

An Incremental RNC Algorithm for Generating All Maximal Independent Sets in Hypergraphs of Bounded Dimension*

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Abstract

We show that for hypergraphs of bounded edge size, the problem of extending a given list of maximal independent sets is *NC*-reducible to the computation of an arbitrary maximal independent set for an induced sub-hypergraph. The latter problem is known to be in *RNC*. In particular, our reduction yields an incremental *RNC* dualization algorithm for hypergraphs of bounded edge size, a problem previously known to be solvable in polynomial incremental time. We also give a similar parallel algorithm for the dualization problem on the product of arbitrary lattices which have a bounded number of immediate predecessors for each element.

1 Introduction

Let $\mathcal{A} \subseteq 2^V$ be a hypergraph (set family) on a finite vertex set V . A vertex set $I \subseteq V$ is called *independent* if I contains no hyperedge of \mathcal{A} . Let $\mathcal{I}(\mathcal{A}) \subseteq 2^V$ denote the family of all maximal independent sets of \mathcal{A} . We assume that \mathcal{A} is given by a list of its hyperedges and consider the problem of incrementally generating $\mathcal{I}(\mathcal{A})$:

MIS(\mathcal{A}, \mathcal{I}): Given a hypergraph \mathcal{A} and a collection $\mathcal{I} \subseteq \mathcal{I}(\mathcal{A})$ of maximal independent sets for \mathcal{A} , either find a new maximal independent set $I \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{I}$, or prove that the given collection is complete: $\mathcal{I} = \mathcal{I}(\mathcal{A})$.

Our objective in this note is to show that for hypergraphs of bounded dimension,

$$\dim(\mathcal{A}) \stackrel{\text{def}}{=} \max\{|A| : A \in \mathcal{A}\} \leq \text{const},$$

problem *MIS*(\mathcal{A}, \mathcal{I}) can be efficiently solved in parallel:

Theorem 1 *MIS*(\mathcal{A}, \mathcal{I}) \in *NC* for $\dim(\mathcal{A}) \leq 3$, and *MIS*(\mathcal{A}, \mathcal{I}) \in *RNC* for $\dim(\mathcal{A}) = 4, 5, \dots$

The statements of Theorem 1 were previously known [4, 23] only for $\mathcal{I} = \emptyset$, when *MIS*(\mathcal{A}, \mathcal{I}) turns into the classical problem of computing a single maximal independent set for \mathcal{A} (see [1, 12, 15, 16, 20, 21, 22, 25]). We show that conversely, *MIS*(\mathcal{A}, \mathcal{I}) can be reduced to the above special case.

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Theorem 2 *If $\dim(\mathcal{A}) \leq \text{const}$, then problem $MIS(\mathcal{A}, \mathcal{I})$ is NC-reducible¹ to problem $MIS(\mathcal{A}', \emptyset)$, where \mathcal{A}' is some induced partial hypergraph of \mathcal{A} .*

(Given a hypergraph $\mathcal{A} \subseteq 2^V$, a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ is called a *partial hypergraph* of \mathcal{A} , while $\{A \cap U \mid A \in \mathcal{A}'\}$ for some $U \subseteq V$ is called an *induced partial hypergraph* of \mathcal{A} .)

Note that if $I \in \mathcal{I}(\mathcal{A})$ is an independent set, the complement $B = V \setminus I$ is a *transversal* to \mathcal{A} , i.e. $B \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, and vice versa. Hence $\{B \mid B = V \setminus I, I \in \mathcal{I}(\mathcal{A})\} = \mathcal{A}^d$, where $\mathcal{A}^d \stackrel{\text{def}}{=} \{B \mid B \text{ minimal transversal to } \mathcal{A}\}$ is the *transversal* or *dual* hypergraph of \mathcal{A} . For this reason, $MIS(\mathcal{A}, \mathcal{I})$ can be equivalently stated as the *hypergraph dualization problem*:

DUAL(\mathcal{A}, \mathcal{B}): Given a hypergraph \mathcal{A} and a collection $\mathcal{B} \subseteq \mathcal{A}^d$ of minimal transversals to \mathcal{A} , either find a new minimal transversal $B \in \mathcal{A}^d \setminus \mathcal{B}$ or show that $\mathcal{B} = \mathcal{A}^d$.

The hypergraph dualization problem has applications in combinatorics [29], graph theory [19, 24, 30, 31], artificial intelligence [13], game theory [17, 18, 28], reliability theory [10, 28], database theory [2, 6, 7, 27, 32], integer programming [6, 7], and learning theory [3]. It is an open question whether problem $DUAL(\mathcal{A}, \mathcal{B})$, or equivalently $MIS(\mathcal{A}, \mathcal{I})$, can be solved in polynomial time for arbitrary hypergraphs. The fastest currently known algorithm [14] for $DUAL(\mathcal{A}, \mathcal{B})$ is quasi-polynomial and runs in time $O(nm) + m^{o(\log m)}$, where $n = |V|$ and $m = |\mathcal{A}| + |\mathcal{B}|$. However, as shown in [13, 5], for hypergraphs of bounded dimension problem $DUAL(\mathcal{A}, \mathcal{B})$ can be solved in polynomial time. Theorem 1 strengthens this result by implying that $DUAL(\mathcal{A}, \mathcal{B}) \in NC$ for $\dim(\mathcal{A}) \leq 3$ and $DUAL(\mathcal{A}, \mathcal{B}) \in RNC$ for $\dim(\mathcal{A}) = 4, 5, \dots$. As mentioned above, Theorem 1 is a corollary of Theorem 2 and the results of [4, 23].

A vertex set S is called a *sub-transversal* of \mathcal{A} if $S \subseteq B$ for some minimal transversal $B \in \mathcal{A}^d$. Our proof of Theorem 2 makes use of a characterization of sub-transversals suggested in [5]. Even though it is NP-hard in general to test whether a given set $S \subseteq V$ is a sub-transversal of \mathcal{A} , for $|S| \leq \text{const}$ the sub-transversal criterion of [5] is in NC . This turns out to be sufficient for the proof of Theorem 2.

The remainder of the paper is organized as follows. In Sections 2 and 3 we recall the sub-transversal criterion of [5] and prove Theorem 2. Then in Section 4 we discuss a generalization of the sub-transversal criterion and Theorem 2 for the dualization problem on the Cartesian products of n lattices. More precisely, given n lattices $\mathcal{P}_1, \dots, \mathcal{P}_n$ and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, we consider the problem of generating all maximal elements in $\mathcal{P} \setminus \mathcal{A}^+$, where \mathcal{A}^+ is the (upper) ideal generated by \mathcal{A} . If $\mathcal{P} = \{0, 1\}^n$ is the product of n chains $\{0, 1\}$, then this problem is equivalent to the generation of the transversal hypergraph for \mathcal{A} . In general, when \mathcal{A} is a set in $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, we define $\dim(\mathcal{A}) = \max\{|\text{Supp}(a)| : a \in \mathcal{A}\}$, where $\text{Supp}(a)$ is the *support* of $a \in \mathcal{P}$, i.e., the set of all non-minimal components of a . Then we show that for $\dim(\mathcal{A}) \leq \text{const}$, the dualization problem on $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$ is NC -reducible to the maximal independent set problem for some hypergraphs of dimension at most $\dim(\mathcal{A})$, provided that the number of immediate predecessors of any element in each factor-lattice \mathcal{P}_i is also bounded by a constant.

2 Characterization of sub-transversals to a hypergraph

Given a hypergraph $\mathcal{A} \subseteq 2^V$, a subset $S \subseteq V$, and a vertex $v \in S$, let $\mathcal{A}_v(S) = \{A \in \mathcal{A} \mid A \cap S = \{v\}\}$ denote the family of all hyperedges of \mathcal{A} whose intersection with S is exactly v . Let further $\mathcal{A}_0(S) = \{A \in \mathcal{A} \mid A \cap S = \emptyset\}$ denote the partial hypergraph consisting of the hyperedges of \mathcal{A} disjoint from S . A selection of $|S|$ hyperedges $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ is called *covering* if there exists a hyperedge $A \in \mathcal{A}_0(S)$, such that $A \subseteq \bigcup_{v \in S} A_v$. Proposition 1 below states that a non-empty set S is a sub-transversal of \mathcal{A} if and only if there exists a non-covering selection for S .

Proposition 1 (cf. [5]) *Let $S \subseteq V$ be a non-empty vertex set in a hypergraph $\mathcal{A} \subseteq 2^V$.*

i) *If S is a sub-transversal for \mathcal{A} , then there exists a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S .*

¹In fact, our reduction is in AC_0

- ii) Given a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S , we can extend S to a minimal transversal of \mathcal{A} by solving problem $MIS(\mathcal{A}', \emptyset)$ for the induced partial hypergraph

$$\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}_0(S)\} \subseteq 2^U, \quad (1)$$

where

$$U = V \setminus \bigcup_{v \in S} A_v. \quad (2)$$

Proof. Let us start with the following observations:

(a) If $S \subseteq B \subseteq V$, then $\mathcal{A}_v(B) \subseteq \mathcal{A}_v(S)$ holds for all $v \in S$.

(b) If B is a transversal to \mathcal{A} , then B is minimal if and only if $\mathcal{A}_v(B) \neq \emptyset$ for all $v \in B$.

Observation (a) follows directly from the definitions of $\mathcal{A}_v(S)$ and $\mathcal{A}_v(B)$. To see (b), note that if $\mathcal{A}_v(B) = \emptyset$ for some $v \in B$, then $B \setminus \{v\}$ is still a transversal to \mathcal{A} .

Proof of (i) Suppose that $\emptyset \neq S \subseteq B$, where $B \in \mathcal{A}^d$ is a minimal transversal. By observations (a) and (b), we have $\emptyset \neq \mathcal{A}_v(B) \subseteq \mathcal{A}_v(S)$ for each $v \in S$. Consider then a selection of the form $\{A_v \in \mathcal{A}_v(B) \mid v \in S\}$. If it covers a hyperedge $A \in \mathcal{A}_0(S)$, then A would be disjoint from B , contradicting the fact that $B \in \mathcal{A}^d$.

Proof of (ii) Suppose we are given a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$. If $\mathcal{A}_0(S) = \emptyset$, then S is obviously a transversal to \mathcal{A} . Hence by (b), S itself is a minimal transversal to \mathcal{A} . Let us assume now that $\mathcal{A}_0(S) \neq \emptyset$ and consider the hypergraph \mathcal{A}' as defined in (1)-(2). Since the given selection is non-covering and $\mathcal{A}_0(S) \neq \emptyset$, we conclude that the vertex and edge sets of \mathcal{A}' are not empty, and \mathcal{A}' contains no empty edges. Let T be a minimal transversal to \mathcal{A}' . (Such a transversal can be computed by letting $T = U \setminus I$, where $I = \text{output}(MIS(\mathcal{A}', \emptyset))$.) It is easy to see that $S \cup T$ is a transversal to \mathcal{A} . Moreover, $S \cup T$ is minimal, since if we delete a vertex $v \in S$, then $A_v \cap [(S \setminus \{v\}) \cup T] = \emptyset$, while deleting a vertex $v \in T$ results in an empty intersection with some $A \in \mathcal{A}_0(S)$. \square

Unfortunately, if the cardinality of S is not bounded, finding a non-covering selection for S (equivalently, testing if S is a sub-transversal) is NP-hard. In fact, this is so even for $\dim(\mathcal{A}) = 2$ (i.e., for graphs and transversals \equiv vertex covers).

Proposition 2 *Given an undirected graph $G = (V, E)$ and a vertex set $S \subseteq V$, it is NP-complete to determine whether S can be extended to a minimal vertex cover.*

Proof. We use a polynomial transformation from the satisfiability problem. Let $C = C_1 \wedge \dots \wedge C_m$ be a conjunctive normal form, and let us consider the graph $G = (V, E)$, where V is the set of all clauses and literals of C , and where E consists of the pairs (x, \bar{x}) of mutually negating literals, and the pairs (C_i, u) formed by a clause and one of its literals. Then the set $S = \{C_1, \dots, C_m\}$ can be extended to a minimal vertex cover of G if and only if C is satisfiable. \square

We close this section with the observation that if the size of S is bounded by a constant, then there are only polynomially many selections $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S . All of these selections, including the non-covering ones, can be easily enumerated in parallel.

Corollary 1 *For any fixed c , there is an NC algorithm which, given a hypergraph $\mathcal{A} \subseteq 2^V$ and a set S of at most c vertices, determines whether S is a sub-transversal to \mathcal{A} and if so, finds a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$.*

Note that Corollary 1 holds for hypergraphs \mathcal{A} of arbitrary dimension.

3 Proof of Theorem 2

We prove the theorem for the equivalent problem $DUAL(\mathcal{A}, \mathcal{B})$, i.e. show that for $\dim(\mathcal{A}) \leq \text{const}$, problem $DUAL(\mathcal{A}, \mathcal{B})$ is NC -reducible to $MIS(\mathcal{A}', \emptyset)$, for some induced partial hypergraph \mathcal{A}' of \mathcal{A} . Our reduction consists of several steps.

Step 1. Delete all hyperedges of \mathcal{A} that contain other hyperedges of \mathcal{A} . Clearly, this does not change the minimal transversals to \mathcal{A} . We assume in the sequel that no hyperedge of \mathcal{A} contains another hyperedge of \mathcal{A} , i.e., that

$$\mathcal{A} \text{ is Sperner.} \quad (3)$$

Note that the dual hypergraph \mathcal{A}^d is Sperner by definition, and hence $\mathcal{B} \subseteq \mathcal{A}^d$ is Sperner as well.

Step 2 (optional). Delete all vertices in V that are not covered by some $A \in \mathcal{A}$ so that we have $V = \bigcup_{A \in \mathcal{A}} A$. If $\bigcup_{B \in \mathcal{B}} B$ is a proper subset of V , a new minimal transversal in $\mathcal{A}^d \setminus \mathcal{B}$ can be found as follows:

- Pick a vertex $u \in V \setminus \bigcup_{B \in \mathcal{B}} B$.
- The set $S = \{u\}$ is a sub-transversal to \mathcal{A} . In view of (3), any hyperedge $A_u \in \mathcal{A}$ such that $u \in A_u$ is a non-covering selection for S .
- Let $u \in T \in \mathcal{A}^d$, then $T \notin \mathcal{B}$, because none of the transversals in \mathcal{B} contains u . By Proposition 1, the problem of extending $S = \{u\}$ to a minimal transversal T is equivalent to that of computing a maximal independent set for hypergraph (1) with $U = V \setminus A_u$.

We can thus assume without loss of generality that

$$\bigcup_{A \in \mathcal{A}} A = \bigcup_{B \in \mathcal{B}} B = V.$$

Step 3. By definition, each set $B \in \mathcal{B}$ is a minimal transversal to \mathcal{A} . This implies that each set $A \in \mathcal{A}$ is transversal to \mathcal{B} . Check whether each $A \in \mathcal{A}$ is a *minimal* transversal to \mathcal{B} . Suppose that some $A^o \in \mathcal{A}$ is not minimal, i.e. there is a vertex $u \in A^o$ such that $A^* = A^o \setminus \{u\}$ is still transversal to \mathcal{B} . Then we can proceed as follows.

- Let $\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}\}$, where $U = V \setminus A^*$.
- By (3), we have $A \cap U \neq \emptyset$ for each hyperedge $A \in \mathcal{A}$. Hence any minimal transversal T to \mathcal{A}' is also a minimal transversal for \mathcal{A} .
- It is easy to see that $T \notin \mathcal{B}$. This is because any set $B \in \mathcal{B}$ intersects A^* whereas T is disjoint from A^* . This reduces the computation of a new element in $\mathcal{A}^d \setminus \mathcal{B}$ to problem $MIS(\mathcal{A}', \emptyset)$.

In the sequel we assume in addition to (3) that each set in \mathcal{A} is a minimal transversal to \mathcal{B} :

$$\mathcal{A} \subseteq \mathcal{B}^d. \quad (4)$$

Before proceeding to the next step of the reduction, we pause to make some observations. Clearly, $(\mathcal{A}^d)^d = \mathcal{A}$ for any Sperner hypergraph \mathcal{A} . Therefore, if $\mathcal{B} \neq \mathcal{A}^d$ then $\mathcal{A} \neq \mathcal{B}^d$. By (4), we then have $\mathcal{B}^d \setminus \mathcal{A} \neq \emptyset$. Hence we arrive at the following duality criterion: $\mathcal{A}^d \setminus \mathcal{B} \neq \emptyset$ if and only if there is a sub-transversal S to \mathcal{B} such that

$$|S| \leq \dim(\mathcal{A}), \quad \text{and} \quad (5)$$

$$S \not\subseteq A \text{ for all } A \in \mathcal{A}. \quad (6)$$

The “if” part is obvious and holds even without assumption (4). To show the “only if” part, consider an arbitrary minimal transversal $T \in \mathcal{B}^d \setminus \mathcal{A}$. Clearly, T satisfies (6). Let S be a minimal subset of T that still satisfies (6) and let v be an arbitrary vertex in S . Since $S \setminus \{v\}$ does not satisfy (6) we have $S \setminus \{v\} \subseteq A$ for some $A \in \mathcal{A}$. Assuming $|S| > \dim(\mathcal{A})$, we obtain $A = S \setminus \{v\}$ by (6). Hence $A \subset S \subseteq T$. However, both A and T are minimal transversals to \mathcal{B} . This contradiction shows (5).

So far, we have not relied on the assumption that $\dim(\mathcal{A})$ is bounded. We need this assumption to guarantee that the next step of our reduction is in NC .

Step 4 (Duality test.) For each set S satisfying (5), (6) and the condition that

$$A \not\subseteq S \text{ for all } A \in \mathcal{A}, \quad (7)$$

check whether or not

$$S \text{ is a sub-transversal to } \mathcal{B}. \quad (8)$$

Recall that by Proposition 1, S satisfies (8) if and only if there is a selection

$$\{B_v \in \mathcal{B}_v(S) \mid v \in S\} \quad (9)$$

which covers no set $B \in \mathcal{B}_0(S)$. Here as before, $\mathcal{B}_0(S) = \{B \in \mathcal{B} \mid B \cap S = \emptyset\}$ and $\mathcal{B}_v(S) = \{B \in \mathcal{B} \mid B \cap S = \{v\}\}$ for $v \in S$.

If conditions (5), (6),(7) and (8) cannot be met, we conclude that $\mathcal{B} = \mathcal{A}^d$ and halt.

Step 5. Suppose we have found a non-covering selection (9) for some set S satisfying (5), (6), (7) (and hence (8)). We claim that the set

$$Z = S \cup \left[V \setminus \bigcup_{v \in S} B_v \right]$$

is independent in \mathcal{A} . Suppose to the contrary that $A \subseteq Z$ for some $A \in \mathcal{A}$. By (6), there is a vertex $u \in S$ such that $u \notin A$. Then $A \cap B_u = \emptyset$, yielding a contradiction.

Note also that Z is transversal to \mathcal{B} because selection (9) is non-covering.

Let $\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}\}$, where $U = V \setminus Z$, and let T be a minimal transversal to \mathcal{A}' . (As before, we can let $T = U \setminus \text{output}(\text{MIS}(\mathcal{A}', \emptyset))$.) Since Z is an independent set of \mathcal{A} , we have $T \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, i.e., T is transversal to \mathcal{A} . Clearly, T is minimal, i.e. $T \in \mathcal{A}^d$. It remains to argue that T is a *new* minimal transversal to \mathcal{A} , i.e., $T \notin \mathcal{B}$. This follows from the fact that Z is transversal to \mathcal{B} and disjoint from T . \square

Remark Theorems 1 and 2 can be extended to so-called *fairly independent sets*. Let $\mathcal{A} \subseteq 2^V$ be a hypergraph and let $t \in \{0, 1, \dots, |\mathcal{A}| - 1\}$ be a given threshold. A vertex set $I \subseteq V$ is called fairly independent or t -independent if I contains at most t hyperedges of \mathcal{A} . For $t = 0$ each fairly independent set is thus an independent set of \mathcal{A} . Let us call a vertex set $U \subseteq V$ a t -union if U contains at least t hyperedges of \mathcal{A} , and let \mathcal{A}_{u_t} denote the hypergraph of all minimal t -unions. It is not difficult to see that a vertex set $I \subseteq V$ is t -independent in \mathcal{A} if and only if I is a standard independent set of $\mathcal{A}_{u_{t+1}}$. Furthermore, if t and $\dim(\mathcal{A})$ are both bounded, then $\mathcal{A}_{u_{t+1}}$ can be constructed in NC and the dimension of $\mathcal{A}_{u_{t+1}}$ is bounded as well. Hence for $t \leq \text{const}$, all maximal t -independent sets in a hypergraph of bounded dimension can be incrementally generated by an RNC algorithm.

The sub-transversal criterion of Proposition 1 can also be extended to t -independent sets. Call a vertex set $T \subseteq V$ a t -transversal to \mathcal{A} if T is disjoint from at most t hyperedges of \mathcal{A} . Note that T is a t -transversal if and only if $I = V \setminus T$ is t -independent in \mathcal{A} . By definition, a vertex set S is a t -sub-transversal if S is a subset of some minimal t -transversal T . Let \mathcal{B} be a selection of $|S|$ subfamilies of hyperedges $\{B_v \subseteq \mathcal{A}_v(S) \mid v \in S\}$ and let $k_v = |\mathcal{B}_v|$. Denote by l the number of hyperedges in \mathcal{A}_0 covered by the union of all sets in \mathcal{B} . Call \mathcal{B} a t -selection if $l \leq t < l + k_v$ for all $v \in S$. Proposition 1 can be generalized as follows: *A non-empty vertex set S is a t -sub-transversal of \mathcal{A} if and only if there exists a t -selection for S .* Note that in this criterion, we can consider only those selections \mathcal{B} for which $k_v \leq t + 1$ for all $v \in S$. In particular, for $t = 0$ we obtain $l = 0$ and $k_v \equiv 1$, which is equivalent to the definition of non-covering selections introduced in Section 2.

4 Maximal independent sets in products of lattices

In this section, we discuss a generalization of problem $MIS(\mathcal{A}, \mathcal{I})$ in which the input hypergraphs \mathcal{A} and \mathcal{I} are replaced by two subsets of a partially ordered set \mathcal{P} . Given a subset $\mathcal{A} \subseteq \mathcal{P}$, let $\mathcal{A}^+ = \{x \in \mathcal{P} \mid x \succ a, a \in \mathcal{A}\}$ and $\mathcal{A}^- = \{x \in \mathcal{P} \mid x \preceq a, a \in \mathcal{A}\}$ denote the ideal and the filter generated by \mathcal{A} . Any element in $\mathcal{P} \setminus \mathcal{A}^+$ is called *independent of \mathcal{A}* . Let $\mathcal{I}(\mathcal{A})$ be the set of all maximal independent elements for \mathcal{A} , then

$$\mathcal{A}^+ \cap \mathcal{I}(\mathcal{A})^- = \emptyset \quad \text{and} \quad \mathcal{A}^+ \cup \mathcal{I}(\mathcal{A})^- = \mathcal{P}.$$

Consider the following problem:

$MIS(\mathcal{P}, \mathcal{A}, \mathcal{B})$: Given a set \mathcal{A} in a poset \mathcal{P} and a collection of maximal independent elements $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$, either find a new maximal independent element $x \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$, or prove that $\mathcal{B} = \mathcal{I}(\mathcal{A})$.

Clearly, the above problem can be efficiently solved in parallel for any explicitly given poset, i.e., when \mathcal{P} is represented by the list of its elements and their precedence graph. If \mathcal{P} is the product of n chains $\{0, 1\}$ and $\mathcal{A} \subseteq \mathcal{P} = \{0, 1\}^n$ is (the set of characteristic vectors of the hyperedges of) a hypergraph on n vertices, we obtain problem $MIS(\mathcal{A}, \mathcal{I})$ stated in the introduction. We are interested in the more general case where $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$ for explicitly given posets $\mathcal{P}_1, \dots, \mathcal{P}_n$. For instance, partially ordered sets as attribute values arise in many data analysis applications, e.g., chains or products of chains in [9, 11, 26], lattices and products of lattices in [8]. Frequently a partially defined monotone binary function $f(x_1, \dots, x_n)$ is sought to explain the data, where the variables x_1, \dots, x_n represent some attributes ranging over such posets. In many applications, $f : \mathcal{P} \rightarrow \{0, 1\}$ is defined by sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$ of positive and negative samples, i.e. $f(x) = 1$ for $x \in \mathcal{A}$ and $f(x) = 0$ for $x \in \mathcal{B}$ is assumed. Due to monotonicity, we can assume without loss of generality that \mathcal{A} and \mathcal{B} are both antichains, and that $\mathcal{A}^+ \cap \mathcal{B}^- = \emptyset$. Now, determining whether f is totally defined and if not, finding a point $x \in \mathcal{P} \setminus (\mathcal{A}^+ \cap \mathcal{B}^-)$, is easily seen to be equivalent to problem $MIS(\mathcal{P}, \mathcal{A}, \mathcal{B})$.

In what follows, we assume that each poset \mathcal{P}_i has a unique minimum element 0_i , and let $\text{Supp}(x) = \{i \mid x_i \succ 0_i\}$ denote the set of non-minimal components of $x = (x_1, \dots, x_n) \in \mathcal{P}$. As mentioned in the introduction, we define $\dim(\mathcal{A}) = \max\{|\text{Supp}(a)| : a \in \mathcal{A}\}$. We also denote by x^\perp the set of immediate predecessors of x , i.e.,

$$x^\perp = \{y \in \mathcal{P} \mid z \preceq x, z \neq x \Rightarrow z \preceq y \text{ for some } y \in x^\perp\},$$

and let $\text{in-deg}(\mathcal{P}) = \max\{|x^\perp| : x \in \mathcal{P}\}$. Clearly, we have $\text{in-deg}(\mathcal{P}) = \sum_{i=1}^n \text{in-deg}(\mathcal{P}_i)$ for $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$. If \mathcal{P} is a lattice, we let $x \vee y$ and $x \wedge y$ denote the maximum and minimum of $x, y \in \mathcal{P}$.

Theorems 1 and 2 admit the following generalizations.

Theorem 1' Let $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, where each poset \mathcal{P}_i is a lattice of in-degree $\leq \text{const}$, and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ be two given sets such that $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$. Then $MIS(\mathcal{P}, \mathcal{A}, \mathcal{B}) \in NC$ for $\dim(\mathcal{A}) \leq 3$, and $MIS(\mathcal{P}, \mathcal{A}, \mathcal{B}) \in RNC$ for $\dim(\mathcal{A}) = 4, 5, \dots$

Theorem 2' Under the assumptions of Theorem 1', $MIS(\mathcal{P}, \mathcal{A}, \mathcal{B})$ is NC-reducible to $MIS(\mathcal{P}', \mathcal{A}', \emptyset)$, where $\mathcal{P}' = \{z\}^+$ for some $z \in \mathcal{P}$ and $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$.

Note that for any $z = (z_1, \dots, z_n) \in \mathcal{P}$, we have $\mathcal{P}' = \{z\}^+ = \{z_1\}^+ \times \dots \times \{z_n\}^+$, i.e. \mathcal{P}' is still the product of n lattices $\mathcal{P}'_i = \{z_i\}^+$ whose in-degrees are bounded by the in-degrees of the original lattices \mathcal{P}_i . Moreover, we have $0'_i = z_i$ in \mathcal{P}'_i , and for this reason the dimension of $\mathcal{A}' \subseteq \mathcal{P}'$ does not exceed the dimension of \mathcal{A} in \mathcal{P} . In addition, it is easy to see that Theorem 2' is indeed a generalization of Theorem 2. If $\mathcal{P} = \{0, 1\}^n$, then z is the characteristic vector of some set $Z \subseteq V = \{1, \dots, n\}$ and $\{Z\}^+$ is the family of all supersets of Z . Furthermore, each element $a \in \mathcal{A}$ is then the characteristic vector of some hyperedge $A \subseteq V$. Under this interpretation, $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$ can be regarded as the hypergraph $\{Z \cup A \mid A \in \mathcal{A}\}$. Problem $MIS(\mathcal{P}', \mathcal{A}', \emptyset)$ calls for computing a set $X \subseteq V$ such that $Z \subseteq X$ and X is a maximal independent set for $\{Z \cup A \mid A \in \mathcal{A}\}$. Letting $U = V \setminus Z$, the latter problem is easily seen to be equivalent to computing a maximal independent set for the induced hypergraph $\{A \cap U \mid A \in \mathcal{A}\}$, as stated in Theorem 2.

In addition to Theorems 1' and 2', we show that if each poset \mathcal{P}_i has a unique minimum element, then problem $MIS(\mathcal{P}, \mathcal{A}, \emptyset)$ can be reduced to the maximal independent set problem for some hypergraphs.

Theorem 3 For each fixed c , there is an NC-algorithm which, given posets $\mathcal{P}_1, \dots, \mathcal{P}_n$ with unique minimum elements $0_i \in \mathcal{P}_i$, $i = 1, \dots, n$, and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$ such that $\dim(\mathcal{A}) \leq c$, reduces $MIS(\mathcal{P}, \mathcal{A}, \emptyset)$ to the maximal independent set problem for some hypergraphs of dimension at most c .

Note that Theorem 3 holds for the posets \mathcal{P}_i of arbitrarily large in-degrees and does not require that these posets be lattices. It is also clear that Theorem 1' is a corollary of Theorems 3 and 2'.

4.1 Characterization of sub-minimal elements of an ideal

Our proof of Theorem 2' makes use of an analogue of Proposition 1. This analogue, Proposition 3 below, assumes that each of the posets \mathcal{P}_i , $i \in V = \{1, \dots, n\}$, is a lower semi-lattice, i.e., for any two elements $x, y \in \mathcal{P}_i$ there is a unique minimum element $x \wedge y$. As before, we denote by x^\perp the set of immediate predecessors of x . Note that if $x = (x_1, \dots, x_n) \in \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, then any element $y \in x^\perp$ has the form $y = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$, where $y_i \in x_i^\perp$ is an immediate predecessor of x_i in \mathcal{P}_i and $i \in \text{Supp}(x)$.

Given a set $\mathcal{B} \subseteq \mathcal{P}$ and a vector $s \in \mathcal{P}$, we say that s is *sub-minimal* for $\mathcal{P} \setminus \mathcal{B}^-$ if $s \preceq x$ for some minimal element x of the ideal $\mathcal{P} \setminus \mathcal{B}^-$. We call a subset $\tilde{\mathcal{B}} \subseteq \mathcal{B}$ a *majorant* for s^\perp if for any $y \in s^\perp$ there is an element $b \in \tilde{\mathcal{B}}$ such that $b \succ y$.

Proposition 3 Let \mathcal{B} be a given set in $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, where each poset \mathcal{P}_i is a lower semi-lattice. A vector $s \in \mathcal{P}$ is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$ if and only if there is a majorant $\tilde{\mathcal{B}} \subseteq \mathcal{B}$ for s^\perp and a vector $z \in \{s\}^+ \cap (\mathcal{P} \setminus \mathcal{B}^-)$ such that

- (a) $z[S]$ is minimal in $\mathcal{P}[S] \setminus \tilde{\mathcal{B}}[S]^-$,
- (b) $z_i = \wedge \{b_i \mid b = (b_1, \dots, b_n) \in \tilde{\mathcal{B}}\}$ for all $i \in V \setminus S$, and
- (c) $|\tilde{\mathcal{B}}| \leq \sum \{\text{in-deg}(\mathcal{P}_i) \mid i \in S\}$,

where $S = \text{Supp}(s)$ and $z[S]$, $\tilde{\mathcal{B}}[S]$ are the restrictions respectively, of z and $\tilde{\mathcal{B}}$ on S .

Let us note that if $|\text{Supp}(s)|$ and all poset in-degrees $\max\{|x^\perp| : x \in \mathcal{P}_i\}$ are bounded, then $|\tilde{\mathcal{B}}| \leq \text{const}$ and hence there are only polynomially many sets $\tilde{\mathcal{B}}$ satisfying condition (c). In addition, (b) and the boundedness of $|\text{Supp}(S)|$ imply that for each $\tilde{\mathcal{B}}$, there are only polynomially many candidate vectors z that can satisfy (a). It is clear that all such sets $\tilde{\mathcal{B}} \subseteq \mathcal{B}$ and vectors $z \in \mathcal{P}$ can be generated and tested efficiently in parallel. We shall also make use of the following fact.

Proposition 4 Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ such that $\mathcal{A}^+ \cap \mathcal{B}^- = \emptyset$. Let us assume further that $s \notin \mathcal{A}^-$ is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$, and let $z \in \mathcal{P}$ be the vector proving this, as in Proposition 3. Then, $z \notin \mathcal{A}^+$.

Proof of Proposition 3. To show the ‘‘only if’’ part, suppose that s is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$, and let x be a minimal element in $\mathcal{P} \setminus \mathcal{B}^-$ such that $s \preceq x$. Denote by $Y \subseteq \mathcal{P}$ the set of all elements of the form $y = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$, where $i \in S = \text{Supp}(s)$ and $y_i \in x_i^\perp$. Clearly, $Y \subseteq x^\perp$ and $|Y| \leq \sum \{\text{in-deg}(\mathcal{P}_i) \mid i \in S\}$. By the minimality of x in $\mathcal{P} \setminus \mathcal{B}^-$, for each $y \in Y$ we can find an element $b = b(y) \in \mathcal{B}$ such that $b \succ y$. Since $s \preceq x$, it follows that any immediate predecessor of s can be majorized by some $y \in Y$. Hence we conclude that $\tilde{\mathcal{B}} = \bigcup \{b(y) \mid y \in Y\}$ is a majorant for s^\perp . By definition, $\tilde{\mathcal{B}}$ satisfies (c). Now letting

$$z_i = \begin{cases} x_i & \text{if } i \in S \\ \wedge \{b_i \mid b = (b_1, \dots, b_n) \in \tilde{\mathcal{B}}\} & \text{if } i \in V \setminus S, \end{cases} \quad (10)$$

we readily obtain (b). To prove (a), let us first show that $x[S] \notin \tilde{\mathcal{B}}[S]^-$. Suppose, to the contrary, that $b[S] \succ x[S]$ for some $b \in \tilde{\mathcal{B}}$. By the definition of $\tilde{\mathcal{B}}$, we have

$$b \succ y = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) \quad (11)$$

for some $i \in S$ and $y_i \in x_i^\perp$. We cannot have $b_i \succ x_i$ because this would imply $x \in \mathcal{B}^-$. Hence $b_i \not\asymp x_i$ for some $i \in S$, and consequently, $b[S] \not\asymp x[S]$. We have thus shown that $x[S] \in \mathcal{P}[S] \setminus \tilde{\mathcal{B}}[S]^-$. Now it is easy to see that $x[S]$ is minimal in $\mathcal{P}[S] \setminus \tilde{\mathcal{B}}[S]^-$, because any immediate predecessor of $x[S]$ in $\mathcal{P}[S]$ can be majorized by the restriction $b[S]$ of some vector $b \in \tilde{\mathcal{B}}$, see (11). By the first line of (10), we have $z[S] = x[S]$ and (a) follows. It remains to show that $z \in \{s\}^+ \cap (\mathcal{P} \setminus \mathcal{B}^-)$. To this end, note that $x \preceq z$, because $x[S] = z[S]$ and for any $b = (b_1, \dots, b_n) \in \tilde{\mathcal{B}}$ and $i \in V \setminus S$ we have $x_i \preceq b_i$, see (11) and the second line of (10). The inequality $x \preceq z$ implies that $z \in \{s\}^+ \cap (\mathcal{P} \setminus \mathcal{B}^-)$, because on the one hand $s \preceq x$, and on the other hand $x \notin \mathcal{B}^-$.

The ‘‘if’’ part of the proof does not require condition (c). Since $z \notin \mathcal{B}^-$, there is a vector $x \notin \mathcal{B}^-$, minimal in $\mathcal{P} \setminus \mathcal{B}^-$, such that $x \preceq z$. We have $x[S] = z[S]$ for this vector, since by (a) and (b), any decrease of z in a coordinate $i \in S$ would yield a vector majorized by some $b \in \tilde{\mathcal{B}} \subseteq \mathcal{B}$. From $x[S] = z[S]$ and $z \succ s$ it follows that $x \succ s$, thus proving that s is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$. \square

Proof of Proposition 4. Suppose, to the contrary, that $z \succ a$ for some $a \in \mathcal{A}$. Let $b \in \tilde{\mathcal{B}}$. By (b), $b[V \setminus S] \succ z[V \setminus S]$. This implies $b[V \setminus S] \succ a[V \setminus S]$. Since $\mathcal{A}^+ \cap \mathcal{B}^- = \emptyset$, it follows that $b[S] \not\asymp a[S]$ for all $b \in \tilde{\mathcal{B}}$, i.e., $a[S] \notin \tilde{\mathcal{B}}[S]^-$. On the other hand, $z[S] \succ a[S]$ by our assumption that $z \succ a$. Now the minimality of $z[S]$ in $\mathcal{P}[S] \setminus \tilde{\mathcal{B}}[S]^-$ implies that $z[S] = a[S]$. However, Proposition 3 also says that $z \succ s$ and hence $a[S] \succ s[S]$. Recalling that $S = \text{Supp}(s)$, we conclude that $a \succ s$, i.e., $s \in \mathcal{A}^-$. \square

4.2 Proof of Theorem 2'

The proof is analogous to that of Theorem 2.

Without loss of generality we can assume that \mathcal{A} is an antichain in \mathcal{P} (cf. Step 1 in Section 3).

If there exists a vector $a \in \mathcal{A}$ which is not minimal in $\mathcal{P} \setminus \mathcal{B}^-$, then we can find an element $z \in \mathcal{A}^+$ such that $z \in \mathcal{P} \setminus \mathcal{B}^-$. This can be done fast in parallel. We can then compute a new maximal independent point $b' \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$ by letting $b' = \text{output}(MIS(\mathcal{P}', \mathcal{A}', \emptyset))$, where $\mathcal{P}' = \{z\}^+$ and $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$ (cf. Step. 3 in the proof of Theorem 2'). It is clear that b' is indeed a new maximal independent element for \mathcal{A} because $b' \succ z$ and $z \notin \mathcal{B}^-$. (Note that if \mathcal{P} is not an upper semi-lattice then the set \mathcal{A}' of all minimal elements of $\{z\}^+ \cap \mathcal{A}^+$ may be exponentially large.)

Let us assume now that

$$\text{Each } a \in \mathcal{A} \text{ is minimal in } \mathcal{P} \setminus \mathcal{B}^-. \quad (12)$$

If

$$\mathcal{I}(\mathcal{A}) \neq \mathcal{B}, \quad (13)$$

then there is a vector $x \in \mathcal{P} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Without loss of generality, we may assume that x is minimal in $\mathcal{P} \setminus \mathcal{B}^-$. We have $x \notin \mathcal{A}^-$ because otherwise $x \in \mathcal{B}^-$ by (12). Let s be a minimal element in $\{x\}^-$ such that $s \notin \mathcal{A}^-$, then $|\text{Supp}(s)|$ does not exceed $\dim(\mathcal{A}) + 1$. Thus, (13) implies that

$$\begin{aligned} \text{There is a vector } s \notin \mathcal{A}^- \text{ such that } |\text{Supp}(s)| &\leq \dim(\mathcal{A}) + 1 \\ \text{and } s \text{ is sub-minimal for } \mathcal{P} \setminus \mathcal{B}^-. \end{aligned} \quad (14)$$

Conversely, (14) implies (13) even without (12) and the assumption that $|\text{Supp}(s)| \leq \dim(\mathcal{A}) + 1$. To see this, observe that if $s \notin \mathcal{A}^-$ is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$, then by Proposition 4 we can find a vector $z \notin \mathcal{A}^+$ which satisfies the conditions of Proposition 3. In particular, $z \notin \mathcal{B}^-$, which implies (13).

As mentioned in Section 4.1, Proposition 3 gives an *NC* test for (14) provided that the dimension of \mathcal{A} and the in-degrees of all posets \mathcal{P}_i are bounded (cf. Step 4 in Section 3). Moreover, if we find an s satisfying (14), then, according to Propositions 3 and 4, we also obtain an element $z \in \mathcal{P} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Letting $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$, any solution to $MIS(\{z\}^+, \mathcal{A}', \emptyset)$ yields a new element in $\mathcal{I}(\mathcal{A})$. \square

4.3 Proof of Theorem 3

Consider the following problem:

MIS($\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset$): Given $2n$ non-empty finite posets $\mathcal{R}_i \subseteq \mathcal{P}_i$, $i \in V = \{1, \dots, n\}$, each of which has a unique minimum element, and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, find a maximal \mathcal{A} -independent element x in $\mathcal{R} = \mathcal{R}_1 \times \dots \times \mathcal{R}_n$.

Denoting by 0_i and r_i the minimum elements of \mathcal{P}_i and \mathcal{R}_i , respectively, we shall assume without loss of generality that

$$r = (r_1, \dots, r_n) \notin \mathcal{A}^+, \quad (15)$$

for otherwise \mathcal{R} contains no \mathcal{A} -independent element. As before, we let $\text{Supp}(a) = \{i \in V \mid a_i \succ 0_i\}$ and let $\dim(\mathcal{A}) = \max\{|\text{Supp}(a)| : a \in \mathcal{A}\}$ denote the dimension of \mathcal{A} . Our goal is to show that for $\dim(\mathcal{A}) \leq c$, problem *MIS*($\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset$) is NC-reducible to the maximal independent set problem for some hypergraphs of dimensions $\leq c$. This will prove Theorem 3 because *MIS*($\mathcal{P}, \mathcal{A}, \emptyset$) is a special case of *MIS*($\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset$) for $\mathcal{R} = \mathcal{P}$. Our reduction iteratively decreases $|\mathcal{A}|$ and the maximum cardinality of the posets \mathcal{R}_i .

Step 1. If $\max\{|\mathcal{R}_i| : i \in V\} = 1$, return $x = r$ and halt.

Step 2. If $\mathcal{A} = \emptyset$, return any maximal point in \mathcal{R} and halt.

Step 3. Let $\text{Supp}_{\mathcal{R}}(a) = \{i \in V \mid a_i \succ r_i\}$. In view of (15), we have $|\text{Supp}_{\mathcal{R}}(a)| \geq 1$ for all $a \in \mathcal{A}$. Remove all points $a \in \mathcal{A}$ with $|\text{Supp}_{\mathcal{R}}(a)| = 1$ and reduce \mathcal{R} accordingly:

$$\mathcal{R}_i \leftarrow \mathcal{R}_i \setminus \bigcup \{a_i^+ \mid \text{Supp}_{\mathcal{R}}(a) = \{i\}, a \in \mathcal{A}\}, \quad \mathcal{A} \leftarrow \{a \in \mathcal{A} : |\text{Supp}_{\mathcal{R}}(a)| \geq 2\}.$$

Step 4. For each $i \in V = \{1, \dots, n\}$, topologically sort poset \mathcal{R}_i , i.e., find a one-to-one mapping $\phi_i : \mathcal{R}_i \rightarrow \{1, \dots, |\mathcal{R}_i|\}$ such that $\phi_i(x) < \phi_i(y)$ whenever $x \prec y$ in \mathcal{R}_i . Let $\mathcal{R}_i^u = \{x \in \mathcal{R}_i \mid \phi_i(x) \geq \lceil |\mathcal{R}_i|/2 \rceil\}$ and let \mathcal{Q}_i denote the antichain consisting of all minimal elements of \mathcal{R}_i^u . Note that \mathcal{R}_i^u and hence \mathcal{Q}_i are not empty for all $i \in V$.

Step 5. Let $U = \bigcup_{i=1}^n \mathcal{Q}_i$, and let $\mathcal{H} \subseteq 2^U$ be the hypergraph whose hyperedges are:

- 1) all pairs of the form $\{x, y\}$, where $x \neq y$ and $x, y \in \mathcal{Q}_i$ for some $i \in V$, and
- 2) all collections H of at most $c = \dim(\mathcal{A})$ elements of U such that H contains at most one element from each \mathcal{Q}_i and $\pi(H) \succ a$ for some $a \in \mathcal{A}$, where $\pi(H) = (\pi_1(H), \dots, \pi_n(H)) \in \mathcal{Q}_1 \times \dots \times \mathcal{Q}_n$ is the vector with the following components:

$$\pi_i(H) = \begin{cases} H \cap \mathcal{Q}_i & \text{if } H \cap \mathcal{Q}_i \neq \emptyset \\ r_i & \text{otherwise.} \end{cases} \quad (16)$$

Step 6. Compute a maximal independent set I for \mathcal{H} . Note that $I \neq \emptyset$ since \mathcal{H} does not contain singletons. Also, by the definition of \mathcal{H} , the independent set I contains at most one element from each antichain \mathcal{Q}_i and the vector $\pi(I) \in \mathcal{Q} = \mathcal{Q}_1 \times \dots \times \mathcal{Q}_n \subseteq \mathcal{R}$ is independent of \mathcal{A} .

Step 7. Go to Step 1 and compute *MIS*($\mathcal{R}' \subseteq \mathcal{P}, \mathcal{A}', \emptyset$), where $\mathcal{R}' = \mathcal{R}'_1 \times \dots \times \mathcal{R}'_n$ is defined as follows:

$$\mathcal{R}'_i = \begin{cases} \mathcal{R}_i & \text{if } |\mathcal{R}_i| = 1 \\ \mathcal{R}_i \cap \{\pi_i(I)\}^+ & \text{if } I \cap \mathcal{Q}_i \neq \emptyset \\ \mathcal{R}_i \setminus \mathcal{R}_i^u & \text{otherwise,} \end{cases}$$

and $\mathcal{A}' = \{a \in \mathcal{A} \mid \{a\}^+ \cap \mathcal{R}' \neq \emptyset\}$.

The correctness of the above iterative procedure can be seen from the following observations:

- Each poset \mathcal{R}'_i still has a unique minimum element r'_i ;
- Since $\pi(I)$ is independent of \mathcal{A} , the new minimum element $r' = (r'_1, \dots, r'_n)$ satisfies (15);

- Let $x \in \mathcal{R}'$ be a maximal \mathcal{A}' -independent element in \mathcal{R}' . Then x is a maximal \mathcal{A} -independent element of $\mathcal{R} \cap \{\pi(I)\}^+$. Hence x is also a maximal \mathcal{A} -independent element of \mathcal{R} , i.e., x solves the original problem $MIS(\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset)$.

Since each iteration almost halves the maximum size of the posets \mathcal{R}_i , our reduction consists of $O(\log(\max\{|\mathcal{R}_i| : i \in V\}))$ iterations and Theorem 3 follows.

Remark It is essential, in the above result, to assume that each poset \mathcal{R}_i has a unique minimum element, for otherwise problem $MIS(\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset)$ becomes NP-hard even for posets \mathcal{P}_i with only 3 elements. To see this, let each poset \mathcal{P}_i be a “ \vee ”, i.e. composed of three elements $\{u, 0, w\}$, where $0 \leq u$ and $0 \leq w$ are the only relations in \mathcal{P}_i . Let $R_i = \{u, w\} \subseteq \mathcal{P}_i$. Now given a disjunctive normal form $D = D_1 \vee \dots \vee D_m$ in n variables x_1, \dots, x_n , let us associate a vector $a^j \in \mathcal{P}$ with every term D_j as follows: a_i^j takes the value u if variable x_i appears in term D_j , the value w if \bar{x}_i appears in D_j , and the value 0 otherwise. Letting $\mathcal{A} = \{a^j \mid j = 1, \dots, m\} \subseteq \mathcal{P}$, it is then easy to see that $\mathcal{R} \subseteq \mathcal{A}^+$ if and only if D is a tautology.

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