathcal{A}_Y .

Step 2. If this number is less than k , re-run algorithm P on (18), and output, from among the minimal transversals of (18), only those that contain S . Then set $S \leftarrow S \cup \{i\}$ and proceed with next i .

Step 3. Otherwise, set $R \leftarrow R \cup \{i\}$ and proceed.

Note that the above procedure produces, *with no repetitions*, either all or k of the elements of \mathcal{A}^d , whichever is smaller in number, in time at most $2n(p(n, k) + 1)$, and requires only $q(N)$ space. \square

Note that it is implicit in the proof of Lemma 3 that, for both graphs $\mathcal{A} \in \mathbb{A}(1, 2)$ and hypergraphs of bounded degree $\mathcal{A} \in \mathbb{A}(c, 0)$, Algorithm GEN is in fact a polynomial delay and polynomial space algorithm for generating \mathcal{A}^d . In particular, Theorem 3 implies the following previously known results [12, 18, 25].

Corollary 4. *For graphs, $\mathcal{A} \in \mathbb{A}(1, 2)$, and also for the hypergraphs of bounded degree, $\mathcal{A} \in \mathbb{A}(c, 0)$, all minimal transversals of \mathcal{A} can be enumerated with polynomial delay and polynomial space.*

4 Generating \mathcal{A}^d Using the Supergraph Approach

Let $\mathcal{A} \subseteq 2^V$ be a hypergraph. In this section, we sketch another algorithm to list all minimal transversals of \mathcal{A} . The algorithm works by building a *strongly connected directed supergraph* $\mathcal{G} = (\mathcal{A}^d, \mathcal{E})$ on the set of minimal transversals, in which a pair of vertices (X, X') forms an edge in \mathcal{E} if and only if X' can be obtained from X by deleting an element from $X \setminus X'$, adding a minimal subset of elements from $X' \setminus X$ to obtain a transversal, and finally reducing the resulting set to a minimal feasible solution in a specified way (say in reverse-lexicographic order). In other words, $(X, X') \in \mathcal{E}$ if and only if $X' \subseteq X \cup Z \setminus \{e\}$, for some $e \in X \setminus X'$ and $Z \subseteq X' \setminus X$, such that Z is minimal with the property that $X \cup Z \setminus \{e\}$ is a transversal.

The strong connectivity of \mathcal{G} can be proved as follows. Given two vertices $X_0, X_l \in \mathcal{A}^d$ of \mathcal{G} , there exists a set $\{X_1, \dots, X_{l-1}\}$ of elements of \mathcal{F} , where for all $i = 1, \dots, l$, X_i is obtained from X_{i-1} by deleting an element $e_i \in X_{i-1} \setminus X_l$ (thus making $X_{i-1} \setminus \{e_i\}$ non-transversal), adding a minimal subset of elements $Z_i \subseteq X_l \setminus X_{i-1}$ to obtain a transversal $X_{i-1} \setminus \{e_i\} \cup Z_i$, and finally, reducing the resulting set to a minimal transversal $X_i \subseteq X_{i-1} \cup Z_i \setminus \{e_i\}$. Note that, for $i = 1, \dots, l$, $|X_i \setminus X_l| < |X_{i-1} \setminus X_l|$ and therefore $l \leq |X_0 \setminus X_l|$. In other words, \mathcal{G} has *diameter* at most n .

The minimal transversals of \mathcal{A} can thus be generated by performing breadth-first search on the vertices of \mathcal{G} , starting from an arbitrary vertex. Such a procedure can be executed in incremental polynomial time if the neighbourhood of every vertex in \mathcal{G} can also be generated in (incremental) polynomial time. Given a hypergraph $\mathcal{A} \in \mathcal{A}(k, r)$, and a minimal transversal $X \in \mathcal{A}^d$, all neighbours of X in \mathcal{G} can be generated in time $O(n^{k+r}|\mathcal{A}^d|^{r+1})$. Indeed, for any $e \in X$, all minimal subsets of vertices Z , such that $X \setminus \{e\} \cup Z$ is a transversal of \mathcal{A} , can be obtained by finding all minimal transversals for the hypergraph $\mathcal{A}_e(X) = \{A \setminus \{e\} : A \in \mathcal{A}, A \cap X = \{e\}\}$. But as noted before, $\mathcal{A}_e(X) \in \mathcal{A}(k, r-1)$ and $|\mathcal{A}_e(X)^d| \leq |\mathcal{A}^d|$. We conclude therefore, as in the proof of Lemma 3, that the time required to produce all the neighbours of X by applying the algorithm recursively on each of the hypergraphs \mathcal{A}_e , for $e \in X$, is $O(n^{k+r}|\mathcal{A}^d|^{r+1})$.

Thus if $k + r \leq \text{const}$, we obtain an total polynomial time algorithm. Such an algorithm can be converted to an incremental one by applying Lemma 4.

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