# Matroid Intersections, Polymatroid Inequalities, and Related Problems *

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**Abstract.** Given $m$ matroids $M_1, \ldots, M_m$ on the common ground set $V$, it is shown that all maximal subsets of $V$, independent in the $m$ matroids, can be generated in quasi-polynomial time. More generally, given a system of polymatroid inequalities $f_1(X) \geq t_1, \ldots, f_m(X) \geq t_m$ with quasi-polynomially bounded right hand sides $t_1, \ldots, t_m$, all minimal feasible solutions $X \subseteq V$ to the system can be generated in incremental quasi-polynomial time. Our proof of these results is based on a combinatorial inequality for polymatroid functions which may be of independent interest. Precisely, for a polymatroid function $f$ and an integer threshold $t \geq 1$, let $\alpha = \alpha(f,t)$ denote the number of maximal sets $X \subseteq V$ satisfying $f(X) < t$, let $\beta = \beta(f,t)$ be the number of minimal sets $X \subseteq V$ for which $f(X) \geq t$, and let $n = |V|$. We show that $\alpha \leq \max\{n, \beta^{(\log t)/c}\}$, where $c = c(n,\beta)$ is the unique positive root of the equation $2^c(n^{c/\log\beta} - 1) = 1$. In particular, our bound implies that $\alpha \leq (n\beta)^{\log t}$. We also give examples of polymatroid functions with arbitrarily large $t, n, \alpha$ and $\beta$ for which $\alpha = \beta^{(1-o(1))\log t/c}$.

## 1 Introduction

Given $m$ matroids $M_1, \ldots, M_m$ on the common ground set $V$ of cardinality $n$, Lawler, Lenstra and Rinnooy Kan [14] in 1980 asked the question of the complexity of generating all maximal sets independent in all the matroids, and gave an exponential-time algorithm whose running time is $O(n^{m+2})$ per each generated maximal independent set. This matroid intersection problem has interesting applications in a variety of fields including combinatorial optimization [13, 19] and symbolic analysis of electrical circuits [10]. In this paper, we show that all maximal sets independent in $m$ matroids can be generated in incremental quasi-polynomial time. More precisely, assume that each matroid $M_i$ is described by an independence oracle, i.e., an algorithm that, given a set $X \subseteq V$, determines whether or not $X$ is independent in $M_i$.

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**Theorem 1.** *Let $M_1, \ldots, M_m$ be $m$ matroids on the common ground set $V$, $|V| = n$, and let $\mathcal{F} \subseteq 2^V$ be the family of all maximal sets independent in all the matroids. Given a partial list $\mathcal{H} \subseteq \mathcal{F}$, either a new element in $\mathcal{F} \setminus \mathcal{H}$ can be computed, or $\mathcal{F} = \mathcal{H}$ can be recognized, in $k^{o(\log k)}$ time and poly$(k)$ calls to the independence oracles, where $k \stackrel{\text{def}}{=} \max\{m, n, |\mathcal{H}|\}$.*

In fact, we shall consider a wider class of problems of which matroid intersection is a special case. Let $V$ be a finite set of cardinality $|V| = n$, let $f : 2^V \mapsto \mathbb{Z}_+$ be a set-function taking non-negative integral values, and let $r = r(f)$ denote the *range* of $f$, i.e., $r(f) = \max\{f(X) \mid X \subseteq V\}$. The set-function $f$ is called *monotone* if $f(X) \leq f(Y)$ whenever $X \subseteq Y$, and *submodular* if

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$$

holds for all subsets $X, Y \subseteq V$. Finally, $f$ is called a *polymatroid function* if it is monotone, submodular and $f(\emptyset) = 0$. Given a system of *polymatroid* inequalities:

$$f_i(X) \geq t_i, \quad i = 1, \ldots, m, \tag{1}$$

where each of the polymatroid functions $f_i : 2^V \mapsto \mathbb{Z}_+$ is defined via an *evaluation oracle*, and $t_1, \ldots, t_m$ are given positive integral thresholds, let $\mathcal{A}$ and $\mathcal{B}$, respectively, denote the family of all maximal infeasible and minimal feasible sets for (1). It is easy to see that $\mathcal{A} = \mathcal{I}(\mathcal{B})$, where $\mathcal{I}(\cdot)$ denotes the family of all maximal independent sets for the hypergraph $(\cdot)$. Consider the following problem:

*GEN*$(\mathcal{B}, \mathcal{H})$: *Given a system of polymatroid inequalities (1) and a collection $\mathcal{H} \subseteq \mathcal{B}$ of minimal feasible sets for (1), either find a new minimal feasible set $H \in \mathcal{B} \setminus \mathcal{H}$ for (1), or show that $\mathcal{H} = \mathcal{B}$.*

Clearly, the matroid intersection problem can be described as a system of polymatroid inequalities (1). [Indeed, let $\rho_i : 2^V \mapsto \{0, 1, \ldots, n\}$ be the rank function of $M_i$. Then the rank function of the dual matroid

$$f_i(X) = \rho_i(V \setminus X) + |X| - \rho_i(V) : 2^V \mapsto \{0, 1, \ldots, n\}$$

is a polymatroid function. Furthermore, a set $X \subseteq V$ is independent in $M_i$ if and only if $f_i(V \setminus X) \geq n - \rho_i(V)$. Letting $\mathcal{B} \stackrel{\text{def}}{=} \{V \setminus X \mid X \in \mathcal{F}\}$, we conclude, therefore, that $\mathcal{B}$ is the family of minimal solutions for the system of polymatroid inequalities $f_i(X) \geq n - \rho_i(V), \;\; i = 1, \ldots, m$.]

The main result of this paper, Theorem 2 below, generalizes Theorem 1 to systems of polymatroid inequalities (1). Let, as before, $\mathcal{B}$ denote the family of all minimal solutions to (1). A generation algorithm for $\mathcal{B}$ is said to run in *incremental quasi-polynomial* time if it can solve problem *GEN*$(\mathcal{B}, \mathcal{H})$ in $2^{\text{polylog} k}$ operations and calls to the evaluation oracles for $f_1, \ldots, f_m$, where $k = \max\{m, n, |\mathcal{H}|\}$.

**Theorem 2.** *Consider a system of polymatroid inequalities (1) in which the right-hand sides are bounded by a quasi-polynomial in the dimension of the system:*

$$\max\{t_1, \ldots, t_m\} \leq 2^{\mathrm{polylog}(nm)}.$$

*Then all minimal solutions to (1) can be generated in incremental quasi-polynomial time.*

Theorem 2 can be complemented with the following negative result.

**Proposition 1.** *There exist polymatroid inequalities $f(X) \geq t$, with polynomial-time computable left-hand side, for which problem $GEN(\mathcal{B}, \mathcal{H})$ is NP-hard for exponentially large $t$.*

The paper is organized as follows. In section 2, we present a combinatorial inequality bounding the number of maximal infeasible sets $\mathcal{A}$ by a quasi-polynomial in $n$ and the number of minimal feasible sets $\mathcal{B}$. We then use this inequality in Section 4 to reduce problem $GEN(\mathcal{B}, \mathcal{H})$, in quasi-polynomial time, to the well-known hypergraph dualization problem, that is, the generation of all maximal independent sets of an explicitly given hypergraph. Since the hypergraph dualization problem can be solved in incremental quasi-polynomial time [9], this will prove Theorem 2 and allow for the efficiently incremental solution of a number of applications, in addition to matroid intersections. Some of these applications are briefly discussed in Section 3, including the generation of minimal feasible solutions to a system of non-negative linear inequalities in Boolean variables (integer programming), minimal infrequent sets of a database (data mining), minimal connectivity ensuring collections of subgraphs from a given list (reliability theory), and minimal spanning collections of subspaces from a given list (linear algebra). The proof of the polymatroid inequality will be given in Sections 5 and 6.

## 2 An inequality for polymatroid functions

Given a polymatroid function $f : 2^V \mapsto \{0, 1, \ldots, r\}$ and an integral threshold $t \in \{1, ..., r\}$, let us denote by $\mathcal{B}_t = \mathcal{B}_t(f)$ the family of all minimal subsets $X \subseteq V$ for which $f(X) \geq t$, and analogously, let us denote by $\mathcal{A}_t = \mathcal{A}_t(f)$ the family of all maximal subsets $X \subseteq V$ for which $f(X) < t$. Throughout the paper we shall use the notation $\alpha = |\mathcal{A}_t(f)|$ and $\beta = |\mathcal{B}_t(f)|$.

**Theorem 3.** *For every polymatroid function $f$ and threshold $t \in \{1, \ldots, r(f)\}$ such that $\beta \geq 2$ we have the inequality*

$$\alpha \leq \beta^{(\log t)/c(n,\beta)}, \tag{2}$$

*where $c(n, \beta)$ is the unique positive root of the equation*[1]

$$2^c(n^{c/\log \beta} - 1) = 1. \tag{3}$$

*In addition, $\alpha \leq n$ holds if $\beta = 1$.*

[1] All logarithms in this paper are assumed to have base 2

Let us first remark that by (3), $1 = n^{-c/\log\beta} + (n\beta)^{-c/\log\beta} \geq 2(n\beta)^{-c/\log\beta}$, and hence $\beta^{1/c(n,\beta)} \leq n\beta$. Consequently, for $\beta \geq 2$ (in which case $n \geq 2$ is implied, too) we can replace (2) by the simpler but weaker inequality

$$\alpha \leq (n\beta)^{\log t}. \tag{4}$$

In fact, (4) holds even in case of $\beta = 1$, because if the hypergraph $\mathcal{B}_t$ consists only of a single hyperedge $X \subseteq V$, then $|\mathcal{A}_t| \leq |X| \leq n$ follows immediately by the relation $\mathcal{A}_t = \mathcal{I}(\mathcal{B}_t)$. On the other hand, for large $\beta$ the bound of Theorem 3 becomes increasingly stronger than (4). For instance, $c(n,n) = \log(1+\sqrt{5})-1 > .694$, $c(n,n^2) > 1.102$, and $c(n,n^\sigma) \sim \log\sigma$ for large $\sigma$.

Let us remark next that the bound of Theorem 3 is reasonably sharp. For instance, let $k$, $l$, and $d$ be positive integers, let $V = V_1 \cup \ldots \cup V_k$ be the disjoint union of $k$ sets of $l$ vertices each, and for $X \subseteq V$, define $f(X) = d^k$ if $|X \cap V_j| \geq d$ for some $j \in \{1,\ldots,k\}$, and $f(X) = d^k - \prod_{i=1}^k (d - |X \cap V_i|)$ otherwise. Then $f$ is a polymatroid function of range $r = d^k$ for which $n = kl$, $|\mathcal{A}_r| = \binom{l}{d-1}^k$, and $|\mathcal{B}_r| = k\binom{l}{d}$. Thus, letting $t = r$, $d = k$, and $l = 2^k$, we obtain an infinite family of polymatroid functions for which $c(n,\beta) = (1-o(1))\log k$ and

$$\alpha = \beta^{(1-o(1))\log t/c(n,\beta)},$$

as $k \to \infty$.

Let us finally note that for many classes of polymatroid functions, $\beta$ cannot be bounded by a quasi-polynomial estimate of the form $(n\alpha)^{\mathrm{poly}\log r}$. Let us consider for instance, a graph $G = t \times K_2$ consisting of $t$ disjoint edges, and let $f(X)$ be the number of edges $X$ intersects, for $X \subseteq V(G)$. Then $f$ is a polymatroid function of range $r = t$, and we have $n = 2t$, $\alpha = |\mathcal{A}_t| = t$ and $\beta = |\mathcal{B}_t| = 2^t$.

Given a non-empty hypergraph $\mathcal{H}$ on the vertex set $V$, a polymatroid function $f : 2^V \mapsto \mathbb{Z}_+$, and a integral positive threshold $t$, the pair $(f,t)$ is called a *polymatroid separator* for $\mathcal{H}$ if $f(H) \geq t$ for all $H \in \mathcal{H}$. We can further strengthen Theorem 3 as follows.

**Theorem 4.** *Let $(f,t)$ be a polymatroid separator for a hypergraph $\mathcal{H}$ of cardinality $|\mathcal{H}| \geq 2$. Then*

$$|\mathcal{A}_t(f) \cap \mathcal{I}(\mathcal{H})| \leq |\mathcal{H}|^{(\log t)/c(n,|\mathcal{H}|)}, \tag{5}$$

*where $\mathcal{I}(\mathcal{H})$ is the family of all maximal independent sets for $\mathcal{H}$.*

Clearly, Theorem 3 is a special case of Theorem 4 for $\mathcal{H} = \mathcal{B}_t(f)$. Since the right-hand side of (5) monotonically increases with $|\mathcal{H}|$, we can assume without loss of generality that $\mathcal{H}$ is Sperner, i.e., none of the hyperedges of $\mathcal{H}$ contains another hyperedge of $\mathcal{H}$.

## 3 Applications

Before proving Theorems 2 and 4, let us consider first some applications.

*Monotone systems of linear inequalities in binary and integer variables:* Consider a system $Ax \geq b$ of $m$ linear inequalities in $n$ Boolean variables, where $A$ is a given non-negative integer $m \times n$-matrix and $b$ is given integer $m$-vector. Since a linear inequality with non-negative integer coefficients is clearly polymatroid, all minimal Boolean vectors $x$ feasible for the system can be generated in quasi-polynomial time by Theorem 2, provided that the right-hand side $b$ is bounded by a quasi-polynomial in $n$ and $m$. In fact, for linear systems the latter condition can be dropped and the bound of Theorem 3 can be strengthened to a linear bound valid even for real $A$ and $b$ and integer $x$ in an arbitrary box $0 \leq x \leq c$. This gives an incremental quasi-polynomial algorithm for enumerating minimal solutions to an arbitrary nonnegative system of linear inequalities in Boolean or integer variables (see [4, 6] for more details). Thus knapsack, generalized knapsack, and set covering problems are all included as special cases. The quasi-polynomial generation of all maximal feasible solutions to a generalized knapsack problem improves on known results, since for instance Lawler, Lenstra and Rinnooy Kan [14] conjectured that the generation of the maximal binary feasible solutions of a generalized knapsack problem cannot be done in incremental polynomial time, unless $P = NP$.

*Minimal infrequent sets for a database:* Given a hypergraph $\mathcal{H} \subseteq 2^V$ (or equivalently, a database with binary attributes), and an integer threshold $t$, a set $X \subseteq V$ is called *t-frequent* if it is contained in at least $t$ hyperedges of $\mathcal{H}$, and is called *t-infrequent* otherwise. The generation of maximal frequent and minimal infrequent sets for are important tasks in knowledge discovery and data mining applications (see, for instance, [1, 2, 18]). Since the function $f(X) \stackrel{\text{def}}{=} |\{H \in \mathcal{H} \mid H \not\supseteq X\}|$ is polymatroid of range $|\mathcal{H}|$, Theorems 3 and 2 imply respectively that the number of maximal frequent sets can be bounded by a quasi-polynomial in the number of minimal infrequent sets and the sizes of $V, \mathcal{H}$, and that the minimal infrequent sets can be generated in quasi-polynomial time. In fact, the bound of Theorem 4 can be strengthened to a sharp linear bound in this case, see [7].

*Connectivity ensuring collections of subgraphs:* Let $R$ be a finite set of $r$ vertices and let $E_1, \ldots, E_n \subseteq R \times R$ be a collection of $n$ graphs on $R$. Given a set $X \subseteq \{1, \ldots, n\}$ define $k(X)$ to be the number of connected components in the graph $(R, \bigcup_{i \in X} E_i)$. Then $k(X)$ is an anti-monotone supermodular function and hence for any integral threshold $t$, the inequality $f(X) = r - k(X) \geq t$ is polymatroid. In particular, $\mathcal{B}_{r-1}(f)$ is the family of all minimal collections of the input graphs $E_1, \ldots, E_n$ which interconnect all vertices in $R$. (If the $n$ input graphs are just $n$ disjoint edges, then $\mathcal{B}_{r-1}$ is the set of all spanning trees in the graph $E_1 \cup \ldots \cup E_n$, see [17].) Since $k(X)$ can be evaluated at any set $X$ in polynomial time, Theorem 2 implies that for each $t \in \{1, \ldots, r\}$, all elements of

$\mathcal{B}_t$ can be enumerated in incremental quasi-polynomial time. This problem has applications in reliability theory [8, 16].

*Spanning a linear space by linear subspaces:* Given a collection $\mathcal{V} = \{\mathcal{V}_1, \ldots, \mathcal{V}_n\}$ of $n$ linear subspaces of $\mathbf{F}^r$, for some field $\mathbf{F}$, consider the problem of enumerating all minimal sub-collections $X$ of $V = \{1, \ldots, n\}$ such that $\text{Span}\langle\bigcup_{i\in X} \mathcal{V}_i\rangle = \mathbf{F}^r$. More generally, consider the polymatroid inequality

$$f(X) = \dim(\bigcup_{i\in X} \mathcal{V}_i) \geq t, \tag{6}$$

where $t \in \{1, \ldots, r\}$ is a given threshold. Then the set $\mathcal{B}_t(f)$ of minimal solutions to (6) is the collection of all minimal subsets of $\mathcal{V}$ the dimension of whose union is at least $t$. Theorem 4 then states that for all $t \in \{1, \ldots, r\}$, the size of $\mathcal{A}_t(f)$ can be bounded by a $\log t$-degree polynomial in $n$ and $|\mathcal{B}_t(f)|$, and thus all sets in $\mathcal{B}_t(f)$ can be enumerated in incremental quasi-polynomial time.

It is worth mentioning that in all of the above examples, generating all maximal infeasible sets for (1) turns out to be NP-hard, see [7, 11, 15].

## 4 Proof of Theorem 2

In this Section we show that Theorem 2 follows from Theorem 4.

Let $\mathcal{B}$ be the set of minimal feasible sets for (1). Clearly, we can incrementally generate all sets in $\mathcal{B}$ by initializing $\mathcal{H} = \emptyset$ and then iteratively solving problem $GEN(\mathcal{B}, \mathcal{H})$ a number of $|\mathcal{B}| + 1$ times. It is easy to see that the first minimal feasible set $H \in \mathcal{B}$ can be found (or $\mathcal{B} = \emptyset$ can be recognized) by evaluating (1) $n + 1$-times. Furthermore, since $\mathcal{I}(\{H\}) = \{V \setminus \{x\} \mid x \in H\}$, the second minimal feasible set can also be identified (or $\mathcal{B} = \{H\}$ can be recognized) in another $n + |H|$ evaluations of (1). Thus, in what follows we can assume without loss of generality that the current set $\mathcal{H} \subseteq \mathcal{B}$ of minimal solutions to (1) has cardinality of at least 2.

By definition, each pair $(f_i, t_i)$ is a polymatroid separator for $\mathcal{H}$, and therefore Theorem 4 implies the inequalities

$$|\mathcal{A}_{t_i}(f_i) \cap \mathcal{I}(\mathcal{H})| \leq |\mathcal{H}|^{(\log t_i)/c(n,|\mathcal{H}|)}, \quad i = 1, \ldots, m.$$

Let $\mathcal{A} = \mathcal{I}(\mathcal{B})$ be the hypergraph of all maximal infeasible sets for (1), then $\mathcal{A} \subseteq \bigcup_{i=1}^{m} \mathcal{A}_{t_i}(f_i)$. Hence we arrive at the following bound:

$$|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})| \leq m|\mathcal{H}|^{(\log t)/c(n,|\mathcal{H}|)},$$

where $t = \max\{t_1, \ldots, t_m\}$. Now, since $t_1, \ldots, t_m$ are bounded by a quasi-polynomial in $n$ and $m$, we conclude that

$$|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})| \leq 2^{\text{polylog} k} \quad \text{where } k = \max\{n, m, |\mathcal{H}|\}. \tag{7}$$

By definition, the family $\mathcal{B} \subseteq 2^V$ of all minimal feasible sets for (1) is a Sperner hypergraph. Furthermore, the hypergraph $\mathcal{B}$ has a simple superset oracle: given a set $X \subseteq V$, we can determine whether or not $X$ contains some set $H \in \mathcal{B}$ by checking the feasibility of $X$ for (1), i.e., by evaluating $f_1(X), \ldots, f_m(X)$. As observed in [3, 11], for any Sperner hypergraph $\mathcal{B}$ defined via a superset oracle, problem $GEN(\mathcal{B}, \mathcal{H})$ reduces in quasi-polynomial time to $|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})|$ instances of the *hypergraph dualization problem*: *Given two explicitly listed Sperner families* $\mathcal{H} \subseteq 2^V$ *and* $\mathcal{G} \subseteq \mathcal{I}(\mathcal{H})$, *either find a new maximal independent set* $X \in \mathcal{I}(\mathcal{H}) \setminus \mathcal{G}$ *or show that* $\mathcal{G} = \mathcal{I}(\mathcal{H})$. (To see this reduction, consider an arbitrary hypergraph $\mathcal{H} \subseteq \mathcal{B}$. Start generating maximal independent sets for $\mathcal{H}$ checking, for each generated set $X \in \mathcal{I}(\mathcal{H})$, whether or not $X$ is feasible for (1). If $X$ is feasible for (1) then $X$ contains a new minimal solution to (1) which can be found by querying the superset oracle at most $|X|+1$ times. If $X \in \mathcal{I}(\mathcal{H})$ is infeasible for (1), then it is easy to see that $X \in \mathcal{I}(\mathcal{B})$, and hence the number of such infeasible sets $X$ is bounded by $|\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{H})|$.)

Combining the above reduction with (7) and the fact that the hypergraph dualization problem can be solved in quasi-polynomial time $poly(n) + (|\mathcal{H}| + |\mathcal{G}|)^{o(\log(|\mathcal{H}|+|\mathcal{G}|)}$ (see [9]), we readily obtain Theorem 2.

## 5 Proper mappings of independent sets into binary trees

Our proof of Theorem 4 makes use of a combinatorial construction which may be of independent interest. Theorem 4 states that for any polymatroid separator $(f, t)$ of a hypergraph $\mathcal{H}$ we have

$$r(f) \geq t \geq |\mathcal{S}|^{c(n,|\mathcal{H}|)/\log(|\mathcal{H}|)},$$

where $\mathcal{S} = \mathcal{I}(\mathcal{H}) \cap \{X \mid f(X) < t\}$, i.e., the range of $f$ must increase with the size of $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$. Thus, to prove the theorem we must first find ways to provide lower bounds on the range of a polymatroid function. To this end we shall show that the number of independent sets which can be organized in a special way into a binary tree structure provides such a lower bound.

Let $\mathbf{T}$ denote a rooted binary tree, $V(\mathbf{T})$ denote its node set, and let $L(\mathbf{T})$ denote the set of its leaves. For every node $v \in V(\mathbf{T})$, let $\mathbf{T}(v)$ be the binary sub-tree rooted at $v$. Obviously, for every two nodes $u, v$ of $\mathbf{T}$ either the sub-trees $\mathbf{T}(u)$ and $\mathbf{T}(v)$ are disjoint, or one of them is a sub-tree of the other. The nodes $u$ and $v$ are called *incomparable* in the first case, and *comparable* in the second case.

Given a Sperner hypergraph $\mathcal{H}$ and a binary tree $\mathbf{T}$, let us consider mappings $\phi : L(\mathbf{T}) \mapsto \mathcal{I}(\mathcal{H})$ assigning maximal independent sets $I_l \in \mathcal{I}(\mathcal{H})$ to the leaves $l \in L(\mathbf{T})$. Let us associate furthermore to every node $v \in V(\mathbf{T})$ the intersection $S_v = \bigcap_{l \in L(\mathbf{T}(v))} I_l$. Let us call finally the mapping $\phi$ *proper* if it is injective, i.e., assigns different independent sets to different leaves, and if the sets $S_u \cup S_v$ are not independent whenever $u$ and $v$ are incomparable nodes of $\mathbf{T}$. Let us point out that the latter condition means that the set $S_u \cup S_v$, for incomparable nodes

$u$ and $v$, must contain a hyperedge $H \in \mathcal{H}$, as a subset. Since the intersection of independent sets is always independent, it follows, in particular that both $S_v$ and $S_u$ are non-empty independent sets (otherwise their union could not be non-independent.) Finally, since all non-root nodes $u \in V(\mathbf{T})$ have at least one incomparable node $v \in V(\mathbf{T})$, we conclude that the sets $S_u$ are non-empty and independent, for all non-root nodes $u$.

**Lemma 1.** *Let us consider a Sperner hypergraph $\mathcal{H}$ and a polymatroid separator $(f,t)$ of it, and let us denote by $\mathcal{S}$ the subfamily of maximal independent sets, separated by $(f,t)$ from $\mathcal{H}$, as before. Let us assume further that $\mathbf{T}$ is a binary tree for which there exists a proper mapping $\phi : L(\mathbf{T}) \mapsto \mathcal{S}$. Then, we have*

$$r(f) \geq t \geq |L(\mathbf{T})|. \tag{8}$$

Let us note that if a proper mapping exists for a binary tree $\mathbf{T}$, then we can associate a hyperedge $H_u \in \mathcal{H}$ to every node $u \in V(\mathbf{T}) \setminus L(\mathbf{T})$ in the following way: Let $v$ and $w$ be the two successors of $u$ in $\mathbf{T}$. Since $v$ and $w$ are incomparable, the union $S_v \cup S_w$ must contain a hyperedge from $\mathcal{H}$. Let us choose such a hyperedge, and denote it by $H_u$. Let us observe next that if $l \in L(\mathbf{T}(v))$ and $l' \in L(\mathbf{T}(w))$, then $S_v \subseteq I_l$ and $S_w \subseteq I_{l'}$, and thus $H_u \subseteq I_l \cup I_{l'}$. In other words, to construct a large binary tree for which there exists a proper mapping, we have to find a way of splitting the family of independent sets, repeatedly, such that the union of any two independent sets, belonging to different parts of the split contains a hyperedge of $\mathcal{H}$. We shall show next that indeed, such a construction is possible.

**Lemma 2.** *For every Sperner hypergraph $\mathcal{H} \subseteq 2^V$, $|\mathcal{H}| \geq 2$, and for every subfamily $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ of its maximal independent sets there exists a binary tree $\mathbf{T}$ and a proper mapping $\phi : L(\mathbf{T}) \mapsto \mathcal{S}$, such that*

$$|L(\mathbf{T})| \geq |\mathcal{S}|^{c(|V|,|\mathcal{H}|)/\log|\mathcal{H}|}. \tag{9}$$

Clearly, Lemmas 1 and 2 imply Theorem 4, which in turn implies Theorem 3. The proof of Lemmas 1 and 2 is given in the next Section.

## 6 Proof of main lemmas

In this section we prove Lemmas 1 and 2, which are the key statements needed to prove our main results.

**Proof of Lemma 1.** Let us recall that $(f,t)$ is a polymatroid separator of the hypergraph $\mathcal{H}$, separating the maximal independent sets $\mathcal{S} = \mathcal{S}(\mathcal{H}, f, t)$ from $\mathcal{H}$, and that to every node $v$ of $\mathbf{T}$ we have associated an independent set $S_v = \bigcap_{l \in L(\mathbf{T}(v))} I_l$, where $I_l \in \mathcal{S}$ denotes the maximal independent set assigned to the leaf $l \in L(\mathbf{T})$ by the proper assignment $\phi$.

To prove the statement of the lemma, we shall show by induction that

$$f(S_w) \leq t - |L(\mathbf{T}(w))| \tag{10}$$

holds for every node $w$ of the tree $\mathbf{T}$. Since $f$ is non-negative, it follows that

$$|L(\mathbf{T}(w))| \leq t \leq r(f)$$

which, if applied to the root of $\mathbf{T}$, proves the lemma. To see (10), let us apply induction by the size of $L(\mathbf{T}(w))$. Clearly, if $w = l$ is a leaf of $\mathbf{T}$, then $|L(\mathbf{T}(l))| = 1$, $S_w = I_l \in \mathcal{S}$, and (10) follows by the assumption that $(f,t)$ is separating $\mathcal{H}$ from $\mathcal{S}$. Let us assume now that $w$ is a node of $\mathbf{T}$ with $u$ and $v$ as its immediate successors. Then $|L(\mathbf{T}(w))| = |L(\mathbf{T}(u))| + |L(\mathbf{T}(v))|$, and $S_w = S_u \cap S_v$. By our inductive hypothesis, and since $f$ is submodular, we have the inequalities

$$\begin{aligned} f(S_u \cup S_v) + f(S_w) \leq f(S_u) + f(S_v) &\leq t - |L(\mathbf{T}(u))| + t - |L(\mathbf{T}(v))| \\ &= 2t - |L(\mathbf{T}(w))|. \end{aligned}$$

Since $\phi$ is a proper mapping, the set $S_u \cup S_v$ contains a hyperedge $H \in \mathcal{H}$, and thus $f(S_u \cup S_v) \geq f(H) \geq t$ by the monotonicity of $f$, and by our assumption that $(f,t)$ is a separator for $\mathcal{H}$. Thus, from the above inequality we get $t + f(S_w) \leq f(S_u \cup S_v) + f(S_w) \leq 2t - |L(\mathbf{T}(w))|$, from which (10) follows. □

For a hypergraph $\mathcal{H}$ and a vertex $v \in V = V(\mathcal{H})$ let us denote by $d_{\mathcal{H}}(v)$ the *degree* of vertex $v$ in $\mathcal{H}$, i.e., $d_{\mathcal{H}}(v)$ is the number of hyperedges of $\mathcal{H}$ containing $v$.

**Lemma 3.** *For every Sperner hypergraph $\mathcal{H} \subseteq 2^V$ on $n = |V| > 1$ vertices, with $m = |\mathcal{H}| \geq n$ hyperedges, there exists a vertex $v \in V$ for which*

$$m\frac{1}{n} \leq d_{\mathcal{H}}(v) \leq m\left(1 - \frac{1}{n}\right).$$

*Proof.* Let us define

$$X = \{v \in V \mid d_{\mathcal{H}}(v) < m\frac{1}{n}\} \quad \text{and} \quad Y = \{v \in V \mid d_{\mathcal{H}}(v) > m(1 - \frac{1}{n})\},$$

and let us assume indirectly that $X \cup Y = V$ forms a partition of the vertex set.

Let us observe first that $|X| < n$ must hold, since otherwise a contradiction

$$m \leq \sum_{H \in \mathcal{H}} |H| = \sum_{v \in X} d_{\mathcal{H}}(v) < n\frac{m}{n} = m,$$

would follow. Let us observe next that $|X| > 0$ must hold, since otherwise

$$\sum_{H \in \mathcal{H}} |H| = \sum_{v \in V} d_{\mathcal{H}}(v) = \sum_{v \in Y} d_{\mathcal{H}}(v) > n \times m(1 - \frac{1}{n}) = m(n-1)$$

follows, implying the existence of a hyperedge $H \in \mathcal{H}$ of size $|H| = n$, i.e., $V \in \mathcal{H}$. Since $\mathcal{H}$ is Sperner, $1 = m < n$ would follow, contradicting our assumptions.

Let us observe finally that the number of those hyperedges which avoid some points of $Y$ cannot be more than $|Y|m/n$, and since $|Y| < n$ by our previous observation, there must exist a hyperedge $H \in \mathcal{H}$ containing $Y$. Thus, all other hyperedges must intersect $X$, and hence we have

$$m - 1 \leq \sum_{H \in \mathcal{H}} |H \cap X| = \sum_{v \in X} d_{\mathcal{H}}(v) < |X|\frac{m}{n} \leq m\frac{n-1}{n}$$

by our first observation. From this $m < n$ would follow, contradicting again our assumption that $m \geq n$. This last contradiction hence proves $X$ and $Y$ cannot cover $V$, and thus follows the lemma. □

For a subset $X \subseteq V$ let $\mathcal{H}^X \stackrel{\text{def}}{=} \{H \in \mathcal{H} \mid H \supseteq X\}$, and let us simply write $\mathcal{H}^v$ if $X = \{v\}$.

**Lemma 4.** *Given a hypergraph $\mathcal{H}$ and a subfamily $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ of its maximal independent sets, $|\mathcal{S}| \geq 2$, there exists a hyperedge $H \in \mathcal{H}$ and a vertex $v \in H$ such that*

$$|\mathcal{S}^v| \geq \frac{|\mathcal{S}|}{n} \text{ and } |\mathcal{S}^{H \setminus v}| \geq \frac{|\mathcal{S}|}{n|\mathcal{H}|}.$$

*Proof.* Let us note first that if $2 \leq |\mathcal{S}| < n$, then the statement is almost trivially true. To see this, let us choose two distinct maximal independent sets $S_1$ and $S_2$ from $\mathcal{S}$, and a vertex $v \in S_2 \setminus S_1$. Since $S_1 \cup \{v\}$ is not independent, there exists a hyperedge $H \in \mathcal{H}$ for which $v \in H \cap S_2$ and $H \setminus \{v\} \subseteq S_1$, implying thus that both $|\mathcal{S}^v|$ and $|\mathcal{S}^{H \setminus v}|$ are at least 1, and the right-hand sides in the claimed inequalities are not more than 1.

Thus, we can assume in the sequel that $|\mathcal{S}| \geq n$. Let us then apply Lemma 3 for the Sperner hypergraph $\mathcal{S}^c \stackrel{\text{def}}{=} \{V \setminus I \mid I \in \mathcal{S}\}$, and obtain that

$$\frac{|\mathcal{S}|}{n} \leq d_{\mathcal{S}^c}(v) \leq |\mathcal{S}|(1 - \frac{1}{n})$$

holds for some $v \in V$, since $|\mathcal{S}| = |\mathcal{S}^c|$ obviously. Thus, from the second inequality we obtain

$$|\mathcal{S}^v| \geq \frac{|\mathcal{S}|}{n}.$$

To see the second inequality of Lemma 4, let us note that members of $\mathcal{S}^c$ are minimal transversals of $\mathcal{H}$, and thus for every $T \in \mathcal{S}^c$, $T \ni v$ there exists a hyperedge $H \in \mathcal{H}$ for which $H \cap T = \{v\}$, by the definition of minimal transversals. Thus,

$$\bigcup_{H \in \mathcal{H}: H \ni v} \{T \in \mathcal{S}^c \mid T \cap H = \{v\}\} \supseteq \{T \in \mathcal{S}^c \mid T \ni v\}$$

holds, from which

$$\sum_{H \in \mathcal{H}: H \ni v} |\mathcal{S}^{H \setminus v}| \geq d_{\mathcal{S}^c}(v) \geq \frac{|\mathcal{S}|}{n}$$

follows. Therefore, since $|\{H \in \mathcal{H} \mid H \ni v\}| = d_{\mathcal{H}}(v) \leq |\mathcal{H}|$ holds obviously, there must exist a hyperedge $H \in \mathcal{H}$, $H \ni v$, for which

$$|\mathcal{S}^{H\setminus v}| \geq \frac{|\mathcal{S}|}{n|\mathcal{H}|}$$

holds, implying thus the lemma. □

**Proof of Lemma 2.** Let us denote by $L(\alpha)$ the maximum number of leaves of a binary tree $\mathbf{T}$ with a proper mapping $\phi : V(\mathbf{T}) \to \mathcal{S}$, where $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ is an arbitrary subfamily of maximal independent sets of $\mathcal{H}$. To simplify notation, let us write $\alpha = |\mathcal{S}|$ and $\beta = |\mathcal{H}|$. To prove the statement, we need to show that

$$L(\alpha) \geq \alpha^{c/\log\beta} \tag{11}$$

where $c = c(n, \beta)$ is as defined in (3).

Let us prove this inequality by induction on $\alpha$. Clearly, if $\alpha = 1$, then $L(1) = 1$ holds, and we have equality in (11).

Let us assume next that we already have verified the claim for all subfamilies of size smaller than $\alpha$, and let us consider a subfamily $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ of size $\alpha = |\mathcal{S}|$. According to Lemma 4, we can choose two disjoint subfamilies $\mathcal{S}', \mathcal{S}'' \subseteq \mathcal{S}$ such that $|\mathcal{S}'| \geq \frac{\alpha}{n}$ and $|\mathcal{S}''| \geq \frac{\alpha}{n\beta}$, and such that for any pair of sets $S' \in \mathcal{S}'$ and $S'' \in \mathcal{S}''$ the union $S' \cup S''$ contains a member of $\mathcal{H}$. Thus, building binary trees with proper mappings separately for $\mathcal{S}'$ and $\mathcal{S}''$, and joining them as two siblings of a common root, we obtain a binary tree with a proper mapping for $\mathcal{S}$. Since the right-hand side of our claim is a monotone function of $\alpha$, we can conclude for the number of leaves in the obtained binary tree that

$$L(\alpha) \geq L(\frac{\alpha}{n}) + L(\frac{\alpha}{n\beta}). \tag{12}$$

Applying now our inductive hypothesis, we get

$$L(\alpha) \geq \left(\frac{\alpha}{n}\right)^{\frac{c}{\log\beta}} + \left(\frac{\alpha}{n\beta}\right)^{\frac{c}{\log\beta}} = \alpha^{\frac{c}{\log\beta}}\left[n^{\frac{-c}{\log\beta}} + (n\beta)^{\frac{-c}{\log\beta}}\right] = \alpha^{c/\log\beta},$$

where the last equality holds by (3). This proves (11), and hence the lemma follows. □

Note that the right-hand side of (11) is the least possible solution of the recursion (12).

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