# Weighted Transversals of a Hypergraph

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**Abstract:** We consider a generalization of the notion of transversal to a finite hypergraph, so called *weighted transversals*. Given a non-negative weight vector assigned to each hyperedge of the input hypergraph, we define a weighted transversal as a minimal vertex set which intersects a collection of hyperedges of sufficiently large total weight. We show that the hypergraph of all weighted transversals is dual-bounded, i.e, the size of its dual hypergraph is polynomial in the number of weighted transversals and the size of the input hypergraph. Our bounds are based on new inequalities of extremal set theory and threshold Boolean logic, which may be of independent interest. We also prove that the problem of generating all weighted transversals for a given hypergraph is polynomial-time reducible to the well-known hypergraph dualization problem. As a corollary, we obtain an incremental quasi-polynomial-time algorithm for generating all weighted transversals for a given hypergraph. This result includes as special case the generation of all the minimal Boolean solutions to a given system of non-negative linear inequalities.

**Keywords: Boolean dualization, hypergraph transversals**

## 1 Introduction

Given a finite set $V$ of $n = |V|$ points, and a hypergraph (set family) $\mathcal{A} \subseteq 2^V$, a subset $B \subseteq V$ is called a *transversal* of the family $\mathcal{A}$ if $B \cap A \neq \emptyset$ for all sets $A \in \mathcal{A}$; it is called a *minimal*

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*transversal* if no proper subset of $B$ is a transversal of $\mathcal{A}$. The hypergraph $\mathcal{A}^d$ consisting of all minimal transversals of $\mathcal{A}$ is called the *dual* (or *transversal*) hypergraph of $\mathcal{A}$. It is easy to see that dropping non-minimal sets from $\mathcal{A}$ will not change $\mathcal{A}^d$. We can assume therefore that the hypergraph $\mathcal{A}$ is Sperner ($\mathcal{A}^d$ is Sperner by definition). If $\mathcal{A}$ is Sperner, then $(\mathcal{A}^d)^d = \mathcal{A}$.

The problem of generating the transversal hypergraph $\mathcal{A}^d$ to a given Sperner hypergraph $\mathcal{A}$, known as *dualization*, has applications in combinatorics [28], graph theory [19, 20, 29, 31], artificial intelligence [13], game theory [15, 16, 27], reliability theory [10, 27], database theory [1, 24, 32] and learning theory [2]. This problem can be stated as follows:

**DUAL**$(\mathcal{A}, \mathcal{B})$**:** Given a complete list of all hyperedges of $\mathcal{A}$ and a set of minimal transversals $\mathcal{B} \subseteq \mathcal{A}^d$, either prove that $\mathcal{B} = \mathcal{A}^d$, or find a new transversal $X \in \mathcal{A}^d \setminus \mathcal{B}$.

Clearly, we can generate all of the hyperedges of $\mathcal{A}^d$ by initializing $\mathcal{B} = \emptyset$ and iteratively solving the above problem $|\mathcal{A}^d| + 1$ times. Note also that in general, $|\mathcal{A}^d|$ can be exponentially large both in $|\mathcal{A}|$ and $|V|$. For this reason, the complexity of dualization is customarily measured in the input and output sizes. In particular, $\mathcal{A}^d$ can be generated in *incremental polynomial time* if problem DUAL$(\mathcal{A}, \mathcal{B})$ can be solved in time polynomial in $|V|, |\mathcal{A}|$ and $|\mathcal{B}|$.

The dualization problem is one of the intriguing open problems, the true complexity of which is not (yet) known. It can be efficiently solved for several special classes of hypergraphs. For example, if the sizes of all the hyperedges of $\mathcal{A}$ are limited by a constant $c$, then problem DUAL$(\mathcal{A}, \mathcal{B})$ can be solved in polynomial time (see e.g. [5, 13]). In addition, for $c = 2$ there are dualization algorithms that run with polynomial delay, i.e. in $poly(|V|, |\mathcal{A}|)$ time for a specific sequence $\emptyset \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots \subset \mathcal{A}^d$, see e.g. [19, 20, 31]). Efficient algorithms exist also for the dualization of 2-monotonic, threshold, matroid, read-bounded, acyclic and some other classes of hypergraphs (see e.g. [3, 9, 11, 12, 22, 23, 25, 26]). Even though no incremental polynomial time algorithm for the dualization of arbitrary hypergraphs is known, an incremental *quasi-polynomial time* one exists (see [14]). This algorithm solves the dualization problem in $O(nm) + m^{o(\log m)}$ time, where $n = |V|$ and $m = |\mathcal{A}| + |\mathcal{B}|$ (see also [18] for more detail). In fact, the algorithm of [14] can solve problem DUAL$(\mathcal{A}, \mathcal{B})$ in $poly(n, m)$ time on $m^{o(\log m)}$ parallel processors. In particular, this means that problem DUAL$(\mathcal{A}, \mathcal{B})$ cannot be NP-hard unless any NP-complete problem of size $s$ can be solved in $poly(s)$ time on $s^{o(\log s)}$ processors.

In [6, 8] we considered *partial* and *multiple* transversals of a hypergraph, and showed that the generation of those are equivalent with dualization. Improving on a combinatorial inequality instrumental in those proofs, we are further generalizing those results in this paper.

We shall consider a natural generalization of minimal transversals, called *weighted transversals*. Given a (not necessarily Sperner) hypergraph $\mathcal{A} \subseteq 2^V$, a non-negative $r$-dimensional weight vector $w(A) \in \mathbb{R}^r_+$ associated with every hyperedge $A \in \mathcal{A}$, and a threshold $r$-vector $t$, a vertex set $X$ is called a *$w, t$-transversal* if $X$ intersects all the hyperedges of $\mathcal{A}$, except for a sub-family of total weight at most $t$:

$$\sum \{w(A) \mid A \in \mathcal{A},\ A \cap X = \emptyset\} \le t. \tag{1}$$

We call minimal $w, t$-transversals *weighted* transversals and let $\mathcal{A}_{w,t}$ denote the set of all weighted transversals for $\mathcal{A}$. Note that for the special case of $r = 1$, $t = 0$, and $w(A) = 1$ for all $A \in \mathcal{A}$, we have $\mathcal{A}_{w,t} = \mathcal{A}^d$, i.e., the set of weighted transversals turns into the transversal hypergraph for $\mathcal{A}$. Let us now consider the problem of generating all weighted transversals for a given hypergraph:

**GEN**$(\mathcal{A}_{w,t}, \mathcal{B})$**:** Given a complete list of all hyperedges of $\mathcal{A}$ along with their weights $w(A) \in \mathbb{R}^r_+$, a threshold vector $t \in \mathbb{R}^r_+$ and a set $\mathcal{B}$ of weighted transversals $\mathcal{B} \subseteq \mathcal{A}_{w,t}$, either prove that $\mathcal{B} = \mathcal{A}_{w,t}$, or find a new transversal $X \in \mathcal{A}_{w,t} \setminus \mathcal{B}$.

As mentioned above, problem GEN$(\mathcal{A}_{w,t}, \mathcal{B})$ includes DUAL$(\mathcal{A}, \mathcal{B})$ as a special case. In fact, these two problems turn out to be polynomially equivalent. In particular, this shows that problem GEN$(\mathcal{A}_{w,t}, \mathcal{B})$ can be solved in quasi-polynomial time.

**Theorem 1** *Problem GEN$(\mathcal{A}_{w,t}, \mathcal{B})$ is polytime reducible to the hypergraph dualization problem.*

Weighted transversals generalize both partial and multiple transversals introduced in [6, 8]. Similarly to those cases, the above equivalence of GEN$(\mathcal{A}_{w,t}, \mathcal{B})$ and DUAL$(\mathcal{A}, \mathcal{B})$ is based on the fact that the hypergraph of weighted transversals is *uniformly dual-bounded*: for any sub-family $\mathcal{H} \subseteq \mathcal{A}_{w,t}$, the number of minimal transversals to $\mathcal{A}_{w,t}$ and $\mathcal{H}$ can be bounded by a polynomial in the size of $\mathcal{H}$ and the length of the input description of $\mathcal{A}_{w,t}$.

**Theorem 2** *Let $\mathcal{H} \subseteq \mathcal{A}_{w,t}$ be an arbitrary non-empty sub-hypergraph of the family of all weighted transversals to a hypergraph $\mathcal{A}$, where $w : \mathcal{A} \to \mathbb{R}^r_+$, and $t \in \mathbb{R}^r_+$. Then we have*

$$|(\mathcal{A}_{w,t})^d \cap \mathcal{H}^d| \leq r \sum_{H \in \mathcal{H}} |\{A \in \mathcal{A} \mid A \cap H \neq \emptyset\}|. \tag{2}$$

*In particular, we have $|(\mathcal{A}_{w,t})^d \cap \mathcal{H}^d| \leq rm|\mathcal{H}|$, where $m = |\mathcal{A}|$. Thus, for the special case of $\mathcal{H} = \mathcal{A}_{w,t} \neq \emptyset$ we obtain $|(\mathcal{A}_{w,t})^d| \leq rm|\mathcal{A}_{w,t}|$.*

Theorem 2 implies that the method of jointly generating the elements of $\mathcal{A}_{w,t}$ and its dual $\mathcal{A}^d_{w,t}$ (see [4, 6, 17]) is equally efficient to generate $\mathcal{A}_{w,t}$, since we will not waste too much time generating the dual according to (2), and hence Theorem 1 will follow.

As we mentioned earlier, the notion of weighted transversals generalize both partial and multiple transversals introduced in [6, 8]. In Section 2 we briefly discuss maximal infeasible and minimal feasible solutions to systems of non-negative linear inequalities in binary and integer variables, which includes multiple transversals (for a more complete analysis see [7]). Section 3 contains two lemmas which present combinatorial inequalities instrumental in the proof of Theorem 2. These two lemmas and Theorem 2 are then proved in Sections 4, 5, and 6, respectively. Finally, Section 7 contains some concluding remarks.

# 2 Maximal infeasible and minimal feasible solutions to systems of non-negative linear inequalities

Consider a system of $r$ linear inequalities in $n$ binary variables

$$Ax \geq b, \quad x \in \{0,1\}^n \tag{3}$$

where $A$ is a given non-negative $r \times n$ -matrix and $b$ is a given $r$-vector. Let $V = \{1, \ldots, n\}$ be the set of column indices of $A$, and let $\mathcal{A}$ be the hypergraph consisting of $m = n$ singletons $A_1 = \{1\}, \ldots, A_n = \{n\}$. Each column of the $r \times n$ matrix $A$ can be interpreted as a non-negative $r$-dimensional weight vector associated with the corresponding hyperedge of $\mathcal{A} = \{\{1\}, \ldots, \{n\}\}$.

Let also $t = Ae - b$, where $e \in \mathbb{R}^n$ is the vector of all ones. Then the characteristic vector of each minimal $w, t$-transversal is a minimal binary solution to (3) and vice versa, the supporting set of any minimal binary solution to (3) is a minimal $w, t$-transversal. It is also easy to see that under this interpretation, the anti-characteristic vector of any set in $(\mathcal{A}_{w,t})^d$ is a maximal infeasible binary vector for (3) and conversely, the complement to the supporting set of any maximal infeasible binary vector for (3) yields a hyperedge of $(\mathcal{A}_{w,t})^d$. From (2) we conclude that for any feasible system (3),

$$\#\ maximal\ infeasible\ x \quad \leq \sum \{p(x) \mid x\ minimal\ feasible\ \}, \tag{4}$$

where $p(x)$ is number of positive components of $x$. In particular,

$$\#\ maximal\ infeasible\ vectors \leq\ rn[\#\ minimal\ feasible\ vectors]. \tag{5}$$

In fact, it can be shown that the above inequalities also hold for *integer* $x$.

**Theorem 3** *Inequalities (4) and (5) hold for any feasible system*

$$Ax \geq b, \quad 0 \leq x \leq c, \quad x\ integer,$$

*where $A$ is a non-negative $r \times n$-matrix, $b$ is an $r$-vector, and $c$ is a non-negative $n$-vector some or all of whose components may be infinite.*

Note that for $c = e$ we can obtain the binary case (3). Note also that inequalities (4) and (5) are sharp for $r = 1$, e.g., for $A = e^T$, $c \geq e$ and $b = 1$. For large $r$, these inequalities are again accurate up to a factor polylogarithmic in $r$. To see this, we can use the same example as in Section 2. Let $n = 2k$ and consider the system of $r = 2^k$ inequalities of the form $x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1$, $i_1 \in \{1, 2\}$, $i_2 \in \{3, 4\}, \ldots, i_k \in \{2k-1, 2k\}$. This system has $2^k$ maximal infeasible binary vectors and only $k$ minimal feasible binary vectors, all of which have 2 positive components. Hence

$$\#\ maximal\ infeasible\ vectors\ = \frac{r}{2\log r} \sum_{x\ min\ feasible} p(x) = \frac{rn}{2(\log r)^2}[\#\ minimal\ feasible\ vectors].$$

# 3 Auxiliary inequalities

Our proof of Theorem 2 makes use of two lemmas which we state in this section. The first lemma is an intersection inequality for two set families $\mathcal{S} = \{S_1, \ldots, S_\alpha\}$ and $\mathcal{T} = \{T_1, \ldots, T_\beta\}$ on a common ground set $U$, say $U = \{1, \ldots, m\}$. We say that $\mathcal{T}$ *covers all pairwise intersections of* $\mathcal{S}$ if $\alpha \geq 2$ and for any $1 \leq i < j \leq \alpha$ there is an index $k \in \{1, \ldots, \beta\}$ such that $S_i \cap S_j \subseteq T_k$. We also say that $\mathcal{S}$ and $\mathcal{T}$ are *threshold separable* if there is a non-negative weight-function $w : U \to \mathbb{R}_+$ and a real $t$ such that $w(T_k) \leq t$ for all $k \in \{1, \ldots, \beta\}$ and $w(S_i) > t$ for all $i \in \{1, \ldots, \alpha\}$, where the weight of a set $X \subseteq U$ is defined in the usual way: $w(\emptyset) = 0$ and $w(X) = \sum\{w_u \mid u \in X\}$ for $X \neq \emptyset$.

**Lemma 4** *Let $\mathcal{S} = \{S_1, \ldots, S_\alpha\}$ and $\mathcal{T} = \{T_1, \ldots, T_\beta\}$ be threshold separable families of subsets of $U = \{1, \ldots, m\}$ such that $\mathcal{T}$ covers all pairwise intersections of $\mathcal{S}$. Then $\alpha \leq \sum_{k=1}^{\beta} |U \setminus T_k|$. In particular, $\alpha \leq m\beta$.*

We note that the threshold separability of $\mathcal{S}$ and $\mathcal{T}$ is essential for the validity of the lemma. In particular, there are examples [8] of Sperner set families $\mathcal{S} = \{S_1, \ldots, S_\alpha\}$ and $\mathcal{T} = \{T_1, \ldots, T_\beta\}$ such that $\mathcal{T}$ covers all pairwise intersections of $\mathcal{S}$ but $\alpha$ is exponentially larger than $\beta$. Let us also remark that Lemma 4 improves on the quadratic inequality shown as Lemma 3 in [8].

To state our second lemma, we need additional notation. For a monotone Boolean function $h : 2^V \to \{0,1\}$, let us denote by $\max[h]$ the family of its maximal "false sets", i.e. $\max[h] = \{X \subseteq V \mid X \textit{ maximal subset such that } h(X) = 0\}$. Analogously, let $\min[h]$ denote the family of its minimal "true sets", i.e. $\min[h] = \{X \subseteq V \mid X \textit{ minimal subset such that } h(X) = 1\}$. Finally, given a hypergraph $\mathcal{A} \subseteq 2^V$, a non-negative scalar weight function $w : \mathcal{A} \to \mathbb{R}$, and a threshold $t \in \mathbb{R}$, we denote by $g_{\mathcal{A},w,t} : 2^V \to \{0,1\}$ the characteristic function of the family of $w,t$-transversals to $\mathcal{A}$, i.e., $g_{\mathcal{A},w,t}(X) = 1$ if and only if $X$ is a $w,t$-transversal of $\mathcal{A}$. By definition, $g_{\mathcal{A},w,t}$ is a monotone Boolean function.

**Lemma 5** *Let $\mathcal{A}$ be a hypergraph on $|V| = n$ vertices, $w(A)$ be a non-negative scalar weight associated to each hyperedge $A \in \mathcal{A}$, and let $t \geq 0$ be a given real. Let us consider an arbitrary monotone Boolean function $h : 2^V \to \{0,1\}$ such that $h \not\equiv 0$ and $g_{\mathcal{A},w,t}(X) \geq h(X)$ for all $X \subseteq V$. Then, we have*

$$|\max[h] \;\cap\; \{X \mid g_{\mathcal{A},w,t}(X) = 0\}| \;\leq \sum_{X \in \min[h]} |\{A \in \mathcal{A} \mid X \cap A \neq \emptyset\}|. \tag{6}$$

Note that if $\mathcal{A}$ consists of $n$ singletons $\{1\}, \ldots, \{n\}$, then $g_{\mathcal{A},w,t} = g_{w,t}$ is a threshold function and we obtain the following inequalities [8]:

$$|\max[h] \cap \{X \mid g_{w,t} \leq t\}| \leq \sum \{|X| : X \in \min[h]\} \leq n|\min[h]|.$$

If the function $h$ is also threshold and $h \equiv g_{w,t}$, then $|\max[h]| \leq n|\min[h]|$ and, by symmetry for threshold functions, $|\min[h]| \leq n|\max[h]$. The latter two inequalities are well known [3, 11, 25, 26].

# 4 Proof of Lemma 4

First of all, since $\mathcal{T}$ covers all pairwise intersection of $\mathcal{S}$, any inclusion of the form $S_i \subseteq S_j$, $i \neq j$ implies $S_i = S_i \cap S_j \subseteq T_k$ for some $k \in \{1, \ldots, \beta\}$. However, $S_i \subseteq T_k$ contradicts the threshold separability of $\mathcal{T}$ and $\mathcal{S}$. We can therefore assume without loss of generality that $\mathcal{S} = \{S_1, \ldots, S_\alpha\}$ is a Sperner hypergraph. In particular, all the sets in $\mathcal{S}$ are distinct. Without loss of generality we can also assume that $\mathcal{T}$ is Sperner as well, for otherwise we can replace $\mathcal{T}$ by the family of all maximal subsets of $\mathcal{T}$.

We first prove the lemma for $\alpha = 2$. Assume without loss of generality that $\beta = 1$ and $S_1 \cap S_2 \subseteq T_1$. If $|T_1| \geq |U| - 1$, then the separability of $\mathcal{S}$ and $\mathcal{T}$ implies $S_1 = S_2 = U$, contradiction. If $|T_1| \leq |U| - 2$, we have $\alpha = 2 \leq |U \setminus T_1|$ and the lemma follows.

We assume henceforth that $\alpha \geq 3$ and prove the lemma by induction on $|U| = m$. Clearly, the lemma holds for $m = 1$ because there does not exist a Sperner hypergraph with one vertex and $\alpha \geq 2$ hyperedges. Let $m \geq 2$.

For $u \in U$, let $\alpha_u$ (respectively $\beta_u$) denote the number of hyperedges in $\mathcal{S}$ (respectively $\mathcal{T}$) containing $u$. We split the inductive proof of the lemma into three cases.

*Case 1:* $\beta_u = \beta$ for some $u \in U$. In this case let us define $U' = U \setminus \{u\}$, $\mathcal{T}' = \{T_1 \setminus \{u\}, \ldots, T_\beta \setminus \{u\}\}$ and $\mathcal{S}' = \{S_1 \setminus \{u\}, \ldots, S_\alpha \setminus \{u\}\}$. Then, the hypergraphs $\mathcal{S}', \mathcal{T}' \subseteq 2^{U'}$ can be separated by the original weight function restricted to $U'$ if we use the threshold value $t' = t - w_u$. By the inductive hypothesis, this gives $\alpha \le \sum_{k=1}^{\beta} |U' \setminus T'_k| = \sum_{k=1}^{\beta} |U \setminus T_k|$.
*Case 2:* $\beta_u < \beta$ for all $u \in U$, and $\alpha_v \le 1$ for some $v \in U$, i.e. there is at most one hyperedge in $\mathcal{S}$ which contains vertex $v$, and no vertex of $U$ is contained in all hyperedges in $\mathcal{T}$. If $\alpha_v = 1$, assume without loss of generality that vertex $v$ belongs to $S_\alpha$. Regardless of whether $\alpha_v = 1$ or $\alpha_v = 0$, define $U'' = U \setminus \{v\}$, $\mathcal{S}'' = \{S_1, \ldots, S_{\alpha-1}\}$, $\mathcal{T}'' = \{T_1 \setminus \{v\}, \ldots, T_\beta \setminus \{v\}\}$. Since $\alpha - 1 \ge 2$, hypergraphs $\mathcal{S}''$ and $\mathcal{T}''$ satisfy the assumptions of the lemma with the original weight function $w$ restricted to $U''$ and with the original threshold value $t$. By induction, we have

$$\alpha - 1 \le \sum_{k=1}^{\beta} |U'' \setminus T''_k| = \sum_{k=1}^{\beta} |U \setminus T_k| - (\beta - \beta_v) \le \sum_{k=1}^{\beta} |U \setminus T_k| - 1,$$

since $\beta > \beta_v$ holds in this case, and thus the lemma follows.
*Case 3:* $\alpha_u \ge 2$ for all $u \in U$, i.e. each vertex $u \in U$ is covered by at least two sets in $\mathcal{S}$. Let $U^{[u]} = U \setminus \{u\}$, $\mathcal{S}^{[u]} = \{S_i \setminus \{u\} \mid u \in S_i,\ i \in \{1, \ldots, \alpha\}\}$, and $\mathcal{T}^{[u]} = \{T_k \setminus \{u\} \mid u \in T_k,\ k \in \{1, \ldots, \beta\}\}$. Since $|\mathcal{S}^{[u]}| = \alpha_u \ge 2$ and hypergraphs $\mathcal{S}^{[u]}$, $\mathcal{T}^{[u]} \subseteq 2^{U^{[u]}}$ satisfy the assumptions of the lemma with the restriction of $w$ to $U^{[u]}$ and $t^{[u]} = t - w_u$, we can apply the inductive hypothesis to get

$$\alpha_u \le \sum_{k:\ u \in T_k} |U^{[u]} \setminus T_k^{[u]}| = \sum_{k:\ u \in T_k} |U \setminus T_k|, \qquad u = 1, \ldots, m.$$

By multiplying the above inequalities by the non-negative weights $w_u$ and summing up the resulting bounds for all $u$, we obtain:

$$\sum_{u=1}^{m} w_u \alpha_u \le \sum_{u=1}^{m} w_u \sum_{k:\ u \in T_k} |U \setminus T_k|.$$

From the separability of $\mathcal{S}$ and $\mathcal{T}$ it follows that

$$t\alpha < \sum_{i=1}^{\alpha} w(S_i) = \sum_{u=1}^{m} w_u \alpha_u \quad \text{and} \quad \sum_{u=1}^{m} w_u \sum_{k:\ u \in T_k} |U \setminus T_k| = \sum_{k=1}^{\beta} w(T_k) |U \setminus T_k| \le t \sum_{k=1}^{\beta} |U \setminus T_k|.$$

Hence $t\alpha < t \sum_{k=1}^{\beta} |U \setminus T_k|$. This completes the proof of the lemma, since we can assume without loss of generality that $t > 0$. $\square$

## 5 Proof of Lemma 5

Denote $\max[h] \cap \{X \mid g_{\mathcal{A},w,t}(X) = 0\}$ by $\mathcal{X}$ and let $\mathcal{X} = \{X_1, \ldots, X_\alpha\}$. Let also $\min[h] = \mathcal{Y} = \{Y_1, \ldots, Y_\beta\}$. Since $h \not\equiv 0$, there is at least one minimal true set for $h$, i.e, $\beta \ge 1$.

Consider an arbitrary set $X_i$ in $\mathcal{X}$. Since $g_{\mathcal{A},w,t}(X_i) = 0$, this set cannot contain a $w, t$-transversal to $\mathcal{A}$, i.e., $\sum\{w(A) \mid X_i \cap A = \emptyset,\ A \in \mathcal{A}\} > t$. Equivalently, we have

$$\sum\{w(A) \mid A \subseteq X_i^c,\ A \in \mathcal{A}\} > t, \quad i = 1, \ldots, \alpha, \tag{7}$$

where $X^c = V \setminus X$ is the complements of $X$.

On the other hand, $h(Y_k) = 1$ for any set $Y_k \in \min[h]$. Since $g_{\mathcal{A},w,t}(X) \geq h(X)$ for all $X$, we conclude that $g_{\mathcal{A},w,t}(Y_k) = 1$ for any $k \in \{1, \ldots, \beta\}$. By the definition of $g_{\mathcal{A},w,t}$, this means that each set $Y_k$ contains a $w, t$-transversal to $\mathcal{A}$, i.e, $\sum\{w(A) \mid Y_k \cap A = \emptyset,\ A \in \mathcal{A}\} \leq t$. Equivalently,

$$\sum\{w(A) \mid A \subseteq Y_k^c,\quad A \in \mathcal{A}\} \leq t, \qquad k = 1, \ldots, \beta. \tag{8}$$

Given a set $X \subseteq V$, let $\phi(X) = \{A \mid A \subseteq X,\ A \in \mathcal{A}\}$ denote the set of those hyperedges of $\mathcal{A}$ which are contained in $X$. Clearly, $\phi$ is a monotonic mapping, i.e., $X \subseteq X' \subseteq V \quad \Rightarrow \phi(X) \subseteq \phi(X')$. In addition, for any sets $X, X' \subseteq V$ we have the identity

$$\phi(X) \cap \phi(X') \equiv \phi(X \cap X'). \tag{9}$$

Denoting the number of hyperedges in $\mathcal{A}$ by $m$, we can view any set $\phi(X)$ as a subset of $U = \{1, \ldots, m\}$. Consider the set families $\mathcal{S} = \{\phi(X_1^c), \ldots, \phi(X_\alpha^c)\}$ and $\mathcal{T} = \{\phi(Y_1^c), \ldots, \phi(Y_\beta^c)\}$. Inequalities (7) and (8) imply that $\mathcal{S}$ and $\mathcal{T}$ are threshold separable: $w(\phi(X_k^c)) > t$ for $k = 1, \ldots, \alpha$ and $w(\phi(Y_k^c)) \leq t$ for $k = 1, \ldots, \beta$. We now split the proof into two cases.

*Case 1:* $\alpha \leq 1$. As mentioned above, $\min[h]$ contains at least one set, say $Y_1$. If $Y_1$ intersects at least one hyperedge of $\mathcal{A}$, we obtain (6) and the lemma follows. Otherwise $Y_1$ is disjoint from all hyperedges of $\mathcal{A}$ and (8) implies $\sum\{w(A) \mid A \subseteq Y_1^c,\ A \in \mathcal{A}\} = \sum\{w(A) \mid A \in A\} \leq t$. This contradicts (7) unless $\mathcal{S}$ is an empty family, i.e. $\alpha = 0$.

*Case 2:* $\alpha \geq 2$. Let us show that $\mathcal{T}$ covers all pairwise intersections of $\mathcal{S}$. Let $X_i, X_j,\ \ 1 \leq i < j \leq \alpha$, be two distinct sets in $\mathcal{X}$. Since $X_i, X_j \in \max[h]$ are *maximal* false sets for $h$, we have $h(X_i \cup X_j) = 1$. Consequently, there is a minimal true point $Y_k \in \min[h]$ such that $Y_k \subseteq X_i \cup X_k$. Equivalently, we can write $X_i^c \cap X_j^c \subseteq Y_k^c$. Hence $\phi(X_i^c \cap X_j^c) \subseteq \phi(Y_k^c)$ by the monotonicity of $\phi$. In view of (9) we now obtain $\phi(X_i^c) \cap \phi(X_j^c) \subseteq \phi(Y_k^c)$, i.e., the intersection of any two distinct sets in $\mathcal{S}$ can be covered by a set in $\mathcal{T}$. We have thus shown that $\mathcal{S}$ and $\mathcal{T}$ satisfy the assumptions of Lemma 4. Hence $\alpha \leq \sum_{k=1}^{\beta}(|\mathcal{A}| - |\phi(Y_k^c)|)$. However, $|\mathcal{A}| - |\phi(Y^c)|$ is exactly the number of hyperedges in $\mathcal{A}$ which have a non-empty intersection with $Y$, and (6) follows. □

## 6 Proof of Theorem 2

Let $\mathcal{H}$ be an arbitrary non-empty sub-hypergraph of $\mathcal{A}_{w,t}$, and let $h : 2^V \to \{0, 1\}$ denote the monotone Boolean function defined by the condition $h(X) = 1 \quad \Leftrightarrow \quad H \subseteq X$ for some $H \in \mathcal{H}$. Note that $h \not\equiv 0$ because $\mathcal{H} \neq \emptyset$. Furthermore, $\mathcal{H} = \min[h]$ because $\mathcal{H}$ and $\mathcal{A}_{w,t}$ are Sperner hypergraphs.

Next, for each component $t_\rho$ of the threshold vector $t = (t_1, \ldots, t_r)$ and for each component $w_\rho$ of the weight function $w = (w_1, \ldots, w_r) : \mathcal{A} \to \mathbb{R}_+^r$ let $\mathcal{A}_{w_\rho,t_\rho}$ denote the hypergraph of all minimal $w_\rho, t_\rho$-transversals of $\mathcal{A}$, and let $g_\rho = g_{\mathcal{A},w_\rho,t_\rho}$ be the associated monotone Boolean function, i.e. $g_\rho(X) = 1$ if and only if $X$ is a $w_\rho, t_\rho$- transversal of $\mathcal{A}$. Consider an arbitrary set $X \subseteq V$ such that $h(X) = 1$. Since $X$ contains a $w, t$-transversal, we have $\sum\{w_\rho(A) \mid A \in \mathcal{A},\ A \cap X = \emptyset\} \leq t_\rho$ for all $\rho = 1, \ldots, r$. Consequently, $X$ contains a $w_\rho, t_\rho$-transversal for each $\rho$. This shows that

$$g_\rho(X) \geq h(X) \text{ for all } X \subseteq V \text{ and } \rho = 1, \ldots, r.$$

Let us now consider an arbitrary set $X \in \mathcal{H}^d$. Clearly, we have $h(X^c) = 0$ because none of the sets $H \in \mathcal{H}$ can be contained in $X^c$. Thus, $X^c$ is a false set for $h$. In fact, it is easy to see that $X^c$ is a maximal false set for $h$, i.e., $h(X^c \cup \{u\}) = 1$ for any vertex $u \notin X^c$. This is because $h(X^c \cup \{u\}) = 0$ would imply that $X \setminus \{u\}$ intersects each hyperedge $H \in \mathcal{H}$ in contradiction with the fact that $X \in \mathcal{H}^d$ is a minimal transversal to $\mathcal{H}$. We have thus shown that $X \in \mathcal{H}^d \;\;\Rightarrow\;\; X^c \in max[h]$. Suppose that we also have $X \in (\mathcal{A}_{w,t})^d$. Then $X^c$ contains no $w,t$-transversal, and consequently $\sum\{w_\rho(A) \mid A \in \mathcal{A},\ A \cap X^c = \emptyset\} > t_\rho$ for some $\rho \in \{1,\ldots,r\}$. This means that $X^c$ cannot contain a $w_\rho, t_\rho$ transversal and therefore $g_\rho(X^c) = 0$. Hence

$$X \in (\mathcal{A}_{w,t})^d \cap \mathcal{H}^d \quad \Rightarrow \quad X^c \in \bigcup_{\rho=1}^{r} \mathcal{X}_\rho,$$

where $\mathcal{X}_\rho \stackrel{\text{def}}{=} \max[h] \cap \{X \mid g_\rho(X) = 0\}$, $\rho = 1,\ldots,r$. In particular, we have

$$|(\mathcal{A}_{w,t})^d \cap \mathcal{H}^d| \le \sum_{\rho=1}^{r} |\max[h] \cap \{X \mid g_\rho(X) = 0\}|. \tag{10}$$

It remains to apply Lemma 5 to each pair of functions $h$ and $g_\rho$ to obtain

$$|\max[h] \cap \{X \mid g_\rho(X) = 0\}| \le \sum_{X \in \min[h]} |\{A \in \mathcal{A} \mid A \cap X \neq \emptyset\}|. \tag{11}$$

Since $\min[h] = \mathcal{H}$, (10) and (11) yield (2). ☐

## 7 NP-hard modifications

As the above results indicate, generating weighted transversals is polytime reducible to dualization and thus can be executed in e.g. incremental quasi-polynomial time. However, even minor modifications in the definition of weighted transversals may lead to NP-hard problems.

**Proposition 6** *[7] Given a hypergraph $\mathcal{A} \subseteq 2^V$ and a threshold $t \in \{0,1,\ldots,|\mathcal{A}|-1\}$, consider the problems of generating the following sets:*
*1) All minimal families of hyperedges whose union contains at least $t$ vertices;*
*2) All minimal vertex sets containing at least $t$ hyperedges;*
*3) All minimal families of hyperedges whose union contains at least $t$ hyperedges;*
*4) All minimal vertex sets $X \subseteq V$ such that $|X| \ge t$, and $X$ is the union of some hyperedges;*
*5) All maximal families of hyperedges whose union contains at most $t$ vertices;*
*6) All maximal vertex sets containing at most $t$ hyperedges;*
*7) All maximal vertex sets $X$ such that $X$ contains at most $t$ hyperedges and $X$ is the union of those hyperedges;*
*8) All maximal families of hyperedges whose union contains at most $t$ hyperedges;*
*9) All maximal subsets $X \subseteq V$ such that $|X| \le t$, and $X$ is the union of some hyperedges;*
*10) All maximal vertex sets contained in the union of (at most) $t$ hyperedges;*
*11) All minimal vertex set not contained in the union of (at most) $t$ hyperedges.*
*Problems (1) and (6) are polynomially reducible to dualization, whereas all of the remaining problems are NP-hard.*

Let us add that if we "delete the words" MAXimal and MINimal in the formulations of problems 1-11 above, then all these problems become solvable in incremental polynomial time.

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