MATH 354:03 LINEAR OPTIMIZATION, SPRING 2012 HANDOUT #1

Goal: In this handout we will go over the underlying algebra of simplex method, basically what we covered in the last 2 weeks of our class.

Tools: Linear algebra, matrix multiplication.

Ok, let's start with our STANDARD form LP.

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array} \tag{1}$$

where $A : m \times n$, $b : m \times 1$, $c : n \times 1$ and $\text{rank}(A) = m$. Let us further assume $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} \neq \emptyset$. Which implies the polyhedron $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ has at least one extreme point (why? 1 points). Let's $x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$ be the BFS corresponding to this extreme point (after possible re-arranging of the columns of A), where B is the basis corresponding to x. Therefore we use the following decomposition of $A = [B, N]$, $c^T = \left[c_B^T, c_N^T\right]$ and $x^T = (x_B^T, x_N^T)$ (let's remember x_B is called the set of basic variables and x_N is called the set of non-basic variables). Let's re-write the problem:

$$\begin{array}{lrrcl} \max & c_B^T x_B & +c_N^T x_N & & \\ \text{s.t.} & Bx_B & +Nx_N & = & b \\ & x_B & & \geq & 0 \\ & & x_N & \geq & 0 \end{array} \tag{2}$$

or equivalently,

$$\begin{array}{lrrcl} \max & c_B^T x_B & +c_N^T x_N & & \\ \text{s.t.} & x_B & +B^{-1}Nx_N & = & B^{-1}b \\ & x_B & & \geq & 0 \\ & & x_N & \geq & 0. \end{array} \tag{3}$$

Let's remember we are doing Gaussian elimination when we multiply the set of equations with B^{-1}. As **Gaussian elimination preserves the solutions to a system of linear equations** (row equivalence), (2) and (3) are equivalent. As we are interested in the set of feasible solutions to the problem, we must have $x_B = B^{-1}b - B^{-1}Nx_N$ for all feasible solutions. Therefore we can substitute x_B in the objective with $B^{-1}b - B^{-1}Nx_N$:

$$\begin{array}{lllll} \max & c_B^T(B^{-1}b - B^{-1}Nx_N) & +c_N^T x_N & & \\ \text{s.t.} & x_B & +B^{-1}Nx_N & = & B^{-1}b \\ & x_B & & \geq & 0 \\ & & x_N & \geq & 0. \end{array} \tag{4}$$

Let's re-arrange the messy objective above,

$$\begin{array}{lllll} \max & c_B^T B^{-1}b & -(c_B^T B^{-1}N - c_N^T)x_N & & \\ \text{s.t.} & x_B & +B^{-1}Nx_N & = & B^{-1}b \\ & x_B & & \geq & 0 \\ & & x_N & \geq & 0. \end{array} \tag{5}$$

We call $c_B^T B^{-1}N - c_N^T$ the vector of reduced costs associated to non-basic variables. Now we reduced the problem into a form, where optimality is easy observed by the following argument:

KEY TO SIMPLEX: If $c_B^T B^{-1}N - c_N^T \geq 0$ (namely **the reduced costs are nonnegative**) then the current BFS $x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$ is optimal.

proof. Indeed. As $\{Ax = b\} = \{x_B + B^{-1}Nx_N = B^{-1}b\}$ (because B is a basis, it's inverse exists and it is a Gaussian elimination operator), if $c_B^T B^{-1}N - c_N^T \geq 0$ then $(c_B^T B^{-1}N - c_N^T)x_N \geq 0$ therefore any other feasible solution has objective at most $c^T x = c_B^T x_B + c_N^T x_N = c_B^T B^{-1}b + c_N^T 0 = c_B^T B^{-1}b$ (which is the objective value of current BFS $x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$).

Let's use the following notation, let $A = [a_1, a_2, \ldots, a_n]$ denote the columns of A, $\bar{b} = B^{-1}b$ and J the set of non-basic indices (The column indices corresponding to N). Also let $y_j = B^{-1}a_j$, $z_j = c_B^T B^{-1}b = c_B^T y_j$ for $j \in J$. Then (5) reduces to the following,

$$\begin{array}{lllll} \max & c_B^T B^{-1}b & -\sum_{j\in J}(z_j - c_j)x_j & & \\ \text{s.t.} & x_B & +\sum_{j\in J} y_j x_j & = & \bar{b} \\ & x_B & & \geq & 0 \\ & & x_j & \geq & 0, \quad j \in J. \end{array} \tag{6}$$

Let $x_B^T = (x_{B_1}, x_{B_2}, \ldots, x_{B_m})$ then we can re-write (6) as follows:

$$\begin{array}{lllll} \max & c_B^T B^{-1}b & -\sum_{j\in J}(z_j - c_j)x_j & & \\ \text{s.t.} & x_{B_i} & +\sum_{j\in J} y_j x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\ & x_B & & \geq & 0 \\ & & x_j & \geq & 0, \quad j \in J. \end{array} \tag{7}$$

Therefore we can define the simplex iteration easily. Let $k \in J$ be an index s.t. $z_k - c_k < 0$. Obviously we can increase x_k as much as possible to profit from the situation (we keep all other non-basic variables at 0, and increase x_k). However how much should we increase x_k?

Let $y_{ik} > 0$, we can re-write ith row of our problem as follows,

$$x_{B_i} + y_{ik}x_k = \bar{b}_i$$

As $y_{ik} > 0$, $x_k \geq 0$ and $\bar{b}_i \geq 0$, we can increase x_k to at most $\frac{\bar{b}_i}{y_{ik}}$. Because at that point $x_{B_i} = 0$ (If $y_{ik} \leq 0$ it is easy to see we can increase x_k as much as we want without violating the non-negativity of x_{B_i}, VERIFY IT!). However we want the non-negativity of all x_B. Clearly implying we should select $x_k = \frac{\bar{b}_r}{y_{rk}} = \min_{1 \leq i \leq m} \{\frac{\bar{b}_i}{y_{ik}} | y_{ik} > 0\} > 0$. Therefore $x_{B_r} = 0$ and $x_k = \frac{\bar{b}_r}{y_{rk}}$ defines a new BFS, we update the current basis by removing the column a_r replaced by a_k and continue with the next iteration of simplex.