An inequality for polymatroid functions and its applications ⋆

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Abstract

An integral-valued set function $f : 2^V \mapsto \mathbb{Z}$ is called polymatroid if it is submodular, non-decreasing, and $f(\emptyset) = 0$. Given a polymatroid function f and an integer threshold $t \geq 1$, let $\alpha = \alpha(f,t)$ denote the number of maximal sets $X \subseteq V$ satisfying $f(X) < t$, let $\beta = \beta(f,t)$ be the number of minimal sets $X \subseteq V$ for which $f(X) \geq t$, and let $n = |V|$. We show that if $\beta \geq 2$ then $\alpha \leq \beta^{(\log t)/c}$, where $c = c(n, \beta)$ is the unique positive root of the equation $1 = 2^c(n^{c/\log\beta} - 1)$. In particular, our bound implies that $\alpha \leq (n\beta)^{\log t}$ for all $\beta \geq 1$. We also give examples of polymatroid functions with arbitrarily large t, n, α and β for which $\alpha \geq \beta^{(.551 \log t)/c}$.

More generally, given a polymatroid function $f : 2^V \mapsto \mathbb{Z}$ and an integral threshold $t \geq 1$, consider an arbitrary hypergraph $\mathcal{H}$ such that $|\mathcal{H}| \geq 2$ and $f(H) \geq t$ for all $H \in \mathcal{H}$. Let $\mathcal{S}$ be the family of all maximal independent sets X of $\mathcal{H}$ for which $f(X) < t$. Then $|\mathcal{S}| \leq |\mathcal{H}|^{(\log t)/c(n,|\mathcal{H}|)}$. As an application, we show that given a system of polymatroid inequalities $f_1(X) \geq t_1, \ldots, f_m(X) \geq t_m$ with quasi-polynomially bounded right hand sides $t_1, \ldots, t_m$, all minimal feasible solutions to this system can be generated in incremental quasi-polynomial time. In contrast to this result, the generation of all maximal infeasible sets is an NP-hard problem for many polymatroid inequalities of small range.

Key words: dualization, incremental algorithm, independent set, matroid, submodular function, polymatroid function, system of polymatroid inequalities, transversal hypergraph.

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1 Introduction

1.1 Main results

Let V be a finite set of cardinality $|V| = n$, let $f : 2^V \mapsto \mathbb{Z}_+$ be a set-function taking non-negative integral values, and let $r = r(f)$ denote the *range* of f, i.e., $r(f) = \max\{f(X) \mid X \subseteq V\}$. The set-function f is called *Boolean* if $r = 1$, *monotone* if $f(X) \leq f(Y)$ whenever $X \subseteq Y$, and *submodular* if

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y) \tag{1}$$

holds for all subsets $X, Y \subseteq V$. Finally, f is called a *polymatroid function* if it is monotone, submodular and $f(\emptyset) = 0$. Let us remark that in general, determining the range of a submodular set function may be an NP-hard problem, while $r(f) = f(V)$ holds for polymatroid functions.

Given a monotone function $f : 2^V \mapsto \{0, 1, \ldots, r\}$, and an integral threshold $t \in \{1, ..., r\}$ let us denote by $\mathcal{B}_t = \mathcal{B}_t(f)$ the family of all minimal subsets $X \subseteq V$ for which $f(X) \geq t$, and analogously, let us denote by $\mathcal{A}_t = \mathcal{A}_t(f)$ the family of all maximal subsets $X \subseteq V$ for which $f(X) < t$. It is easy to see that $\mathcal{A}_t = \mathcal{I}(\mathcal{B}_t)$, where $\mathcal{I}(\cdot)$ denotes the family of all maximal independent sets for the hypergraph $(\cdot)$. Throughout the paper we shall use the notation $\alpha = |\mathcal{A}_t(f)|$ and $\beta = |\mathcal{B}_t(f)|$.

Theorem 1 *For every polymatroid function f and threshold $t \in \{1, \ldots, r(f)\}$ such that $\beta \geq 2$ we have the inequality*

$$\alpha \leq \beta^{(\log t)/c(n,\beta)}, \tag{2}$$

where $c(n, \beta)$ is the unique positive root of the equation

$$2^c(n^{c/\log\beta} - 1) = 1. \tag{3}$$

In addition, $\alpha \leq n$ holds if $\beta = 1$.

Let us first remark that by (3), $1 = n^{-c/\log\beta} + (n\beta)^{-c/\log\beta} \geq 2(n\beta)^{-c/\log\beta}$, and hence $\beta^{1/c(n,\beta)} \leq n\beta$. Consequently, for $\beta \geq 2$ (in which case $n \geq 2$ is implied, too) we can replace (2) by the simpler but weaker inequality

$$\alpha \leq (n\beta)^{\log t}. \tag{4}$$

In fact, (4) holds even in case of $\beta = 1$, because if the hypergraph $\mathcal{B}_t$ consists only of a single hyperedge $X \subseteq V$, then $|\mathcal{A}_t| \leq |X| \leq n$ follows immediately by the relation $\mathcal{A}_t = \mathcal{I}(\mathcal{B}_t)$. On the other hand, for large β the bound of

Theorem 1 becomes increasingly stronger than (4). For instance, $c(n, n) = \log(1 + \sqrt{5}) - 1 > .694$, $c(n, n^2) > 1.102$, and $c(n, n^\sigma) \sim \log \sigma$ for large σ.

Let us remark next that the bound of Theorem 1 is reasonably sharp. As we shall show in Section 3, for any positive integers k and l there exists a polymatroid function f of range $r = 2^k$ for which $n = kl$, $|\mathcal{A}_r| = l^k$, and $|\mathcal{B}_r| = kl(l-1)/2$. Thus, letting $t = r$ and $l = 2^k$, we obtain an infinite family of polymatroid functions for which

$$\alpha \geq \beta^{(.551 \log t)/c(n,\beta)} \quad \text{and} \quad \alpha \geq (n\beta)^{(\frac{1}{3}+o(1)) \log t}, \tag{5}$$

as $t = r \to \infty$, see Section 3 for more detail. In Section 3 we also show that our lower bounds (5) can be achieved within the subclass of rank functions defined on the subsets of some linear space. Namely, we can construct kl subspaces $V_{ij} \subseteq \mathbb{R}^{2^k}$, $i = 1, \ldots, k$, $j = 0, \ldots, l-1$ of dimension 2^{k-1} each, such that for any i and $j \neq j'$ we have $\dim(V_{ij} \cup V_{ij'}) = 2^k$, while for every $(j_1, j_2, ..., j_k) \in \{0, 1, ..., l-1\}^k$ the inequality $\dim(\bigcup_{i=1}^k V_{ij_i}) < 2^k$ holds.

Let us finally note that for many classes of polymatroid functions, β cannot be bounded by a quasi-polynomial estimate of the form $(n\alpha)^{\text{poly} \log r}$. Let us consider for instance, a graph $G = t \times K_2$ consisting of t disjoint edges, and let $f(X)$ be the number of edges X intersects, for $X \subseteq V(G)$. Then f is a polymatroid function of range $r = t$, and we have $n = 2t$, $\alpha = |\mathcal{A}_t| = t$ and $\beta = |\mathcal{B}_t| = 2^t$.

We will strengthen Theorem 1 as follows. Given a non-empty hypergraph $\mathcal{H}$ on the vertex set V, a polymatroid function $f : 2^V \mapsto \mathbb{Z}_+$, and a integral positive threshold t, the pair (f, t) is called a *polymatroid separator* for $\mathcal{H}$ if $f(H) \geq t$ for all $H \in \mathcal{H}$.

Theorem 2 *Let (f, t) be a polymatroid separator for a hypergraph $\mathcal{H}$ of cardinality $|\mathcal{H}| \geq 2$. Then*

$$|\mathcal{A}_t(f) \cap \mathcal{I}(\mathcal{H})| \leq |\mathcal{H}|^{(\log t)/c(n,|\mathcal{H}|)}, \tag{6}$$

where $\mathcal{I}(\mathcal{H})$ is the family of all maximal independent sets for $\mathcal{H}$.

In particular, if (f, t) is a polymatroid separator for a non-empty hypergraph $\mathcal{H}$, then $|\mathcal{A}_t(f) \cap \mathcal{I}(\mathcal{H})| \leq (n|\mathcal{H}|)^{\log t}$.

Clearly, Theorem 1 is a special case of Theorem 2 for $\mathcal{H} = \mathcal{B}_t(f)$. Since the right-hand side of (6) monotonically increases with $|\mathcal{H}|$, we will only need to prove Theorem 2 for *Sperner* hypergraphs $\mathcal{H}$, i.e., under the assumption that that none of the hyperedges of $\mathcal{H}$ contains another hyperedge of $\mathcal{H}$.

Let $\mathcal{H}$ be a non-empty Sperner hypergraph on the vertex set V. A polymatroid separator (f,t) is called *exact for* $\mathcal{H}$ if $\mathcal{H}=\mathcal{B}_t(f)$ and consequently, $\mathcal{I}(\mathcal{H})=\mathcal{A}_t(f)$. It is immediate to see that for exact separators (f,t), Theorem 2 implies $r(f)\geq t\geq |\mathcal{I}(\mathcal{H})|^{c(n,|\mathcal{H}|)/\log|\mathcal{H}|}$ whenever $|\mathcal{H}|\geq 2$. As we shall see in Section 2.1, any Sperner hypergraph $\mathcal{H}$ has an exact polymatroid separator (f,t) such that $r(f)=t=|\mathcal{I}(\mathcal{H})|$ holds.

A polymatroid separator (f,t) is called *linear* if the set-function f is modular, i.e., if equality holds in (1) for all $X,Y\subseteq V$. As shown in [9,12], if (f,t) is a linear separator for a non-empty hypergraph $\mathcal{H}$, then $|\mathcal{A}_t(f)\cap\mathcal{I}(\mathcal{H})|\leq n|\mathcal{H}|$. Unlike (6), this bound does not depend on t and holds even for real-valued linear separators.

1.2 Generating minimal feasible sets for systems of polymatroid inequalities

Before proceeding further we shall discuss some algorithmic implications of Theorem 2 related to the complexity of enumerating all minimal solutions to a system of polymatroid inequalities

$$f_i(X)\geq t_i,\quad i=1,\ldots,m, \tag{7}$$

over the subsets $X\subseteq V$. Specifically, letting $\mathcal{B}$ denote the family of all minimal feasible sets for (7), we consider the following problem:

$GEN(\mathcal{B},\mathcal{H})$: *Given a system of polymatroid inequalities (7) and a collection $\mathcal{H}\subseteq\mathcal{B}$ of minimal feasible sets for (7), either find a new minimal feasible set $H\in\mathcal{B}\setminus\mathcal{H}$ for (7), or show that $\mathcal{H}=\mathcal{B}$.*

In what follows, we assume that each of the polymatroid functions $f_i:2^V\mapsto\mathbb{Z}$ is defined via a (quasi) polynomial-time evaluation oracle, and that $t_1,\ldots,t_m$ are given positive integral thresholds. Let us also note that since $\min\{f,t\}$ is a polymatroid function whenever f is polymatroid, we could also assume without loss of generality that $t_i=range(f_i)$.

Clearly, we can incrementally generate all sets in $\mathcal{B}$ by initializing $\mathcal{H}=\emptyset$ and then iteratively solving the above generation problem $|\mathcal{B}|+1$ times. It is easy to see that the first minimal feasible set $H\in\mathcal{B}$ can be found (or $\mathcal{B}=\emptyset$ can be recognized) by evaluating (7) $n+1$-times. Furthermore, since $\mathcal{I}(\{H\})=\{V\setminus\{x\}\mid x\in H\}$, the second minimal feasible set can also be identified (or $\mathcal{B}=\{H\}$ can be recognized) in another $n+|H|$ evaluations of (7). Thus, in what follows we can assume $|\mathcal{H}|\geq 2$ without any loss of generality.

By definition, each pair (f_i, t_i) is a polymatroid separator for $\mathcal{H}$, and therefore Theorem 2 implies the inequalities

$$|\mathcal{A}_{t_i}(f_i) \cap \mathcal{I}(\mathcal{H})| \leq |\mathcal{H}|^{(\log t_i)/c(n,|\mathcal{H}|)}, \quad i = 1, \ldots, m.$$

Let $\mathcal{A} = \mathcal{I}(\mathcal{B})$ be the hypergraph of all maximal infeasible sets for (7), then $\mathcal{A} \subseteq \bigcup_{i=1}^{m} \mathcal{A}_{t_i}(f_i)$. Hence we arrive at the following bound.

Corollary 1 *Let $\mathcal{B}$ be the family of all minimal feasible sets for the system of polymatroid inequalities (7) and let $\mathcal{H} \subseteq \mathcal{B}$ be an arbitrary subfamily of $\mathcal{B}$ of size $|\mathcal{H}| \geq 2$. Then*

$$|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})| \leq m|\mathcal{H}|^{(\log t)/c(n,|\mathcal{H}|)} \leq m(n|\mathcal{H}|)^{\log t}, \tag{8}$$

where $t = \max\{t_1, \ldots, t_m\}$. In particular, $|\mathcal{I}(\mathcal{B})| \leq m|\mathcal{B}|^{(\log t)/c(n,|\mathcal{B}|)}$.

As mentioned earlier, if the functions f_i in (7) are linear, then (8) can be improved to show that $|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})| \leq mn|\mathcal{H}|$ (see [9,12]).

Corollary 1 shows that if the right hand sides of (7) are quasi-polynomially bounded, i.e., $t = \max\{t_1, \ldots, t_m\} \leq 2^{\text{poly}\log nm}$, then for any non-empty hypergraph $\mathcal{H} \subseteq \mathcal{B}$ we have

$$|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})| \leq \text{quasi-poly}(n, m, |\mathcal{H}|). \tag{9}$$

By definition, the family $\mathcal{B} \subseteq 2^V$ of all minimal feasible sets for (7) is a Sperner hypergraph. Furthermore, the hypergraph $\mathcal{B}$ has a (quasi) polynomial-time superset oracle, i.e., given a set $X \subseteq V$, we can determine in (quasi) polynomial time whether or not X contains some set $H \in \mathcal{B}$ (this is equivalent to checking the feasibility of X for (7)). As shown in [6,9,12,17], for any Sperner hypergraph $\mathcal{B}$ defined via a (quasi) polynomial-time superset oracle, problem $GEN(\mathcal{B}, \mathcal{H})$ reduces in (quasi) polynomial time to $|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})|$ instances of the *hypergraph dualization problem*:

Given two explicitly listed Sperner families $\mathcal{H} \subseteq 2^V$ and $\mathcal{G} \subseteq \mathcal{I}(\mathcal{H})$, either find a new maximal independent set $X \in \mathcal{I}(\mathcal{H}) \setminus \mathcal{G}$ or show that $\mathcal{G} = \mathcal{I}(\mathcal{H})$.

To see this reduction in our case, consider a hypergraph $\mathcal{H} \subseteq \mathcal{B}$ of minimal feasible solutions to (7). Start generating maximal independent sets for $\mathcal{H}$ checking, for each generated set $X \in \mathcal{I}(\mathcal{H})$, whether or not X is feasible for (7). If X is feasible for (7) then X contains a new minimal solution to (7) which can be found by querying the feasibility oracle at most $|X| + 1$ times. If $X \in \mathcal{I}(\mathcal{H})$ is infeasible for (7), then it is easy to see that $X \in \mathcal{I}(\mathcal{B})$, and hence the number of such infeasible sets X is bounded by $|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})|$.

The hypergraph dualization problem can be solved in quasi-polynomial time $poly(n) + (|\mathcal{H}| + |\mathcal{G}|)^{o(\log(|\mathcal{H}|+|\mathcal{G}|)}$, see [16]. (Moreover, for the hypergraphs of bounded edge-sizes the dualization problem can be efficiently solved in parallel, see [7].) Hence from the above reduction we readily obtain the following result.

Theorem 3 *Consider a system of polymatroid inequalities (7) in which the right hand sides are bounded by a quasi-polynomial in the dimension of the system. Suppose further that (7) has a quasi-polynomial-time feasibility oracle. Then problem $GEN(\mathcal{B}, \mathcal{H})$ can be solved in quasi-polynomial time, and hence all minimal feasible sets for (7) can be enumerated in incremental quasi-polynomial time.*

We discuss some special cases and applications of Theorem 3 in Section 2. It is worth mentioning that in most of our examples, generating all maximal infeasible sets for (7) turns out to be NP-hard.

1.3 Proper mappings of independent sets into binary trees

Our proof of Theorem 2 makes use of a combinatorial construction which may be of independent interest. Theorem 2 states that for any polymatroid separator (f, t) of a hypergraph $\mathcal{H}$ we have

$$r(f) \geq t \geq |\mathcal{S}|^{c(n,|\mathcal{H}|)/\log(|\mathcal{H}|)},$$

where $\mathcal{S} = \mathcal{I}(\mathcal{H}) \cap \{X \mid f(X) < t\}$, i.e., the range of f must increase with the size of $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$. Thus, to prove the theorem we must first find ways to provide lower bounds on the range of a polymatroid function. To this end we shall show that the number of independent sets which can be organized in a special way into a binary tree structure provides such a lower bound.

Let $\mathbf{T}$ denote a binary tree, $V(\mathbf{T})$ denote its node set, and let $L(\mathbf{T})$ denote the set of its leaves. For every node $v \in V(\mathbf{T})$ let $\mathbf{T}(v)$ be the binary sub-tree rooted at v. Obviously, for every two nodes u, v of $\mathbf{T}$ either the sub-trees $\mathbf{T}(u)$ and $\mathbf{T}(v)$ are disjoint, or one of them is a sub-tree of the other one. The nodes u and v are called *incomparable* in the first case, and *comparable* in the second case.

Given a Sperner hypergraph $\mathcal{H}$ and a binary tree $\mathbf{T}$, let us consider mappings $\phi : L(\mathbf{T}) \mapsto \mathcal{I}(\mathcal{H})$ assigning maximal independent sets $I_l \in \mathcal{I}(\mathcal{H})$ to the leaves $l \in L(\mathbf{T})$. Let us associate furthermore to every node $v \in V(\mathbf{T})$ the intersection $S_v = \bigcap_{l \in L(\mathbf{T}(v))} I_l$. Let us call finally the mapping ϕ *proper* if it is injective, i.e., assigns different independent sets to different leaves, and if the sets $S_u \cup S_v$ are not independent whenever u and v are incomparable nodes of

$\mathbf{T}$. Let us point out that the latter condition means that the set $S_u \cup S_v$, for incomparable nodes u and v, must contain a hyperedge $H \in \mathcal{H}$, as a subset. Since the intersection of independent sets is always independent, it follows, in particular that both S_v and S_u are non-empty independent sets (otherwise their union could not be non-independent.) Finally, since all non-root nodes $u \in V(\mathbf{T})$ have at least one incomparable node $v \in V(\mathbf{T})$, we get that the sets S_u, for $u \in V(\mathbf{T}) \setminus \{s\}$ are all non-empty and independent.

Lemma 1 *Let us consider a Sperner hypergraph $\mathcal{H}$ and a polymatroid separator (f, t) of it, and let us denote by $\mathcal{S}$ the subfamily of maximal independent sets, separated by (f, t) from $\mathcal{H}$, as before. Let us assume further that $\mathbf{T}$ is a binary tree for which there exists a proper mapping $\phi : L(\mathbf{T}) \mapsto \mathcal{S}$. Then, we have*

$$r(f) \geq t \geq |L(\mathbf{T})|. \tag{10}$$

Let us note that if a proper mapping exists for a binary tree $\mathbf{T}$, then we can associate a hyperedge $H_u \in \mathcal{H}$ to every node $u \in V(\mathbf{T}) \setminus L(\mathbf{T})$ in the following way: Let v and w be the two successors of u. Since v and w are incomparable, the union $S_v \cup S_w$ must contain a hyperedge from $\mathcal{H}$. Let us choose such a hyperedge, and denote it by H_u. Let us observe next that if $l \in L(\mathbf{T}(v))$ and $l' \in L(\mathbf{T}(w))$, then $S_v \subseteq I_l$ and $S_w \subseteq I_{l'}$, and thus $H_u \subseteq I_l \cup I_{l'}$. In other words, to construct a large binary tree for which there exists a proper mapping, we have to find a way of splitting the family of independent sets, repeatedly, such that the union of any two independent sets, belonging to different parts of the split must contain a hyperedge of $\mathcal{H}$. We shall show next that indeed, such a construction is possible.

Lemma 2 *For every Sperner hypergraph $\mathcal{H} \subseteq 2^V$, $|\mathcal{H}| \geq 2$, and for every subfamily $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ of its maximal independent sets there exists a binary tree $\mathbf{T}$ and a proper mapping $\phi : L(\mathbf{T}) \mapsto \mathcal{S}$, such that*

$$|L(\mathbf{T})| \geq |\mathcal{S}|^{c(n,|\mathcal{H}|)/\log|\mathcal{H}|}, \tag{11}$$

where $n = |V|$.

Clearly, Lemmas 1 and 2 imply Theorem 2, which in turn implies Theorem 1. Lemmas 1 and 2 will be proved in Section 4.

1.4 Related work

For the special case when the Sperner hypergraph $\mathcal{H} = E(G)$ is the edge set of a connected graph G and $\mathcal{S} = \mathcal{I}(\mathcal{H})$, Lemma 2 follows from the results by

Balas and Yu (1989) (see also [2,29]): Theorem 4 of [5] claims that

$$2^p \leq |\mathcal{I}(\mathcal{H})| \leq \delta^p + 1, \tag{12}$$

where δ is the number of pairs of vertices in V at distance 2 (in particular, $\delta < n^2/2$), and p is the cardinality of a maximum induced matching in G. Any such matching can be used to construct a proper mapping $\phi : L(\mathbf{T}) \mapsto \mathcal{I}(\mathcal{H})$ for a uniform binary tree $\mathbf{T}$ of depth p, i.e., for which $|L(\mathbf{T})| = 2^p$. Namely, let $e_i = (v_0^i, v_1^i)$ for $i = 1, ..., p$ denote the edges of the induced matching. For each binary vector $x = (x_1, x_2, ..., x_p) \in \{0,1\}^p$ let us associate a subset $\tilde{I}_x$ defined by $\tilde{I}_x = \{v_{x_i}^i \mid i = 1, ..., p\}$. Finally, let $I_x \in \mathcal{I}(\mathcal{H})$ be a maximal independent set of G containing $\tilde{I}_x$, for $x \in \{0,1\}^p$. (Since the edges e_1, ..., e_p form an induced matching, all sets $\tilde{I}_x$, $x \in \{0,1\}^p$ are independent.) Thus, the sets I_x, $x \in \{0,1\}^p$ are pairwise distinct maximal independent sets of G by the above construction, and $e_i \subseteq I_x \cup I_y$ whenever $x_i \neq y_i$. Naturally, the leaves of the binary tree $\mathbf{T}$ of depth p are also encoded by the binary sequences of length p. Thus a mapping ϕ is defined by assigning I_x to the leaf coded with x, and it is not difficult to verify that this mapping is proper (and that the edge e_i will be the associated hyperedge at each node of $\mathbf{T}$ at depth $i-1$, for $i = 1, ..., p$).

By these definitions, $\log |L(\mathbf{T})| = p$ and $\log |\mathcal{I}(\mathcal{H})| \leq p \log \delta$, according to (12), and hence

$$|L(\mathbf{T})| \geq |\mathcal{I}(\mathcal{H})|^{1/\log \delta} \geq |\mathcal{I}(\mathcal{H})|^{1/(2 \log n)},$$

whereas the bound of Lemma 2 gives:

$$|L(\mathbf{T}) \geq |\mathcal{I}(\mathcal{H})|^{c(n,|\mathcal{H}|)/\log |\mathcal{H}|}.$$

For this reason, Lemma 2 can be viewed as an extension of Theorem 4 of [5] from graphs to hypergraphs (even though we could not generalize directly the notion of an induced matching).

Let us remark here that according to the above results, the existence of a "large" induced matching in a graph, or more generally, the existence of a "large" binary tree with a proper mapping in a hypergraph can be viewed as the reason for the existence of "many" maximal independent sets. These reasons, however, may not be easy to exhibit for a given graph or hypergraph. The problem of finding the maximum size induced matching in a graph is known to be NP-hard (see e.g. [31]) and it is even hard to approximate it well (see [15]). The complexities of the corresponding problems of finding or approximating well the largest binary tree with a proper mapping are open.

The problem of generating all maximal independent sets for a general independence system was shown to be NP-hard in [23]. Some special classes of independence systems were also discussed in the same paper. In particular, it was conjectured that, for linear systems (7), problem $GEN(\mathcal{B}, \mathcal{H})$ cannot be solved in polynomial time unless P=NP. Furthermore, an algorithm was given, to generate all maximal independent sets in the intersection of m matroids, whose running time was exponential in m. In contrast to these results, Corollary 1 implies that these two problems can be solved in quasi-polynomial time.

Other algorithms for generating combinatorial structures, described by polymatroid inequalities (e.g., spanning trees, maximal cliques and cycles in a graph) can be found in [20,30,32], see also Section 2 below for other examples. It should be mentioned also that, even though generating all maximal infeasible sets for (7) is NP-hard, there is a polynomial randomized scheme for nearly uniform sampling from the set of *all* infeasible sets for linear systems (7) ([21,28]). On the other hand, by using the amplification technique of [19] it is easy to show that a similar randomized scheme for nearly uniform sampling from within the set of all minimal feasible sets for a given system (7) would imply that any NP-complete problem can be solved in polynomial time by a randomized algorithm with arbitrary small one-sided failure probability (see [17] for more detail). The complexity of counting the number of minimal solutions for linear systems (7) is studied in [14].

Finally, we remark that similar inequalities for some special cases of Theorem 1 appear in [4,8,9,11,13], see also Section 2; many other intersection inequalities for hypergraphs can be found in [3], chapter 4.

2 Some polymatroid separators

In this Section we discuss a number of polymatroid separators as well as some stronger forms of Theorems 1 and 2 for specific cases.

2.1 Partial unions and transversals of a hypergraph. Fairly independent sets.

Let $\mathcal{H}$ be an explicitly given hypergraph on a finite vertex set V. For a set $X \subseteq V$, define

$$f(X) = |\{H \in \mathcal{H} \mid X \cap H \neq \emptyset\}| \tag{13}$$

to be the number of hyperedges of $\mathcal{H}$ which have a non-empty intersection with X. The function f is polymatroid and $r(f) = |\mathcal{H}|$. For a given threshold $t \in \{1, \ldots, |\mathcal{H}|\}$, the hypergraph $\mathcal{B}_t$ consists of all *partial t-transversals*, i.e., all minimal vertex sets which intersect at least t hyperedges of $\mathcal{H}$. (Equivalently, the complement to each partial t-transversal is a *fairly independent set*, i.e., a maximal vertex set containing at most $|\mathcal{H}| - t$ hyperedges of $\mathcal{H}$.)

For $t = |\mathcal{H}|$ the set family $\mathcal{B}_t$ is identical to the transversal hypergraph $\mathcal{H}^d$ of $\mathcal{H}$. This readily implies the following claim.

Proposition 1 *Every non-empty Sperner hypergraph $\mathcal{H}$ has an exact polymatroid separator of the form (13).*

Proof. Since $\mathcal{H}^{dd} = \mathcal{H}$ holds for any Sperner hypergraph $\mathcal{H}$, applying definition (13) to $\mathcal{H}^d$ we obtain a polymatroid function f' of range $r(f') = |\mathcal{H}^d|$ such that the family of all minimal feasible sets for $f'(X) \geq r$ is exactly $\mathcal{H}$. $\square$

We mention in passing that each hypergraph $\mathcal{H}$ also admits a monotone *supermodular* separator $h(X) = |\{H \in \mathcal{H} \mid H \subseteq X\}|$ of range $|\mathcal{H}|$.

Returning to the polymatroid function f defined by (13), we can identify the hypergraph $\mathcal{A}_t = \mathcal{A}_t(f)$ with the family of all maximal subsets of V that avoid at least $|\mathcal{H}| - t + 1$ hyperedges of $\mathcal{H}$. Equivalently, the complement to each set $X \in \mathcal{A}_t$ is a *partial* $(|\mathcal{H}| - t + 1)$-*union*, that is a minimal vertex set containing at least $|\mathcal{H}| - t + 1$ hyperedges of $\mathcal{H}$.

As shown in [10,11], the inequality of Theorem 1 can be strengthened for the class of polymatroid functions (13) as follows:

$$|\mathcal{A}_t| \leq t|\mathcal{B}_t| \quad \text{for any } t \in \{1, \ldots, r\}. \tag{14}$$

Moreover, the above bounds may be sharp for arbitrarily large hypergraphs $\mathcal{H}$ and $t \in \{1, \ldots, |\mathcal{H}|\}$. (Theorem 2 can also be strengthened to a similar sharp linear bound, see [10,11].)

Given a list of hyperedges of $\mathcal{H}$ and a threshold $t \in \{1, \ldots, |\mathcal{H}|\}$, we can easily check whether or not a given vertex set X contains a partial t-transversal (we only need to check if X intersects at least t hyperedges of $\mathcal{H}$). This gives a polynomial-time feasibility oracle for $f(X) \geq t$. From Theorem 3 we thus conclude that all partial t-transversals $X \in \mathcal{B}_t$ for a given hypergraph $\mathcal{H}$ can be generated in incremental quasi-polynomial time.

Since problem $GEN(\mathcal{B}_t, \mathcal{X})$ can be solved in quasi-polynomial time, it is unlikely to be NP-hard. In contrast to this result, problem $GEN(\mathcal{A}_t, \mathcal{X})$ is known to be NP-hard. Specifically, it is NP-complete to decide if $\mathcal{X} \neq \mathcal{A}_t$ for an explicitly given set family $\mathcal{X} \subseteq \mathcal{A}_t$ (see [26] and also [11] for more detail).

2.2 Maximal frequent and minimal infrequent sets for binary matrices

The notion of frequent sets in data-mining can be related naturally to the polymatroid separator (13) considered above. Let $\mathcal{D} : \mathcal{R} \times \mathcal{C} \mapsto \{0,1\}$ be a given $m \times n$ binary matrix. To each subset of columns $X \subseteq \mathcal{C}$, we associate the subset $R(X) \subseteq \mathcal{R}$ of all those rows $r \in \mathcal{R}$ for which $\mathcal{D}(r,c) = 1$ in all columns $c \in X$. The cardinality of $R(X)$ is called the *support* of X and is easily seen to be an anti-monotone supermodular function of X. Hence

$$f(X) = m - |R(X)| \tag{15}$$

is a polymatroid function of range m. In fact, the above definition is identical to that given in (13) if we let $\mathcal{H}$ to be the hypergraph defined by the anti-incidence matrix $\mathcal{D}$.

A column set X is called *t-frequent* if $|R(X)| \geq t$ and *maximal t-frequent* if no superset of X is t-frequent. The generation of (maximal) frequent sets of a given binary matrix is an important task of knowledge discovery and data mining (see, for instance, [1]). Let $\mathcal{F}_t(\mathcal{D})$ be the family of all maximal frequent sets for $\mathcal{D}$, then $\mathcal{F}_t(\mathcal{D})$ is exactly the hypergraph $\mathcal{A}_{m-t+1}(f)$ for the submodular function f defined by (15). Denoting by $\mathcal{I}_t(\mathcal{D}) = \mathcal{B}_{m-t+1}(f)$ the family of all minimal columns sets X for which $|R(X)| < t$, i.e., all minimal *t-infrequent sets*, we can conclude that

$$|\mathcal{F}_t(\mathcal{D})| \leq (m-t+1)|\mathcal{I}_t(\mathcal{D})|, \quad t \in \{1,\ldots,m\}. \tag{16}$$

As for (14), these inequalities are best possible. For instance, they are sharp when $\mathcal{D}$ is an $m \times (m-t+1)$ matrix, in which every entry is 1, except the diagonal entries in the first $m-t+1$ rows, which are 0. In addition, (16) stays accurate, up to a factor of $\log m$, even when $m \gg n$ and $|\mathcal{F}_t|$ and $|\mathcal{I}_t|$ are arbitrarily large, see [11] for more detail. It is also worth mentioning that $|\mathcal{F}_t|$ cannot be bounded by a quasi-polynomial in $|\mathcal{I}_t|$ and n, the number of columns of $\mathcal{D}$.

2.3 Weighted transversals

Extending the notion of partial t-transversals to weighted hypergraphs we arrive at weighted transversals [12]. Given a hypergraph $\mathcal{H} \subseteq 2^V$, we assign an m-dimensional non-negative integral weight vector $w = w(H) \in \mathbb{Z}_+^m$ to each hyperedge $H \in \mathcal{H}$ and consider the system of m polymatroid inequalities

$$f(X) = \sum\{w(H) \mid X \cap H \neq \emptyset,\ H \in \mathcal{H}\} \geq t \tag{17}$$

where $t \in \mathbb{Z}_+^m$ is a given threshold vector. The minimal solutions $X \subseteq V$ to the above system are called *weighted* transversals for $(\mathcal{H}, w)$. Let $\mathcal{B}$ be the set of all weighted transversals for (17) and let $\mathcal{A} = \mathcal{I}(\mathcal{B})$ denote the hypergraph of all maximal infeasible sets for (17). Considering that each weighted hypergraph $\mathcal{H}$ can be regarded as a collection of m multi-hypergraphs $(\mathcal{H}_1, \ldots, \mathcal{H}_m)$, where multi-hypergraph $\mathcal{H}_i$ contains $w_i(H)$ copies of edge $H \in \mathcal{H}$, we conclude from (14) that $|\mathcal{A}| \leq (t_1 + \ldots + t_m)|\mathcal{B}|$. This bound depends on the threshold vector and can be arbitrarily large with respect to n and m when the range of (17) becomes high. However, it can be shown [12] that regardless of the weights used in the definition of the hypergraph of weighted transversals, for any non-empty hypergraph $\mathcal{X} \subseteq \mathcal{B}$ we have the inequality

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{B})| \leq m \sum_{X \in \mathcal{X}} |\{H \in \mathcal{H} \mid H \cap X \neq \emptyset\}| \tag{18}$$

where as before $\mathcal{I}(\mathcal{X})$ denotes the family of all maximal independent sets for $\mathcal{X}$. In particular, it follows that $|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{B})| \leq m|\mathcal{H}||\mathcal{X}|$, which for $\mathcal{X} = \mathcal{B}$ gives

$$|\mathcal{I}(\mathcal{B})| \leq m|\mathcal{H}||\mathcal{B}|. \tag{19}$$

As earlier, Theorem 3 implies an incremental quasi-polynomial algorithm for generating all minimal weighted transversals for a given weighted hypergraph $(\mathcal{H}, w)$ and threshold vector t. Generating all maximal infeasible vectors to (17) is NP-hard already for scalar unit weights $w(H) \equiv 1$.

2.4 Monotone systems of Boolean and integer inequalities

This example can be viewed as a special case of weighted transversals. Consider a system of m linear inequalities in n binary variables

$$Ax \geq b, \quad x \in \{0,1\}^n \tag{20}$$

where A is a given non-negative $m \times n$ -matrix and b is a given m-vector. Let $\mathcal{C}$ be the set of columns of A, and let $\mathcal{H}$ be the hypergraph on the vertex set $\mathcal{C}$ whose hyperedges are the n singletons (columns). We have $|\mathcal{H}| = n$ and each column of the $m \times n$ matrix A can now be interpreted as a non-negative m-dimensional weight vector assigned to the corresponding hyperedge of $\mathcal{H}$. Under this interpretation, (20) is a special case of (17) and we can identify the hypergraph $\mathcal{B}$ of weighted transversals with the set of all minimal Boolean solutions to (20). Accordingly, the hypergraph $\mathcal{A} = \mathcal{I}(\mathcal{B})$ can be viewed as the set of all maximal infeasible vectors for (20). From (18) we now conclude that for any non-empty set $\mathcal{X} \subseteq \mathcal{B}$ of minimal feasible solutions to (20) we have the inequalities

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{B})| \leq m \sum_{x \in \mathcal{X}} p(x) \leq nm|\mathcal{X}|, \tag{21}$$

where $p(x)$ is number of positive components of x. In particular, for any feasible system (20) we obtain

$$|\mathcal{I}(\mathcal{B})| \leq nm|\mathcal{B}|. \tag{22}$$

The above bounds are sharp when $m = 1$, for instance, for the inequality $x_1 + \ldots + x_n \geq n$. For large m, these bounds are accurate up to a factor polylogarithmic in m. For instance, for any positive integer k, the system of $m = 2^k$ inequalities in $n = 2k$ binary variables

$$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1, \ i_1 \in \{1, 2\}, \ i_2 \in \{3, 4\}, \ldots, i_k \in \{2k - 1, 2k\}$$

has 2^k maximal infeasible binary vectors and only k minimal feasible binary vectors, i.e.,

$$|\mathcal{I}(\mathcal{B})| = \frac{nm}{2(\log m)^2}|\mathcal{B}|.$$

As shown in [9,10], inequalities (21) and (22) actually hold for any *monotone* system of inequalities (20) in binary variables, i.e., under the assumption that for any feasible vector $x \in \{0, 1\}^n$, any binary vectors $y \geq x$ is also feasible for (20). (For instance, (20) is monotone if the matrix A is non-negative.) In fact, inequalities (21) and (22) also hold for any monotone system of m linear inequalities in n *integer* variables

$$Ax \geq b, \quad x \in \mathcal{C} \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^n \mid \ 0 \leq x \leq c\}, \tag{23}$$

where c is a given n vector some or all components of which may be infinite. All minimal feasible integer solutions to a given monotone system of integer inequalities (23) can be generated in incremental quasi-polynomial time [8,12], which should be contrasted with the conjecture of E. Lawler, Lenstra and Rinnooy Kan [23] that for non-negative A and $d = (1, \ldots, 1)$ this problem cannot be solved in incremental polynomial time unless P=NP. On the other hand, the problem of generating all maximal infeasible binary vectors for (20) is NP-hard already for binary matrices A and $b = (2, 2, \ldots, 2)$, see [25] and also [9] for more detail.

2.5 Spanning a linear space by linear subspaces.

The transversal hypergraph problem is equivalent to the following set covering problem: Given an r-element ground set $\mathcal{R}$ and a family $\mathcal{V}$ of n subsets of $\mathcal{R}$, enumerate all minimal subfamilies of $\mathcal{V}$ which cover the entire set $\mathcal{R}$. Replacing

$\mathcal{R}$ by the vector space $\mathbf{F}^r$ over some field $\mathbf{F}$, and replacing each given subset of $\mathcal{V}$ by a linear subspace of $\mathbf{F}^r$, we arrive at the following *space covering* problem:

Given a collection $\mathcal{V} = \{V_1, \ldots, V_n\}$ *of* n *linear subspaces of* $\mathbf{F}^r$, *enumerate all minimal subsets* X *of* $V = \{1, \ldots, n\}$ *such that* $\mathrm{Span}\langle\bigcup_{i\in X} V_i\rangle = \mathbf{F}^r$.

Generalizing further, consider the polymatroid inequality

$$f(X) = \dim(\bigcup_{i\in X} V_i) \geq t, \tag{24}$$

where $t \in \{1, \ldots, r\}$ is a given threshold. Note that when each subspace V_i is spanned by a subset R_i of vectors from some fixed basis of $\mathbf{F}^r$, the value of $f(X)$ is just the size of $\bigcup_{i\in X} R_i$, and hence (24) includes the class of polymatroid inequalities discussed in Sections 2.1 and 2.2. This shows that generating all maximal infeasible sets $\mathcal{A}_t = \mathcal{A}_t(f)$ for (24) is NP-hard.

Let $\mathcal{B}_t = \mathcal{B}_t(f)$ be the collection of all minimal solutions $X \subseteq V$ to (24). When $t = r$ and the given subspaces V_i are all lines, i.e.,

$$V_i = \mathrm{Span}\langle b_i\rangle \quad \text{for given vectors } b_i \in \mathbf{F}^r, \quad i = 1, \ldots, n,$$

the set $\mathcal{B}_t$ can be identified with the collection of all column bases of the $r \times n$ matrix $[b_1, \ldots, b_n]$. Accordingly, we then have $|\mathcal{A}_r| \leq r|\mathcal{B}_r|$, where $\mathcal{A}_r = \mathcal{I}(\mathcal{B}_r)$ is the collection of all hyperplanes of the vectorial matroid $M = \{b_1, \ldots, b_n\}$. It is an open problem whether there exists an efficient randomized algorithm for nearly uniform sampling from within the set of bases of a given matrix. However, for any matroid M defined by a polynomial-time independence oracle, all bases of M can be enumerated in incremental polynomial time by using depth-first search on $\mathcal{B}_r$ with the obvious adjacency relation: two bases $X, X' \in \mathcal{B}_r$ are adjacent if $|X \cap X'| \geq r - 1$, where r is the rank of M. (All hyperplanes of M can also be enumerated in incremental polynomial time by using the characteristic property of hyperplanes:

(*) If H and H' are two distinct hyperplanes in M and $b \notin H \cup H'$ then there exists a hyperplane H'' such that $b \cup (H \cap H') \subseteq H''$.

If $\mathcal{H} = \{H_1, \ldots, H_k\}$ is an incomplete collection of hyperplanes closed with respect to (*) then $\mathcal{H}$ defines a matroid M' of rank $r' < r$, which can be checked by constructing a base for M', i.e., a minimal transversal to the collection of co-circuits $M \setminus H_1, \ldots, M \setminus H_k$.)

It is easy to show that if the dimension of each input subspace V_i, $i = 1, \ldots, n$, is bounded by some constant d, then

$$|\mathcal{A}_r| \leq rn^{d-1}|\mathcal{B}_r|, \tag{25}$$

i.e., the size of $\mathcal{A}_r$ can still be bounded by a polynomial in n and the size of $\mathcal{B}_r$ (The bound of (25) follows from the fact that each element $X \in \mathcal{A}_r$ can be completely defined by a minimal subset $Y \subseteq X$ that spans the same subspace as X, and that Y can be transformed into an element of $\mathcal{B}_r$ by adding one and then deleting at most $d-1$ of the input subspaces V_i.) Theorems 1 and 3 state that for all $t \in \{1, \ldots, r\}$, the size of $\mathcal{A}_t$ can be bounded by a $\log t$-degree polynomial in n and $|\mathcal{B}_t|$, and that all sets in $\mathcal{B}_t$ can be enumerated in incremental quasi-polynomial time, regardless of the dimensions of the input subspaces V_i.

It is interesting to mention that even though the space covering problem can be solved in incremental quasi-polynomial time for any input subspaces $V_1, \ldots, V_n$, the following close modification of the problem is NP-hard: Enumerate all minimal subsets $X \subseteq V$ such that $\text{Span}\langle \bigcup_{i \in X} V_i \rangle$ contains a given linear subspace V_0. In fact, the above *subspace* covering problem is NP-hard even when V_0 is a line and $\dim(V_i) = 2$ for all $i = 1, \ldots, n$. If all the input subspaces are lines, i.e., $V_i = \text{Span}\langle b_i \rangle$, $i = 0, 1, \ldots, n$, the problem calls for enumerating all minimal dependent sets ($\equiv$ circuits) containing b_0 in the vectorial matroid $M = \{b_0, b_1, \ldots, b_n\}$. The latter problem can be done in incremental polynomial time for any matroid M defined by a polynomial-time independence oracle. (First, we can enumerate all circuits of M in incremental polynomial time because any circuit of M is a co-circuit of the dual matroid M^* and vice versa. Second, assuming without loss of generality that M is connected, by Lehman's theorem ([33], p.74) any circuit in M can be expressed via two circuits containing b_0, and hence (# circuits of M) $\leq$ (# circuits through b_0)^2. More efficient cycle generation algorithm for the special case of graphs can be found in [30].)

It is also worth mentioning the NP-hardness of the *conic* variant of the space covering problem: Given a collection of r-dimensional polyhedral cones K_1, …, K_n defined by their rational generators, enumerate all minimal sets $X \subseteq \{1, \ldots, n\}$ such that $\text{Cone}\langle \bigcup_{i \in X} K_i \rangle$ spans the entire space. The following problem is also NP-hard [22]: Given a rational r-vector b and a collection of n dihedral cones $K_i = \text{Cone}\langle a_i, a_i' \rangle$, $i = 1, \ldots, n$, enumerate all minimal sets $X \subseteq \{1, \ldots, n\}$ for which $b \in \text{Cone}\langle \bigcup_{i \in X} K_i \rangle$. Replacing the input dihedral cones by rays we obtain the *vertex enumeration* problem: Enumerate all vertices of a given polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. The incremental complexity of the vertex enumeration problem is not known. In particular, it is not known whether there exists a quasi-polynomial-time algorithm which, given a rational polytope $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ and a collection Q of the vertices of P, can determine if $P = \text{Conv.hull}\langle Q \rangle$.

2.6 Spanning collections of graphs.

Let R be a finite set of r vertices and let $E_1, \ldots, E_n \subseteq R \times R$ be a collection of n graphs on R. Given a set $X \subseteq \{1, \ldots, n\}$ define $k(X)$ to be the number of connected components in the graph $(R, \bigcup_{i \in X} E_i)$. Then $k(X)$ is an anti-monotone supermodular function and hence for any integral threshold t, the inequality

$$f(X) = r - k(X) \geq t$$

is polymatroid. In particular, $\mathcal{B}_{r-1}$ is the family of all minimal collections of the input graphs $E_1, \ldots, E_n$ which interconnect all vertices in R. (If the n input graphs are just n disjoint edges, then $\mathcal{B}_{r-1}$ is the set of all spanning trees in the graph $E_1 \cup \ldots \cup E_n$.) Since $k(X)$ can be evaluated at any set X in polynomial time, Theorem 3 implies that for each $t \in \{1, \ldots, r\}$, all elements of $\mathcal{B}_t$ can be enumerated in incremental quasi-polynomial time. In particular, given a collection of n *equivalence relations* (partitions) on R, we can enumerate in incremental quasi-polynomial time all minimal subsets of the given relations whose transitive closure puts all elements of R in one equivalence class (or produces at most $r - t$ equivalence classes).

Interestingly, enumerating all minimal collections of $E_1, \ldots, E_n$ connecting two distinguished vertices $s, t \in R$ turns out to be NP-hard even if the input sets $E_1, \ldots, E_n$ are disjoint and contain at most 2 edges each, see [17]. Needless to say that as before, generating all maximal collections of $E_1, \ldots, E_n$ for which the number of connected components of $(R, \bigcup_{i \in X} E_i)$ exceeds a given threshold remains NP-hard.

2.7 Matroid Intersections

Let $M_1, \ldots, M_m$ be m matroids defined on a common ground set V by m polynomial-time independence oracles. Lawler, Lenstra and Rinnooy Kan [23] considered the following *matroid intersection problem*: incrementally generate all maximal subsets $Y \subseteq V$ independent for all matroids $M_1, \ldots, M_m$. Let $\rho_1, \ldots, \rho_m : 2^V \mapsto \{0, 1, \ldots, |V|\}$ be the rank functions of $M_1, \ldots, M_m$, then the matroid intersection problem calls for the enumeration of all maximal feasible solutions to the system of m inequalities

$$\rho_i(Y) \geq |Y|, \quad i = 1, \ldots .m, \quad Y \subseteq V. \tag{26}$$

Letting $X = V \setminus Y$, (26) can equivalently be stated as follows: enumerate all minimal solutions X to the system of polymatroid inequalities

$$f_i(X) \stackrel{\text{def}}{=} |X| + \rho_i(V \setminus X) - \rho_i(V) \geq |V| - \rho_i(V), \quad i = 1, \ldots .m, \quad Y \subseteq V. \tag{27}$$

Since the polymatroid functions $f_i(X)$ can be evaluated at any set X in polynomial time, Theorem 3 implies that the matroid intersection problem can be solved in incremental quasi-polynomial time. This substantially improves the algorithm suggested in [23], whose running time is exponential in m.

Several other examples of polymatroid functions can be found, for instance, in Lovász [24] and Welsh [33]. Let M be a matroid on a ground set U with the rank function $\rho : 2^U \mapsto \{0, 1, \ldots\}$, and let $U_1, \ldots, U_n$ be some subsets of U. For each $X \subseteq V \stackrel{\text{def}}{=} \{1, \ldots, n\}$, let $f(X) = \rho(\bigcup_{i \in X} U_i)$. Then f is a polymatroid function. In fact, every polymatroid function arises by this construction from some matroid, see [18,27], and also [24].

3 A lower bound for $|\mathcal{A}_t|$

In this section we demonstrate that inequality (2) of Theorem 1 is reasonably tight.

3.1 A hypergraph example

In our first example, let $\mathcal{H}$ be the edge set of the graph $G = k \times K_l$ consisting of k pairwise disjoint copies of a clique on l vertices. In this (hyper)graph, the number of vertices is $n = |V| = kl$, the number of (hyper)edges is $|\mathcal{H}| = k\binom{l}{2}$, the number of maximal independent sets is $|\mathcal{I}(\mathcal{H})| = l^k$, and we can prove the following statement.

Lemma 3 *For the hypergraph $\mathcal{H}$ defined above, there exists an exact polymatroid separator (f, t), such that $t = r(f) = 2^k$.*

Proof. For $X \subseteq V$, define $f(X)$ by

$$f(X) = \begin{cases} 2^k, & \text{if } X \text{ contains an edge of } \mathcal{H} \\ 2^k - 2^{\gamma(X)}, & \text{otherwise,} \end{cases}$$

where $\gamma(X)$ denotes the number of l-cliques of G disjoint from X. In particular, $f(X) = 2^k - 1$ if (and only if) X is a maximal independent set of $\mathcal{H}$, and $f(X) = 2^k$ if X contains an edge of $\mathcal{H}$. Let us also note that f is obviously monotone, by the above definition. Thus, with $t = 2^k$ the pair (f, t) is indeed an exact separator of $\mathcal{H}$.

It remains to show that f is submodular. For this, let X and Y be two arbitrary subsets of the vertex set V. If both X and Y contain an edge of $\mathcal{H}$, then (1) holds trivially, since we have 2^{k+1} on the right hand side, and we have $f(Z) \leq 2^k$ for all subsets $Z \subseteq V$ by definition. Furthermore, if one of these sets contains an edge of $\mathcal{H}$, say X, then of course $X \cup Y$ does too, and hence (1) reduces to $f(X \cap Y) \leq f(Y)$ which holds again trivially by the monotonicity of f. Let us assume next that neither X nor Y contain an edge of $\mathcal{H}$, but $X \cup Y$ does. In this case $\gamma(X \cap Y) \geq 1 + \max(\gamma(X), \gamma(Y))$ holds, implying $2^{\gamma(X \cap Y)} \geq 2^{\gamma(X)} + 2^{\gamma(Y)}$, from which (1) follows. Let us assume finally that $X \cup Y$ does not contain an edge of $\mathcal{H}$. In this case $\gamma(X \cap Y) \geq \max(\gamma(X), \gamma(Y))$ and $\gamma(X \cap Y) + \gamma(X \cup Y) \geq \gamma(X) + \gamma(Y)$ both hold, implying $2^{\gamma(X \cap Y)} + 2^{\gamma(X \cup Y)} \geq 2^{\gamma(X)} + 2^{\gamma(Y)}$, from which (1) follows again. $\square$

The above lemma now implies that with $t = r(f) = 2^k$ we have $\beta = |\mathcal{H}| = k\binom{l}{2}$, $\alpha = l^k$ and $n = |V| = kl$. For $l = 2^k$ and $k \to \infty$, we thus obtain $\log n / \log \beta \to 1/2$ and hence $c(n, \beta)$ converges to the root of the equation $2^c(2^{c/2} - 1) = 1$. This gives $c = 1.102...$, and consequently

$$\alpha > \beta^{(.551 \log t)/c(n,\beta)}$$

for k sufficiently large.

3.2 A rank function example

Let us next show that the polymatroid function f defined above can be realized as the rank function of some linear subspaces of the vector space $\mathbf{F}^r$, $r \in \mathbb{Z}_+$ over a (possibly large) field $\mathbf{F}$.

For a positive integer l let $\mathbf{F}$ be a field with $l \leq |\mathbf{F}|$ (we shall use, as customary, $+$ and $\times$ to denote the two field operations, and we write 0 and 1 for the unit elements of these operations, respectively). Furthermore, let $n = kl$, let $G = k \times K_l$ be the graph, as above, and let $\mathcal{H}$ be again the edge set of G. Let us introduce the notations $K = \{1, 2, ..., k\}$ and $L = \{0, 1, ..., l-1\}$, and let us denote the vertex set of G by $V = K \times L$. We shall associate to each vertex $(i, j) \in V$ a linear subspace V_{ij} of $\mathbf{F}^r$, where $r = 2^k$. These subspaces will be chosen in such a way that every two subspaces corresponding to the same clique of G intersect only in the origin (and hence generate the whole space $\mathbf{F}^r$), while the intersection of arbitrary s subspaces ($1 \leq s \leq k$), each corresponding to distinct cliques of G, is of dimension 2^{k-s}.

Let $\{b_x \mid x \in \{0,1\}^k\}$ be an arbitrary basis in $\mathbf{F}^r$, indexed by the $r = 2^k$ elements of the binary cube of dimension k, and let $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 \ldots, \lambda_{l-1}$

be distinct elements of $\mathbf{F}$ (hence the requirement $l \leq |\mathbf{F}|$). For every $z \in L^k$, and every index vector $x \in \{0,1\}^k$, define the product

$$\Lambda_z(x) = \prod_{i:x_i=1} \lambda_{z_i} \prod_{i:x_i=0} (1-\lambda_{z_i}).$$

It is easily verified that $\sum_{x\in\{0,1\}^k} \Lambda_z(x) = 1$ for all $z \in L^k$, and that for any two binary vectors $x, y \in \{0,1\}^k$, we have $\Lambda_x(y) = 1$ if $x = y$, and $\Lambda_x(y) = 0$ otherwise. Let us now associate a (unique) vector

$$b_z = \sum_{x\in\{0,1\}^k} \Lambda_z(x) b_x \tag{28}$$

of $\mathbf{F}^r$ to every $z \in L^k$. Observe that for $z = x \in \{0,1\}^k$, we get a basis element $b_z = b_x$ by our selection of $\lambda_0 = 0, \lambda_1 = 1$.

Let us next define the linear subspace V_{ij}, for $(i,j) \in V$, to be the subspace generated by the vectors $b_z \in \mathbf{F}^r$ whose index vector z has value j in its i-th coordinate:

$$V_{ij} = \langle b_z \mid z_i = j \rangle.$$

We will show below that this construction has the announced properties. To simplify notation, we shall need a few more definitions.

For index vectors $x, y \in L^k$, and a subset $S \subseteq K$, we denote by $x[S]$ the restriction of x to S, by $y[\overline{S}]$ the restriction of y to the complementary set $\overline{S} = K \setminus S$, and by $z = x[S], y[\overline{S}]$ the vector defined by

$$z_j = \begin{cases} x_j \text{ if } j \in S, \\ y_j \text{ if } j \notin S. \end{cases}$$

Let us note that $\Lambda_{a[S],b[\overline{S}]}(x[S], y[\overline{S}]) = \Lambda_{a[S]}(x[S])\Lambda_{b[\overline{S}]}(y[\overline{S}])$ holds by the above definitions, for all $a, b \in L^k$, $x, y \in \{0,1\}^k$ and $S \subseteq K$.

Lemma 4 *For any $S \subseteq K$ and $w = (j_i \mid i \in S) \in L^S$, the set of vectors*

$$\left\{ \sum_{y\in\{0,1\}^S} \Lambda_w(y) b_{y,x} \ : \ x \in \{0,1\}^{\overline{S}} \right\} \tag{29}$$

forms a basis for the vector space $V_{S,w} \stackrel{\text{def}}{=} \langle b_z \mid z[S] = w \rangle$. In particular, $\dim(V_{S,w}) = 2^{k-|S|}$.

Proof. First, let us observe that for each $x \in \{0,1\}^{\overline{S}}$, the vector $a = \sum_{y\in\{0,1\}^S} \Lambda_w(y) b_{y,x}$ lies in the space $V_{S,w}$ since $a = b_z$ with $z[S] = w$ and $z[\overline{S}] = x$. Let us observe next that these vectors are linearly independent: suppose, on the contrary, that there exist scalars $\mu_x \in \mathbf{F}$, $x \in \{0,1\}^{\overline{S}}$, not all zero, such that

$$0 = \sum_{x \in \{0,1\}^{\overline{S}}} \mu_x \left(\sum_{y \in \{0,1\}^S} \Lambda_w(y) b_{y,x} \right).$$

Then, by the linear independence of the basis $\{b_x \mid x \in \{0,1\}^k\}$ of $\mathbf{F}^r$, we obtain that

$$\mu_x \Lambda_w(y) = 0, \quad \text{for all} \quad x \in \{0,1\}^{\overline{S}} \quad \text{and} \quad y \in \{0,1\}^S. \tag{30}$$

But summing up equations (30) for a particular $x \in \{0,1\}^{\overline{S}}$ over all $y \in \{0,1\}^S$, and using $\sum_{y \in \{0,1\}^S} \Lambda_w(y) = 1$, we get $\mu_x = 0$, for all $x \in \{0,1\}^{\overline{S}}$, a contradiction, proving that (29) is indeed a family of linearly independent vectors. Let us note finally that these vectors span the entire subspace $V_{S,w}$, since any vector b_z with $z[S] = w$ in this subspace can be written as:

$$\begin{aligned} b_z &= \textstyle\sum_{u \in \{0,1\}^k} \Lambda_z(u) b_u \\ &= \textstyle\sum_{u \in \{0,1\}^k} \Lambda_{z[\overline{S}]}(u[\overline{S}]) \Lambda_{z[S]}(u[S]) b_u \\ &= \textstyle\sum_{x \in \{0,1\}^{\overline{S}}} \Lambda_{z[\overline{S}]}(x) \left(\sum_{y \in \{0,1\}^S} \Lambda_w(y) b_{y,x} \right). \end{aligned}$$

The lemma follows from the above observations. $\square$

For $z', z'' \in L^k$, let us denote by $[z', z'']$ the set of all those vectors $z \in L^k$ for which $z_i \in \{z'_i, z''_i\}$ for $i = 1, \ldots, k$.

Lemma 5 *Let $z', z'' \in L^k$ be such that $z'_i \neq z''_i$ for all $i \in K$. Then the set $B_{z',z''} \stackrel{\text{def}}{=} \{b_z \mid z \in [z', z'']\}$ forms a basis for $\mathbf{F}^r$.*

Proof. Let $M_{z',z''} \stackrel{\text{def}}{=} (\Lambda_z(x))_{x,z}$ be the $2^k \times 2^k$-matrix whose rows are indexed by the vectors $x \in \{0,1\}^k$, and whose columns are indexed by the vectors $z \in [z', z'']$. To prove that the set $B_{z',z''}$ is linearly independent, it is enough by (28) to show that the matrix $M_{z',z''}$ is non-singular. Indeed, we claim that

$$|\det(M_{z',z''})| = \prod_{i=1}^{k} (\lambda_{z'_i} - \lambda_{z''_i})^{2^{k-1}}, \tag{31}$$

from which the lemma will follow by the distinctness of $\lambda_0, \lambda_1, \ldots, \lambda_{l-1}$. To prove (31), we first observe that the left hand side is a polynomial in $\mathbf{F}[\lambda_{z'_1}, \ldots, \lambda_{z'_k}, \lambda_{z''_1}, \ldots, \lambda_{z'_k}]$, of degree 2^{k-1} in each variable λ_{z_i}. Let $i \in K$ and let $u, v \in L^k$ be such that $u_i = z'_i$, $v_i = z''_i$, and $u[K \setminus \{i\}] = v[K \setminus \{i\}] = w \in L^{K \setminus \{i\}}$. Then for any $x \in \{0,1\}^k$, it is easy to see that $\Lambda_u(x) - \Lambda_v(x) = (-1)^{x_i} (\lambda_{z'_i} - \lambda_{z''_i}) \Lambda_w(x[K \setminus \{i\}])$. In particular, if we subtract the two columns of $M_{z',z''}$

indexed by u, v, we obtain $\lambda_{z'_i} - \lambda_{z''_i}$ as a factor for the determinant expression in (31). Repeating this argument for every $i \in K$ and every $w \in L^{K\setminus\{i\}}$, we conclude that the right hand side of (31) is a divisor of the left hand side. Since both polynomials are of the same degree in all variables by our earlier observation, and since they attain the same value at, say, $z' = (0, \ldots, 0)$, $z'' = (1, \ldots, 1)$, (31) follows. $\square$

Lemma 6 *For all $i \in K$, and for all $j, j' \in L$, $j \neq j'$, the subspaces V_{ij} and $V_{ij'}$ span the entire space $\mathbf{F}^r$, i.e., $\dim(V_{ij} \cup V_{ij'}) = 2^k$.*

Proof. Let $z', z'' \in L^k$ be such that $z'_i \neq z''_i$ for all $i \in K$, $z'_i = j$, and $z''_i = j'$. Since the basis set $B_{z',z''}$ is contained in $V_{ij} \cup V_{ij'}$, the lemma follows. $\square$

Lemma 7 *Let $z', z'' \in L^k$ be such that $z'_i \neq z''_i$ for all $i \in K$. Then for any $i \in K$, we have $V_{i,z'_i} = \langle b_z \mid z \in [z', z''],\ z_i = z'_i \rangle$.*

Proof. From Lemma 4, we have $\dim(V_{i,z'_i}) = 2^{k-1}$, and from Lemma 5, the set $\{b_z \mid z \in [z', z''],\ z_i = z'_i\}$ is linearly independent. Since this set is contained in V_{i,z'_i} by definition, the lemma follows. $\square$

Lemma 8 *For $S \subseteq K$ and $w = (j_i \mid i \in S) \in L^S$, we have*

$$\dim(\bigcup_{i \in S} V_{i,j_i}) = 2^k - 2^{k-|S|}. \tag{32}$$

Proof. Fix $z', z'' \in L^k$ such that $z'_i \neq z''_i$ for all $i \in K$, and $z'[S] = w$, and let $B = B_{z',z''}$ be the basis set defined by these two vectors. For $i \in S$ let $B^i \stackrel{\text{def}}{=} B \cap V_{i,j_i}$, and let $B^{S,w} = \bigcup_{i \in S} B^i$. It is then immediate from the definitions and Lemma 7 that $\bigcup_{i \in S} V_{i,j_i} = \langle B^{S,w} \rangle$, and thus it is enough, by Lemma 5, to count the number of elements in the set $B^{S,w}$. Using the inclusion-exclusion formula, we obtain

$$\dim(\textstyle\bigcup_{i \in S} V_{i,j_i}) = |B^{S,w}| = \sum_{\substack{Q \subseteq S \\ Q \neq \emptyset}} (-1)^{|Q|-1} |\bigcap_{i \in Q} B^i|.$$

Therefore, since $|\bigcap_{i \in Q} B^i| = |\{b_z \in B_{z',z''} \ : \ z[Q] = w[Q]\}| = 2^{k-|Q|}$, we get

$$\dim(\bigcup_{i \in S} V_{i,j_i}) = \sum_{\substack{Q \subseteq S \\ Q \neq \emptyset}} (-1)^{|Q|-1} 2^{k-|Q|} = \sum_{m=1}^{|S|} (-1)^{m-1} \binom{|S|}{m} 2^{k-m} = 2^k - 2^{k-|S|},$$

implying the Lemma. $\square$

We are now ready to verify that our construction indeed has the desired properties. For a subset $X \subseteq V = K \times L$ let us define

$$g(X) = \dim(\bigcup_{(i,j) \in X} V_{ij}),$$

and let us set $t = 2^k$. It follows by Lemma 6 that if X contains an edge of the graph G, then $g(X) = 2^k$, i.e, that $\mathcal{B}_t(g) = \mathcal{H}$. It also follows by Lemma 8 that $g(X) = 2^k - 2^{k-|X|} \leq 2^k - 1$ for any independent set $X \subset V$. i.e., $\mathcal{A}_t(g) = \mathcal{I}(\mathcal{H})$. In other words, g is the same set-function as the function f described in the previous subsection.

4 Proofs

In this section we prove Lemmas 1 and 2, which are the key statements needed to prove our main results.

Proof of Lemma 1. Let us recall that (f, t) is a polymatroid separator of the hypergraph $\mathcal{H}$, separating the maximal independent sets $\mathcal{S} = \mathcal{S}(\mathcal{H}, f, t)$ from $\mathcal{H}$, and that to every node v of $\mathbf{T}$ we have associated an independent set $S_v = \bigcap_{l \in L(\mathbf{T}(v))} I_l$, where $I_l \in \mathcal{S}$ denotes the maximal independent set assigned to the leaf $l \in L(\mathbf{T})$ by the proper assignment ϕ.

To prove the statement of the lemma, we shall show by induction that

$$f(S_w) \leq t - |L(\mathbf{T}(w))| \tag{33}$$

holds for every node w of the binary tree $\mathbf{T}$. Since f is non-negative, it follows that

$$|L(\mathbf{T}(w))| \leq t \leq r(f)$$

which, if applied to the root of $\mathbf{T}$, proves the lemma. To see (33), let us apply induction by the size of $L(\mathbf{T}(w))$. Clearly, if $w = l$ is a leaf of $\mathbf{T}$, then $|L(\mathbf{T}(l))| = 1$, $S_w = I_l \in \mathcal{S}$, and (33) follows by the assumption that (f, t) is separating $\mathcal{H}$ from $\mathcal{S}$. Let us assume now that w is a node of $\mathbf{T}$ with u and v as its immediate successors. Then $|L(\mathbf{T}(w))| = |L(\mathbf{T}(u))| + |L(\mathbf{T}(v))|$, and $S_w = S_u \cap S_v$. By our inductive hypothesis, and since f is submodular, we have the inequalities

$$\begin{aligned} f(S_u \cup S_v) + f(S_w) &\leq f(S_u) + f(S_v) \\ &\leq t - |L(\mathbf{T}(u))| + t - |L(\mathbf{T}(v))| \\ &= 2t - |L(\mathbf{T}(w))|. \end{aligned}$$

Since ϕ is a proper mapping, the set $S_u \cup S_v$ contains a hyperedge $H \in \mathcal{H}$, and thus $f(S_u \cup S_v) \geq f(H) \geq t$ by the monotonicity of f, and by our assumption that (f,t) is a separator for $\mathcal{H}$. Thus, from the above inequality we get $t + f(S_w) \leq f(S_u \cup S_v) + f(S_w) \leq 2t - |L(\mathbf{T}(w))|$, from which (33) follows. $\square$

For a hypergraph $\mathcal{H}$ and a vertex $v \in V = V(\mathcal{H})$ let us denote by $d_{\mathcal{H}}(v)$ the *degree* of vertex v in $\mathcal{H}$, i.e., $d_{\mathcal{H}}(v)$ is the number of hyperedges of $\mathcal{H}$ containing v.

Lemma 9 *For every Sperner hypergraph $\mathcal{H} \subseteq 2^V$ on $n = |V| > 1$ vertices, with $m = |\mathcal{H}| \geq n$ hyperedges, there exists a vertex $v \in V$ for which*

$$m\frac{1}{n} \leq d_{\mathcal{H}}(v) \leq m\left(1 - \frac{1}{n}\right).$$

Proof. Let us define

$$X = \{v \in V \mid d_{\mathcal{H}}(v) < m\frac{1}{n}\}$$

and

$$Y = \{v \in V \mid d_{\mathcal{H}}(v) > m(1 - \frac{1}{n})\},$$

and let us assume indirectly that $X \cup Y = V$ forms a partition of the vertex set.

Let us observe first that $|X| < n$ must hold, since otherwise a contradiction

$$m \leq \sum_{H \in \mathcal{H}} |H| = \sum_{v \in X} d_{\mathcal{H}}(v) < n\frac{m}{n} = m,$$

would follow.

Let us observe next that $|X| > 0$ must hold, since otherwise

$$\sum_{H \in \mathcal{H}} |H| = \sum_{v \in V} d_{\mathcal{H}}(v) = \sum_{v \in Y} d_{\mathcal{H}}(v) > n \times m(1 - \frac{1}{n}) = m(n-1)$$

follows, implying the existence of a hyperedge $H \in \mathcal{H}$ of size $|H| = n$, i.e., $V \in \mathcal{H}$. Since $\mathcal{H}$ is Sperner, $1 = m < n$ would follow, contradicting our assumptions.

Let us observe finally that the number of those hyperedges which avoid some points of Y cannot be more than $|Y|m/n$, and since $|Y| < n$ by our previous observation, there must exist a hyperedge $H \in \mathcal{H}$ containing Y. Thus, all other hyperedges must intersect X, and hence we have

$$m-1 \leq \sum_{H\in\mathcal{H}} |H\cap X| = \sum_{v\in X} d_{\mathcal{H}}(v) < |X|\frac{m}{n} \leq m\frac{n-1}{n}$$

by our first observation. From this $m < n$ would follow, contradicting again our assumption that $m \geq n$. This last contradiction hence proves X and Y cannot cover V, and thus follows the lemma. $\square$

For a subset $X \subseteq V$ let $\mathcal{H}^X \stackrel{\text{def}}{=} \{H \in \mathcal{H} \mid H \supseteq X\}$, and let us simply write $\mathcal{H}^v$ if $X = \{v\}$.

Lemma 10 *Given a hypergraph $\mathcal{H}$ and a subfamily $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ of its maximal independent sets, $|\mathcal{S}| \geq 2$, there exists a hyperedge $H \in \mathcal{H}$ and a vertex $v \in H$ such that*

$$|\mathcal{S}^v| \geq \frac{|\mathcal{S}|}{n} \text{ and } |\mathcal{S}^{H\setminus v}| \geq \frac{|\mathcal{S}|}{n|\mathcal{H}|}.$$

Proof. Let us note first that if $2 \leq |\mathcal{S}| < n$, then the statement is almost trivially true. To see this, let us choose two distinct maximal independent sets S_1 and S_2 from $\mathcal{S}$, and a vertex $v \in S_2 \setminus S_1$. Since $S_1 \cup \{v\}$ is not independent, there exists a hyperedge $H \in \mathcal{H}$ for which $v \in H \cap S_2$ and $H \setminus \{v\} \subseteq S_1$, implying thus that both $|\mathcal{S}^v|$ and $|\mathcal{S}^{H\setminus v}|$ are at least 1, and the right hand sides in the claimed inequalities are not more than 1.

Thus, we can assume in the sequel that $|\mathcal{S}| \geq n$. Let us then apply Lemma 9 for the Sperner hypergraph $\mathcal{S}^c \stackrel{\text{def}}{=} \{V \setminus I \mid I \in \mathcal{S}\}$, and obtain that

$$\frac{|\mathcal{S}|}{n} \leq d_{\mathcal{S}^c}(v) \leq |\mathcal{S}|(1-\frac{1}{n})$$

holds for some $v \in V$, since $|\mathcal{S}| = |\mathcal{S}^c|$ obviously. Thus, from the second inequality we obtain

$$|\mathcal{S}^v| \geq \frac{|\mathcal{S}|}{n}.$$

To see the second inequality of Lemma 10, let us note that members of $\mathcal{S}^c$ are minimal transversals of $\mathcal{H}$, and thus for every $T \in \mathcal{S}^c$, $T \ni v$ there exists a hyperedge $H \in \mathcal{H}$ for which $H \cap T = \{v\}$, by the definition of minimal transversals. Thus,

$$\bigcup_{H\in\mathcal{H}:H\ni v} \{T \in \mathcal{S}^c \mid T \cap H = \{v\}\} \supseteq \{T \in \mathcal{S}^c \mid T \ni v\}$$

holds, from which

$$\sum_{H\in\mathcal{H}:H\ni v} |\mathcal{S}^{H\setminus v}| \geq d_{\mathcal{S}^c}(v) \geq \frac{|\mathcal{S}|}{n}$$

follows. Therefore, since $|\{H \in \mathcal{H} \mid H \ni v\}| = d_{\mathcal{H}}(v) \le |\mathcal{H}|$ holds obviously, there must exist a hyperedge $H \in \mathcal{H}$, $H \ni v$, for which

$$|\mathcal{S}^{H \setminus v}| \ge \frac{|\mathcal{S}|}{n|\mathcal{H}|}$$

holds, implying thus the lemma. $\square$

Proof of Lemma 2. Let us denote by $L(\alpha)$ the maximum number of leaves of a binary tree $\mathbf{T}$ with a proper mapping $\phi : V(\mathbf{T}) \to \mathcal{S}$, where $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ is an arbitrary subfamily of maximal independent sets of $\mathcal{H}$. To simplify notation, let us write $\alpha = |\mathcal{S}|$ and $\beta = |\mathcal{H}|$. To prove the statement, we need to show that

$$L(\alpha) \ge \alpha^{c/\log\beta} \tag{34}$$

where $c = c(n, \beta)$ is as defined in (3).

Let us prove this inequality by induction on α. Clearly, if $\alpha = 1$, then $L(1) = 1$ holds, and we have equality in (34).

Let us assume next that we already have verified the claim for all subfamilies of size smaller than α, and let us consider a subfamily $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ of size $\alpha = |\mathcal{S}|$. According to Lemma 10, we can choose two disjoint subfamilies $\mathcal{S}', \mathcal{S}'' \subseteq \mathcal{S}$ such that $|\mathcal{S}'| \ge \frac{\alpha}{n}$ and $|\mathcal{S}''| \ge \frac{\alpha}{n\beta}$, and such that for any pair of sets $S' \in \mathcal{S}'$ and $S'' \in \mathcal{S}''$ the union $S' \cup S''$ contains a member of $\mathcal{H}$. Thus, building binary trees with proper mappings separately for $\mathcal{S}'$ and $\mathcal{S}''$, and joining them as two siblings of a common root, we obtain a binary tree with a proper mapping for $\mathcal{S}$. Since the right hand side of our claim is a monotone function of α, we can conclude for the number of leaves in the obtained binary tree that

$$L(\alpha) \ge L(\frac{\alpha}{n}) + L(\frac{\alpha}{n\beta}). \tag{35}$$

Applying now our inductive hypothesis, we get

$$\begin{aligned} L(\alpha) &\ge \left(\frac{\alpha}{n}\right)^{c/\log\beta} + \left(\frac{\alpha}{n\beta}\right)^{c/\log\beta} \\ &= \alpha^{c/\log\beta}\left[n^{-c/\log\beta} + (n\beta)^{-c/\log\beta}\right] \\ &= \alpha^{c/\log\beta}, \end{aligned}$$

where the last equality holds by (3). This proves (34), and hence the lemma follows. $\square$

Note that the right hand side of (34) is the least possible solution of the recursion (35).

5 Generating minimal integer solutions of a system of polymatroid inequalities in an integral box

In this section, we discuss a generalization of the previous results which replaces polymatroid set-functions by polymatroid functions defined on integral boxes. Let $\mathcal{C} \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^n \mid \ 0 \le x \le c\}$ be an integral box, where $c \in \mathbb{Z}_+^n$ is a given finite n-vector. A function $f : \mathcal{C} \mapsto \{0, 1, \ldots, r\}$, where $r \in \mathbb{Z}_+$, is said to be submodular if

$$f(x \vee y) + f(x \wedge y) \le f(x) + f(y)$$

holds for all $x, y \in \mathcal{C}$, where $\vee$ and $\wedge$ denote, as usual, the join and meet operators over $\mathcal{C}$:

$$\begin{aligned} x \vee y &= (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\}), \\ x \wedge y &= (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}). \end{aligned}$$

As before, f is said to be monotone if $f(x) \le f(y)$ whenever $x \le y$, and is called polymatroid if it is monotone, submodular, and $f(0) = 0$.

Given a polymatroid function f with range r and an integral threshold $t \in \{1, ..., r\}$ let us denote by $\mathcal{B}_t = \mathcal{B}_t(f)$ the family of all minimal vectors $x \in \mathcal{C}$ for which $f(x) \ge t$, and by $\mathcal{A}_t = \mathcal{A}_t(f)$ the family of all maximal vectors $x \in \mathcal{C}$ for which $f(x) < t$. It follows that $\mathcal{A}_t = \mathcal{I}(\mathcal{B}_t)$, where $\mathcal{I}(\mathcal{B}) \stackrel{\text{def}}{=} \{\text{maximal } x \in \mathcal{C} \mid x \not\ge b, \text{for all } b \in \mathcal{B}\}$ is the family of all maximal independent vectors for $\mathcal{B} \subseteq \mathcal{C}$. As in the Boolean case $\mathcal{C} = \{0, 1\}^n$, we shall use the notation $\alpha = |\mathcal{A}_t|$ and $\beta = |\mathcal{B}_t|$.

Theorems 1 and 2 admit the following generalizations:

Theorem 4 *For every polymatroid function $f : \mathcal{C} \mapsto \{0, 1, \ldots, r\}$ and threshold $t \in \{1, \ldots, r\}$ such that $\beta \ge 2$ we have the inequality*

$$\alpha \le \beta^{(\log t)/c(2n,\beta)}. \tag{36}$$

Let $\mathcal{H} \subseteq \mathcal{C}$ be a set of integral n-vectors, $f : \mathcal{C} \mapsto \{0, 1, \ldots, r\}$ a polymatroid function and $t \in \{1, \ldots, r\}$ a threshold. As before, we say that (f, t) is a separator for $\mathcal{H}$ if $f(x) \ge t$ for all $x \in \mathcal{H}$.

Theorem 5 *Let (f,t) be a polymatroid separator for a set $\mathcal{H}$ of at least two points in $\mathcal{C}$. Then*

$$|\mathcal{A}_t(f) \cap \mathcal{I}(\mathcal{H})| \leq |\mathcal{H}|^{(\log t)/c(2n,|\mathcal{H}|)}. \tag{37}$$

Using the same argument as that preceding Theorem 3, and the fact that the dualization problem on boxes can still be solved in quasi-polynomial time, see [8], we readily arrive at the following generalization of Theorem 3.

Theorem 6 *Consider a system of polymatroid inequalities*

$$f_i(x) \geq t_i, \; i = 1, \ldots, m, \qquad x \in \mathcal{C},$$

in which the right hand sides are bounded by a quasi-polynomial function in the size of the system. Suppose further that this system has a quasi-polynomial time feasibility oracle. Then all minimal feasible solutions for the system can be enumerated in incremental quasi-polynomial time.

Our proof of Theorem 5 makes use of a generalization of Lemma 2. To state this generalization, we need first to extend the notion of proper mappings of maximal independent sets to binary trees. Call a family $\mathcal{A} \subseteq \mathcal{C}$ an antichain if no two elements are comparable in $\mathcal{A}$. Given a binary tree $\mathbf{T}$, an antichain $\mathcal{B} \subseteq \mathcal{C}$, and a collection $\mathcal{A} \subseteq \mathcal{I}(\mathcal{B})$ of maximal independent elements of $\mathcal{B}$, let us consider again mappings $\phi : L(\mathbf{T}) \mapsto \mathcal{I}(\mathcal{B})$ that assign a maximal independent element $a^l \in \mathcal{A}$ to each leaf l of $\mathbf{T}$. To each node v of the tree $\mathbf{T}$, we associate the element $x^v = \bigwedge_{l \in L(\mathbf{T}(v))} a^l$. The mapping ϕ will be called proper if it assigns different independent elements to different leaves, and if the element $x^u \vee x^v$ is not independent whenever u and v are incomparable nodes of $\mathbf{T}$. The latter condition implies that for every pair of incomparable nodes $u, v \in V(\mathbf{T})$, there exists an element $a \in \mathcal{A}$ for which $a \leq x^u \vee x^v$.

Lemma 11 *Let $\mathcal{B} \subseteq \mathcal{C}$ be an antichain of size $|\mathcal{B}| \geq 2$ in an n-dimensional integral box $\mathcal{C}$ and let $\mathcal{A} \subseteq \mathcal{I}(\mathcal{B})$. Then there exists a binary tree $\mathbf{T}$ and a proper mapping $\phi : L(\mathbf{T}) \mapsto \mathcal{A}$, such that*

$$|L(\mathbf{T})| \geq |\mathcal{A}|^{c(2n,|\mathcal{B}|)/\log|\mathcal{B}|}. \tag{38}$$

Proof. By induction on $|\mathcal{A}|$. For $b \in \mathcal{B}$ and $i \in V \stackrel{\text{def}}{=} \{1, \ldots, n\}$, let

$$\begin{aligned} \mathcal{A}_b^i \;\; &\stackrel{\text{def}}{=} \{a \in \mathcal{A} \mid a_i \geq b_i\}, \text{ and} \\ \mathcal{A}_b^{V \setminus i} &\stackrel{\text{def}}{=} \{a \in \mathcal{A} \mid a_j \geq b_j \text{ for all } j \in V \setminus i\}. \end{aligned}$$

We shall make use of the following lemma.

Lemma 12 *For every antichain $\mathcal{B} \subseteq \mathcal{C}$ and every $\mathcal{A} \subseteq \mathcal{I}(\mathcal{B})$, there exist $b \in \mathcal{B}$ and an $i \in V$ such that*

$$|\mathcal{A}_b^i| \geq \frac{|\mathcal{A}|}{2n} \text{ and } |\mathcal{A}_b^{V \setminus i}| \geq \frac{|\mathcal{A}|}{2n|\mathcal{B}|}.$$

Then, to build a binary tree $\mathbf{T}$ on $\mathcal{A}$, let us associate to its root the pair $i \in V$, $b \in \mathcal{B}$ which satisfies the conditions of Lemma 12, and let us use members of $\mathcal{A}_a^i$ to label all left-leaves of $\mathbf{T}$, and use $\mathcal{A}_a^{V \setminus i}$ to label all right-leaves. Clearly, $0 < |\mathcal{A}_b^i| < |\mathcal{A}|$, $0 < |\mathcal{A}_b^{V \setminus i}| < |\mathcal{A}|$, and both are collections of maximal independent elements of $\mathcal{B}$. Therefore, we can conclude by induction that there exist binary trees $\mathbf{T}_1$ and $\mathbf{T}_2$ of sufficiently large number of leaves, and proper mappings $\phi_1 : L(\mathbf{T}_1) \mapsto \mathcal{A}_b^i$ and $\phi_2 : L(\mathbf{T}_2) \mapsto \mathcal{A}_b^{V \setminus i}$ such that

$$\bigwedge_{l \in L(\mathbf{T}_1)} a_i^l \geq b_i \qquad \text{and} \qquad \bigwedge_{l \in L(\mathbf{T}_2)} a^l[V \setminus i] \geq b[V \setminus i],$$

and thus the join of these two elements is at least b. The lemma then follows by the argument used in the proof of Lemma 2. $\square$

For $\mathcal{A} \subseteq \mathcal{C}$ and $x \in \mathbb{Z}$, let us define

$$d_{\mathcal{A}}(x) \stackrel{\text{def}}{=} |\{a \in \mathcal{A} \mid a_i \geq x\}|.$$

Lemma 13 *Let $\mathcal{A} \subseteq \mathcal{C}$ be an antichain in an n-dimensional integral box $\mathcal{C}$, and let $\epsilon > 0$ be a given constant. If $n > 1$ and $m = |\mathcal{A}| \geq 1 + 1/\epsilon$, then there exists an $x \in \mathbb{Z}$ such that $0 \leq x \leq \max\{c_1, \ldots, c_n\}$ and*

$$m \frac{1}{(1+\epsilon)n} \leq d_{\mathcal{A}}(x) \leq m \left(1 - \frac{1}{(1+\epsilon)n}\right).$$

Proof. Let us define the vector $y \in \mathbb{C}$ by setting

$$y_i = \max\{x \in \mathbb{Z} \mid 0 \leq x \leq c_i,\ d_{\mathcal{A}}(x) > m(1 - \frac{1}{(1+\epsilon)n})\},$$

for $i = 1, \ldots, n$, and let $Y = \{i \in V \mid y_i = c_i\}$. Assume indirectly that for every $i \in V$, and every $x \in \mathbb{Z}$ such that $y_i < x \leq c_i$, we have $d_{\mathcal{A}}(x) < m/(1+\epsilon)n$. Let us observe first that $Y \neq V$ must hold, since otherwise

$$\begin{aligned} |\bigcup_{i=1}^n \{a \in \mathcal{A} \mid a_i < y_i\}| &\leq \sum_{i=1}^n |\{a \in \mathcal{A} \mid a_i < y_i\}| \\ &= \sum_{i=1}^n (m - d_{\mathcal{A}}(y_i)) < n \frac{m}{(1+\epsilon)n} < m, \end{aligned}$$

implying that there exists an $a^o \in \mathcal{A}$ such that $a^o \geq y = c$. But this would imply that $m = 1$, since $\mathcal{A}$ is an antichain, contradicting our assumptions. Now since there is an $a^o \in \mathcal{A}$ such that $a^o \geq y$ and $\mathcal{A}$ is an antichain, it follows that, for all other elements $a \in \mathcal{A} \setminus \{a^o\}$, there must exist an $i \in V$ for which $a_i > y_i$. Consequently,

$$m - 1 \leq \sum_{a \in \mathcal{A}} |\{i \in V \mid a_i > y_i\}| = \sum_{i \notin Y} d_{\mathcal{A}}(y_i + 1) < n \frac{m}{(1+\epsilon)n},$$

and therefore we get $m < 1 + 1/\epsilon$, again a contradiction. $\square$

Proof of Lemma 12. Assume without loss of generality that $|\mathcal{A}| \geq 2$. Let us apply Lemma 13 with $\epsilon = 1$ for the antichain $\mathcal{A} \subseteq \mathcal{I}(\mathcal{B})$, and obtain an $i \in V$ and an $0 \leq x \leq c_i$ such that

$$|\mathcal{A}| \frac{1}{2n} \leq d_{\mathcal{A}}(x) \leq |\mathcal{A}| \left(1 - \frac{1}{2n}\right).$$

Let us next note that for every $a \in \mathcal{A}$ for which $a_i < x$, there exists an element $b \in \mathcal{B}$ such that $b_i \leq x$, and $b_j \leq a_j$ for all $j \neq i$, by the maximality of the independent element a. Thus,

$$\frac{|\mathcal{A}|}{2n} \leq |\mathcal{A}| - d_{\mathcal{A}}(x) \leq |\bigcup_{b \in \mathcal{B}} \{a \in \mathcal{A} \mid a_i < x, a_j \geq b_j \text{ for all } j \neq i\}|$$

holds, from which we conclude that there must exist a $b \in \mathcal{B}$ such that

$$|\{a \in \mathcal{A} \mid a_i < x, a_j \geq b_j \text{ for all } j \neq i\}| \geq \frac{|\mathcal{A}|}{2n|\mathcal{B}|}.$$

This gives immediately $|\mathcal{A}_b^{V \setminus i}| \geq |\mathcal{A}|/(2n|\mathcal{B}|)$. On the other hand, since $b_i \leq x$ and $d_{\mathcal{A}}(x) \geq |\mathcal{A}|/(2n)$, we obtain $|\mathcal{A}_b^i| = d_{\mathcal{A}}(b_i) \geq |\mathcal{A}|/(2n)$, and the first inequality of the lemma follows. $\square$

We mention in closing that Theorem 2 can also be generalized for polymatroid functions $f : \mathcal{L}_1 \times \cdots \times \mathcal{L}_n \mapsto \{0, 1, \ldots\}$, where $\mathcal{L}_1, \cdots, \mathcal{L}_n$ are arbitrary lattices (i.e. partially ordered sets with meet $\wedge$ and join $\vee$ operations). Let (f, t) be a polymatroid separator for a set $\mathcal{X} \subseteq \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$ of size $|\mathcal{X}| \geq 2$, then it can be shown that

$$|\mathcal{A}_t(f) \cap \mathcal{I}(\mathcal{X})| \leq |\mathcal{X}|^{(\log t)/c(2Q, |\mathcal{X}|)}, \tag{39}$$

where $Q = \sum_{i=1}^n |\mathcal{L}_i|$. In particular, $|\mathcal{A}_t(f)| \leq \max(Q, \mathcal{B}_t(f)^{(\log t)/c(2Q, \mathcal{B}_t(f))})$. Note, however, that for the case where each lattice $\mathcal{L}_i$ is a chain, (39) is weaker than (37).

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