

LINEAR OPTIMIZATION HW 4 solutions (sketch)

1. *The IP below is an IP formulation of the maximum matching problem ...*
 - (a) *Prove that the IP is indeed a formulation of the maximum matching problem.*

It is easy to show a one-to-one correspondence between the feasible/optimal solutions to the IP and matchings/maximum matchings in the graph: each variable corresponds to an edge, and we set a variable to one if and only if the corresponding edge is contained in the matching.
 - (b) *Prove that the constraints $x_i \leq 1$ can be deleted from the LP relaxation without changing the optimal solution of the LP.* Removing the constraints $x_i \leq 1$ doesn't change the set of feasible solutions, because $x_j \geq 0$ and $x_i + x_j \leq 1$ imply $x_i \leq 1$. The optimal solutions are, hence, unaffected.
2. *The IP below is an IP formulation of the minimum vertex cover problem ...*
 - (a) *Prove that the IP is indeed a formulation of the minimum vertex cover problem.*

It is easy to show a one-to-one correspondence between the feasible/optimal solutions to the IP and vertex covers/minimum vertex covers in the graph: each variable corresponds to a vertex, and we set a variable to one if and only if the corresponding vertex is contained in the vertex cover.
 - (b) *Prove that the constraints $x_i \leq 1$ can be deleted from the LP relaxation without changing the optimal solution of the LP.* Removing the constraints $x_i \leq 1$ may change the set of feasible solutions in this case, but $x_i > 1$ is impossible in any *optimal* solution to the simplified LP relaxation. This is because for every feasible solution with $x_i > 1$ we can decrease the objective function value by changing x_i to 1, and this maintains the feasibility of the solution.
3. *What's the dual of the simplified LP relaxation of the maximum matching problem? What's the dual of the simplified LP relaxation of the minimum vertex cover problem?*

The matrix of the two LPs are the incidence matrix of the graph, and its transpose, respectively. Using the table we always use to write the dual problems of a primal it is then easy to verify that the two problems are the duals of each other.
4. *Prove that for bipartite graphs the size of the maximum matching equals to the size of the minimum vertex cover. Show by example that the the size of the maximum matching may not be equal to the size of the minimum vertex cover in a non-bipartite graph.* An example of the second claim is the triangle: the maximum matching contains one edge, while the smallest vertex cover has two vertices. Pretty much any small nonbipartite graph you may think of will serve as a counterexample, actually. The proof of the first claim: the incidence matrix of a bipartite graph is TU, so for bipartite graphs the given IPs are equivalent to their LP relaxations, and (from 1b) and 2b)) also to their simplified LP relaxations. In particular, the

optimal objective function values of the IPs are the same as that of the corresponding simplified LP relaxations. But the two simplified LP relaxations also have the same optimal objective function values, by strong duality, because they are primal and dual problems. So for bipartite graphs all the six problems have the same optimal objective function value, and the claim follows.

Using the same framework (combining total unimodularity and strong duality), one can prove pretty much all of the most famous theorems in combinatorics which are of the form “the maximum of something is equal to the minimum of something else”.