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Even-hole-free and Balanced Circulants

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Abstract. In this paper some well-known conjectures about the even-hole-free graphs and balanced graphs are verified under the additional assumption of circular symmetry.
Keywords: balanced graph, circulant, circular symmetry, even hole, even-hole-free graph

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1 Introduction

A *circulant graph* (or *circulant*) is a graph on n vertices where the i^{th} vertex is connected to the j^{th} vertex if $|i-j|$ belongs to the list of *edge lengths* L. We denote circulants by $\{l_1, \ldots, l_m\}_n$, where n is the number of vertices and $\{l_1, \ldots, l_m\}$ is the list of edge lengths. Such graphs have circular symmetry. Figure 1 shows the circulant $\{1,2\}_6$:

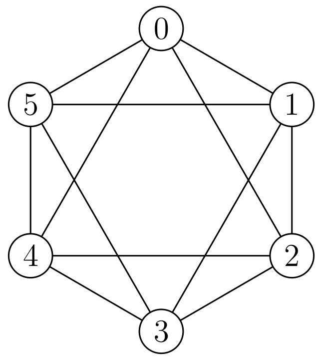

Figure 1: Circulant $\{1,2\}_6$.

Many standard properties can be reformulated for circulants in arithmetic terms. For example, the circulant $\{l_1, \ldots, l_m\}_n$ is connected iff $\mathrm{GCD}(n, l_1, \ldots, l_m) = 1$ and it is bipartite iff n is even, while all $l_1, \ldots, l_m$ are odd.

Such simplifications are naturally used in computer experiments allowing to study significantly larger graphs. Given a conjecture C in the form: "all graphs from a family F have a property P", it may be difficult to analyze C in general (for the whole family F), but much easier for the circulants of F. For example, it was conjectured in [8] that each connected kernel-less digraph, except for odd dicycles, contains an edge which can be deleted and the remaining graph is still kernel-less (see also [3]). Yet, in [1] it was shown that the directed circulant $\{1,7,8\}_{47}$ is a counterexample.

It can be also much easier to prove P for the circulants from F than for the whole family F; see e.g. [2, 4, 10, 7, 9, 11], where $C =$ SPGC, Berge's Strong Perfect Graph Conjecture. Let us remark that two different approaches were applied: in [10] F is the family of graphs without induced odd holes and antiholes (the so-called Berge graphs) and P means that a graph is perfect, while in [2, 4, 7, 9, 11] F is the family of partitionable graphs and P means that a graph is not Berge. Respectively, the Berge and partitionable circulants are studied in [2, 4, 7, 10, 9, 11].

In this paper we consider balanced and even-hole-free circulants. We will always assume that $\mathrm{GCD}(n, l_1, \ldots, l_m) = 1$; in other words, we restrict ourselves to connected circulants. It is easy to see that the circulant $(tl_1, \ldots, tl_m)_{tn}$ is just t vertex disjoint copies of the circulant $(l_1, \ldots, l_m)_n$.

2 Even-Hole-Free Graphs

A *hole* is a chordless cycle of length at least 4. A graph is *even-hole-free* if it contains no hole of even length.

Conjecture 2.1. (Hoàng). Every even-hole-free graph is 3-divisible.

Conjecture 2.2. (Hoàng). If G is an even-hole-free graph then $\chi(G) \leq 2\omega(G) - 1$.

Conjecture 2.3. (Hayward and Reed). An even-hole-free graph contains a vertex whose neighborhood can be partitioned into two cliques.

These conjecture are presented in [12] together with a short proof that Conjecture 2.3 implies Conjecture 2.2 which implies Conjecture 2.1. Indeed, let G be an even-hole-free graph, Conjecture 2.3 implies that each induced subgraph H of G has a vertex of degree at most $2\omega(H) - 2$, and therefore $\chi(G) \leq 2\omega(G) - 1$. Since any graph F is $\frac{\chi(F)}{\omega(F)-1}$-divisible, G is 3-divisible.

Definition 2.1. Given integers $k \geq 1$ and $m \geq 0$, let us introduce a graph $G_E(k, m) = (V, E)$ with circular symmetry as follows: $V = \{0, ..., n-1\}$, where $n = k(2m+1)$, and $(i, j) \in E \iff i - j = \{-1, 0, +1\} \pmod{(2m+1)}$.

For convenience, the loops $i = j$ are included. For example if $k = 1$, the circulant is an odd cycle of size $2m+1$. If $m = 0$ (or 1), the circulant is a complete graph of size k (or $3k$). Figure 2 illustrates the example where $k = 2, m = 2$ then $n = 10$ and $(i, j) \in E \iff i - j \pmod 5 \in \{9, 0, 1; 4, 5, 6\}$.

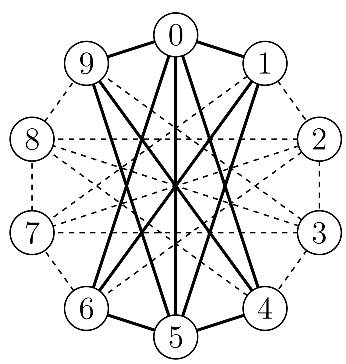

Figure 2: Circulant $\{1, 4, 5\}_{10}$.

We now prove that graphs $G_E(k, m)$ are even-hole-free and then verify for them the three conjectures above.

Definition 2.2. Let ℓ_-, ℓ_0, ℓ_+ be *edge types* of $G_E(k, m)$ corresponding to edges (i, j) where $i - j = \{-1, 0, +1\} + (2m+1)t$ $(t \in \mathbb{N})$, respectively.

Lemma 2.1. *Given a graph $G_E(k, m)$, any cycle of size greater than 3 that contains two consecutive edges of different types or two consecutive edges of type ℓ_0 has a chord.*

Proof. Let l_1, l_2 be the two consecutive edges of different types (or two consecutive edges of type ℓ_0) in the cycle. Summing the lengths of these edges we have: $\delta + (2m+1)\alpha$ ($\alpha \in \mathbb{N}$), where δ is the contribution of the $\{-1, 0, +1\}$ term, and $(2m+1)\alpha$ is the contribution of the term $(2m+1)t$.

Notice that $\delta \in \{-1, 0, +1\}$, because the edges are of different type (or $\delta = 0$ if both edges are of type ℓ_0). Then, $l_3 = l_1 + l_2 = \delta + (2m+1)\alpha$ is a chord creating a chordless cycle of size 3. □

Theorem 2.1. *The graph $G_E(k, m)$ has no even holes. Moreover, every chordless cycle in $G_E(k, m)$ is of size 3 or $2m+1$.*

Proof. Assume by contradiction that there is an even-hole. By Lemma 2.1 we know that such hole cannot contain 2 consecutive edges of different type, or 2 consecutive edges of type ℓ_0, because such hole would have a chord and form a triangle.

Thus, if there is an even-hole it must consist only of edge lengths ℓ_- or only ℓ_+. So assume there is an even-hole consisting of only lengths of type ℓ_+, let $H = \{h_1, , h_2, \ldots, h_p\}$ (p even) be such hole, then we have:

$$\text{sum of lengths of } H = \sum_{i=1}^{p} h_i = p + \alpha(2m+1), \ \ (\alpha \in \mathbb{N})$$

since H is a hole, $p + \alpha(2m+1)$ must be a multiple of $2m+1$, therefore p must be a multiple of $2m+1$. Moreover, since p is even, $p \geq 2(2m+1)$.

Now, consider the first subgraph of H containing the first $2m$ edges:

$$\text{sum of first } 2m \text{ lengths } = \sum_{i=1}^{2m} h_i = 2m + \beta(2m+1) = \gamma(2m+1) - 1, \ \ (\beta, \gamma \in \mathbb{N})$$

and then, by the definition of G_E, there is a chord of type ℓ_+ that forms an odd-hole of size $2m+1$.

The proof is analogous for a hole consisting of edges of type ℓ_-. □

We now present a different way of defining the graphs $G_E(k, m)$, as well as an alternate proof for Theorem 2.1.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, let G^k be a graph constructed from G by the following operations:

1. Take k *copies* of each vertex in $V(G)$, i.e. $V(G^k) = \{v_1^1, \ldots, v_n^1, v_1^2, \ldots, v_n^2, \ldots, v_1^k, \ldots, v_n^k\}$;
2. For each edge $(u, v) \in E(G)$, connect $\{u^1, \ldots, u^k, v^1, \ldots, v^k\}$ in a $2k$ clique;

Figure 3 illustrates an example of this operation applied to a cycle of length 5 (C_5). The resulting graph is the circulant $\{1, 4, 5\}_{10}$.

Theorem 2.2. *A graph G is even-hole-free if and only if G^k is even-hole-free.*

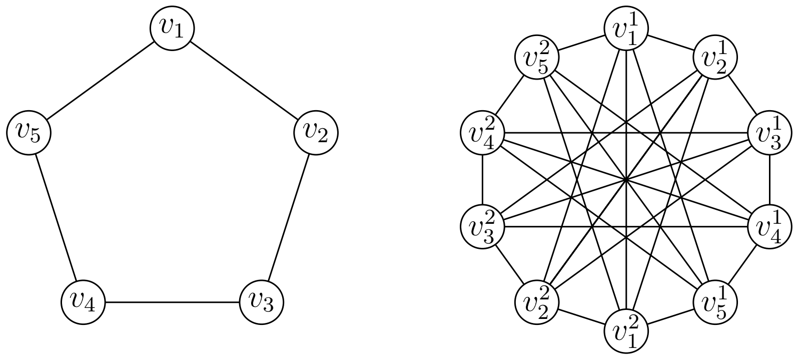

Figure 3: C_5 transformed into C_5^2 (which is the circulant $\{1,4,5\}_{10}$).

Proof. ($\Longleftarrow$) If G^k is even-hole-free, then all its induced subgraphs are even-hole-free, and G is clearly an induced subgraph of G^k.

($\Longrightarrow$) Assume that G is even-hole-free but G^k is not. Then, there is an even-hole $H = \{v_{i_1}^{k_1}, v_{i_2}^{k_2}, \dots, v_{i_m}^{k_m}\}$. Clearly $i_1 \neq i_2 \dots \neq i_m$, otherwise there is a chord in H. But since $i_1 \neq i_2 \dots \neq i_m$ the induced subgraph on $\{v_{i_1}^{k_1}, \dots, v_{i_m}^{k_m}\}$ in G^k is the same as the induced subgraph on $\{v_{i_1}, \dots, v_{i_m}\}$ in G. Therefore, there is no even-hole on G^k. □

Lemma 2.2. *Let G be an odd cycle of length $2m+1$, then G^k is $G_E(k,m)$.*

Proof. Immediate from the definition of $G_E(k,m)$. □

It is easy to see that Theorem 2.1 follows from Theorem 2.2 and Lemma 2.2

Proposition 2.1. *If $m > 1$ the maximum cliques of $G_E(k,m)$ are of size $2k$, otherwise if $m = 0$ or 1 the maximum clique is $n = k(2m+1)$.*

Proof. If $m = 0$ (or 1), $G_E(k,m)$ is a complete graph on n vertices, therefore the maximum clique is n.

Now assume $m > 1$. Since the graph has circular symmetry, every maximum clique has not only the same size, but also the same structure. So consider an initial vertex u: it is connected to $3k-1$ vertices. Let's divide those vertices into 3 types: v_-, v_0, v_+, corresponding to vertices connected to u by an ℓ_-, ℓ_0, ℓ_+ edge, respectively (notice also that u is a v_0 type vertex).

And by the definition of $G_E(k,m)$, the $2k$ vertices v_0, v_+ form a clique (and by symmetry, so do the $2k$ vertices v_-, v_0). Moreover, no vertex v_- is connected to any vertex v_+, and the vertices v_0 are not connected to any other vertices on the graph. So, $2k$ is the maximum. □

Proposition 2.2. *If $m \geq 1$, the maximum independent sets of $G_E(k,m)$ are of size m.*

Proof. Notice that to find the maximum independent set, it suffices to consider an interval of $2m+1$ consecutive vertices (say, from 0 to $2m$), because every vertex outside this interval is *equivalent*[1] to some vertex within this interval.

[1]Equivalent in the sense that it is connected to the same set of vertices. More specifically, a vertex v ($2m+1 \leq v < n$) is equivalent to $\mathrm{mod}(v, 2m+1)$.

Now, let H be the induced subgraph of an interval of $2m+1$ consecutive vertices of G_E. It is easy to see that H is a cycle on $2m+1$ vertices, and therefore its maximum independent set is m. □

Proposition 2.3. *The chromatic number of $G_E(k,m)$ is at most $2k + \lfloor k/m \rfloor + 2$.*

Proof. Follows from Proposition 2.2. For every group of $2m$ consecutive vertices there are two disjoint independent sets of size m, so we assign one color to each. To the next group of $2m$ vertices we assign two new colors, and so on.

The graph has a total of $n = 2km + k$ vertices. Assigning the coloring proposed above, the chromatic number $\chi(G_E) \geq 2\frac{2km+k}{2m} = 2k + k/m$. If $\mathrm{mod}(k, 2m) = 0$, $\chi(G_E) = 2k + k/m$, if $\mathrm{mod}(k, 2m) = 1$, $\chi(G_E) = 2k + \lfloor k/m \rfloor + 1$, otherwise $\chi(G_E) = 2k + \lfloor k/m \rfloor + 2$. □

Now from Propositions 2.1 and 2.2, Conjectures 2.1 and 2.2 follow immediately. Yet, to prove all 3 conjectures, is suffices to prove Conjecture 2.3. This is done by the following theorem.

Theorem 2.3. *An even-hole-free circulant $G_E(k,m)$ contains a vertex whose neighborhood can be partitioned into two cliques.*

Proof. Since a circulant has circular symmetry, it does not matter which vertex we pick. The theorem shall work for every vertex $v \in G_E(k,m)$.

Let v_-, v_0, v_+ be defined as above, and assign type v_0 to the chosen vertex v. The neighborhood of v consists of k vertices of type v_+, k vertices of type v_- and $k-1$ vertices of type v_0. By definition, all vertices v_+ are connected among themselves, and so are all vertices v_-. The v_0 vertices are connected to both types.

Then, one possible partition into two cliques is: one clique contains all neighbors of type v_+ and v_0, and the other clique contains all neighbors of type v_-. □

Conjecture 2.4. *Every non-empty even-hole-free circulant is isomorphic to a $G_E(k,m)$.*

This conjecture was tested by computer for circulants up to 60 vertices.

3 Balanced Graphs

A graph is *balanced* if every induced cycle has length $0 \pmod 4$. Obviously, balanced graphs are bipartite.

Conjecture 3.1. (Conforti, Rao) In every balanced graph, there exists an edge that can be deleted, so that the resulting graph remains balanced.

Conjecture 3.2. (Conforti, Cornuejóls, Rao) Every balanced graph is either basic or has a 2-join or a skew-partition.

Recall that a balanced graph G is called *basic* if all its vertices on one side of the bipartition have degree at most 2 or G contains a hole H such that the vertices of $G \setminus H$ induce a complete bipartite graph.

The definitions of 2-join and skew-partition can be found in literature, see e.g. [5]. In fact, they are not needed here. We will only make use of the concept of star-cutset which is a special case of skew-partition. This notion was introduced by Chvátal in [6] as follows.

A graph is a *star* if it has a vertex incident to all other vertices. Given a graph $G = (V, E)$, a vertices-set $S \subseteq V$ is a *star-cutset* if the induced subgraph $G[S]$ is a star and $G[V \setminus S]$ is not connected.

Definition 3.1. Given integer $k \geq 1$ and $m \geq 1$, let us introduce a graph $G_B(k,m) = (V, E)$ with circular symmetry as follows: $V = \mathbf{Z}_n = \{0, 1, ..., n-1\}$, where $n = 4km$, and $(i,j) \in E \iff i - j = \{-1, +1\} \pmod{4m}$, for some integer t.

For example, if $k = 1$, then the circulant is a cycle of length $4m$. Figure 4 illustrates the case where $k = 2, m = 1$; then $n = 8$ and $(i,j) \in E \iff i - j \pmod 8 \in \{1, 3, 5, 7\}$.

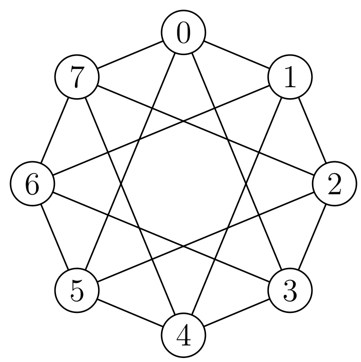

Figure 4: Circulant $\{1, 3\}_8$.

First, we will prove that graphs $G_B(k,m)$ are balanced and then verify for them the above 2 conjectures.

Definition 3.2. Let ℓ_-, ℓ_+ be *edge types* of $G_B(k,m)$ corresponding to edges (i,j) where $i - j = \{-1, +1\} + (4m)t$ $(t \in \mathbb{N})$, respectively.

Lemma 3.1. *Any cycle of size greater than 4 in a given graph $G_B(k,m)$ that contains two consecutive edges of different types has a chord.*

Proof. Let l_1, l_2 be the two consecutive edges of different type in the cycle. Summing the lengths of this edges we have: $+1 - 1 + 4m\alpha$ $(\alpha \in \mathbb{N})$, where $4m\alpha$ is the contribution of the term $(4m)t$.

Let l_3 be an edge of any type following l_2, summing up $l_1 + l_2 + l_3$ we have $\delta + 4m\beta$ $(\beta \in \mathbb{N})$. Notice that $\delta \in \{-1, +1\}$, therefore $l_4 = l_1 + l_2 + l_3$ is a chord, closing a hole of length 4 in the cycle. $\square$

Theorem 3.1. *The graph $G_B(k,m)$ is balanced. Moreover, every hole in $G_B(k,m)$ has size 4 or $4m$.*

Proof. Assume for contradiction that the graph is not balanced. By Lemma 3.1 we know that such hole cannot contain edges of different types, since there will be a chord forming a hole of size 4.

Thus, if there is a cycle of length different than 0 (mod 4), it contains only edges of type ℓ_-, or only of type ℓ_+. Assume it contains only ℓ_+edges, let $H = \{h_1, \ldots, h_p\}$ ($p \neq 0$ (mod 4)) be the hole, then we have:

$$\text{sum of lengths of } H = \sum_{i=1}^{p} h_i = p + 4\alpha m, (\alpha \in \mathbb{N})$$

but since H is a hole, $p + 4\alpha m$ must be a multiple of $4km$, which implies that p must be 0 (mod $4m$).

Now, taking the sum of the lengths of $4m - 1$ edges of type ℓ_+, we have: $4m - 1 + 4m\beta = -1 + 4m\beta$ ($\beta \in \mathbb{N}$), then, by definition of G_B, there is and ℓ_+ edge e such that $e + -1 + 4m\beta = t(4km)$ ($t \in \mathbb{N}$), which implies that $p = 4m$.

The proof is analogous for a hole consisting of edges of type ℓ_-. □

Theorem 3.2. *Every balanced circulant $G_B(k,m)$ is either basic or has a star-cutset.*

Proof. If $k = 1$, then G_B is a cycle of length $4m$ which is basic. If $k > 1$, G_B is not basic, however it has a star-cutset. Indeed, pick any vertex u in G_B, and partition all other vertices in three sets V_-, V_0, V_+, respectively, $v \in V_-, V_0, V_+$ if $u - v = \{-1, 0, +1\}$ (mod $4m$).

Then $S = \{u\} \cup V_- \cup V_+$ is a star-cutset. Indeed, clearly S is a star with the center u and S is a cutset, since $G[V_0]$ consists of isolated vertices. □

Clearly, this theorem implies Conjecture 3.2, since star-cutset is a special case of skew-partition.

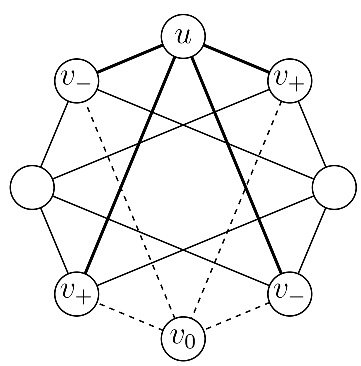

Figure 5: Illustration of Theorem 3.2 on Circulant $\{1,3\}_8$.

Figure 5 illustrates Theorem 3.2 on Circulant $\{1,3\}_8$. The *star cutset* consists of u and the four vertices of type v_- and v_+. The vertex v_0 is disconnected after the removal of the cutset.

Lemma 3.2. *Given a graph $G_B(k, m)$, we have:*

(i) Every cycle of length $4i + 2$ has at least two chords.

(ii) A $(4i + 2)$-cycle has exactly two chords if and only if $m > 1$, $i = 1$, that is the cycle is of length 6, and the edge type sequence is $\ell_+, \ell_+, \ell_+, \ell_-, \ell_-, \ell_-$.

Proof. Notice that a $(4i + 2)$-cycle must contain at least one edge of each type, otherwise, the sum of the edge lengths will not be a multiple of $n = 4km$ (in other words, it would not be a cycle).

Let l_1, l_2 be the consecutive edges of different type, l_0 be the edge preceding l_1, and l_3 be the edge following l_2 in the cycle. Then, by Lemma 3.1, $c_1 = l_0 + l_1 + l_2$ is a chord, and $c_2 = l_1 + l_2 + l_3$ is a chord. This proves (i).

To prove (ii), we start by proving that no $(4i+2)$-cycle of length greater than 6 can have 2 chords. Let us split into 2 cases:

1. There is 1 edge of one type and $4i + 1$ edges of the other. Assume the 1 edge is of type ℓ_- and the remaining of type ℓ_+. Let l_0, l_1, l_2, l_3, l_4 be consecutive edges of type $\ell_+, \ell_+, \ell_-, \ell_+, \ell_+$, respectively. By Lemma 3.1, $c_1 = l_0 + l_1 + l_2$, $c_2 = l_1 + l_2 + l_3$ and $c_3 = l_2 + l_3 + l_4$ are chords. The same reasoning can be applied with the edge types reverted. The graph on the left on Figure 6 illustrates this case.

2. There are at least 2 different edges of each type. This implies that there are 2 distinct pairs of consecutive edges with different types. Each of these pairs causes the cycle to have 2 chords. Furthermore, those chords do not coincide because the cycle has length greater than 6. So in this case, there are at least 4 chords in the cycle. The graph on the right on Figure 6 illustrates this case.

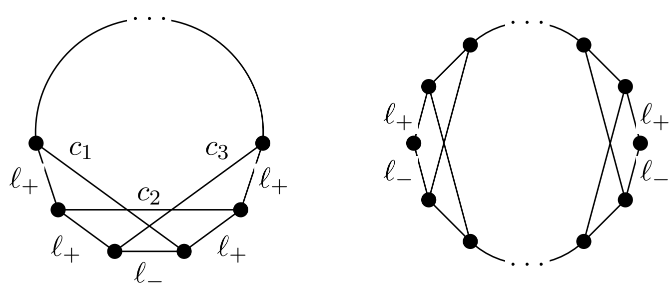

Figure 6: Illustration of cases (1) and (2) above.

Assume now that we have a 6-cycle. First, notice that there are only 3 distinct chords that divided the 6-cycle into two 4-cycles. We split into 4 cases:

1. There is 1 edge of one type (say ℓ_-) and 5 of the other (ℓ_+). Notice that this case can only happen when $m = 1$, because if we sum the edge lengths we have $\sum_{i=1}^{6} l_i = 4m\alpha + 4$, $(\alpha \in \mathbb{N})$ and this can only be a multiple of $4mk$ if $m = 1$. Then it is easy to see that the 6-cycle has 3 chords. The graph on the left on Figure 7 illustrates this case.

2. There are 3 edges of each type, and 3 distinct pairs of consecutive edges of different types. Notice that there are 2 ways (w.l.o.g.) that it can happen: $(\ell_+, \ell_-, \ell_+, \ell_-, \ell_+, \ell_-)$ or $(\ell_+, \ell_-, \ell_-, \ell_+, \ell_-, \ell_+)$. In both cases, it can be seen that the 6-cycle has 3 chords, by arguments similar to those on Lemma 3.1. The graph in the middle on Figure 7 illustrates this case.

3. There are 3 edges of each type, and the edge type distribution is $(\ell_+, \ell_+, \ell_+, \ell_-, \ell_-, \ell_-)$. Here there are only two distinct pairs of consecutive edges of different types. This gives 2 chords. The third chord exists if and only if $m = 1$, and the argument is the same as presented on case (1). The graph on the right on Figure 7 illustrates this case, the dashed edge is the chord that may or may not exist (depending if $m = 1$ or $m > 1$).

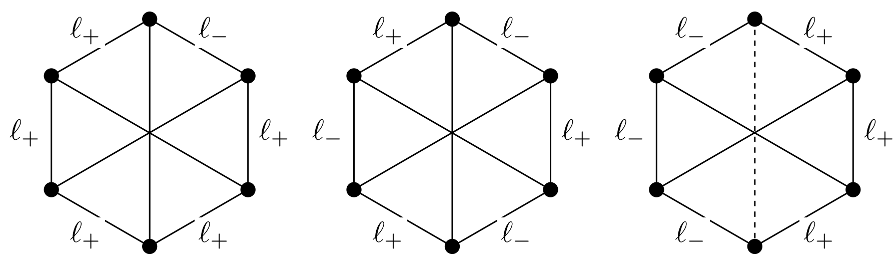

Figure 7: Illustration of cases (1), (2) and (3) above.

This concludes the proof of (ii). □

Theorem 3.3. *In every balanced circulant $G_B(k, m)$, there exists an edge that can be deleted, so that the resulting graph remains balanced.*

Proof. Every balanced graph is bipartite and obviously remains bipartite after deleting an edge. Further, according to Lemma 3.2 (i) every $(4i+2)$-cycle has at least two chords, hence it cannot become chordless after deleting only one edge. □

Conjecture 3.3. *Every balanced circulant is isomorphic to a $G_B(k, m)$.*

This conjecture was tested by computer for circulants up to 60 vertices.

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