Vladimir Gurvich

Triangle inequality for resistances

In the Moscow Institute of Physics and Technology they teach five semesters of general physics. The third one is Electricity. So, in December of 1970, I had to take the exam. Traditionally, it begins with a topic chosen by the student. I chose to compute the resistance of a n-node network that contains $\binom{n}{2}$ passive resistances, only. From high-school, I was sure that there should be a formula. Now, I could derive it, having the concept of a determinant in my possession.

It was not too difficult. Just take the $n \times n$ matrix L whose entry $\lambda_{i,j}$ is the conductance of the edge $e_{i,j}$ between the nodes i and j (if there is none, $\lambda_{i,j} = 0$) and $\lambda_{i,i}$ is the sum of all conductances in the row (or column) i.

The rest is simple: Eliminate the row i and column j (respectively, two rows and two columns i and j) from L and denote the obtained matrix by $L'_{i,j}$ (respectively, by $L''_{i,j}$). Then the effective conductance $\Lambda_{i,j}$ between the poles i and j is just the ratio of the corresponding two determinants, more precisely,

$$\Lambda_{i,j} = |\det L'_{i,j} / \det L''_{i,j}|. \tag{1}$$

I still do not know who was the first to derive this formula; perhaps, Kirchhoff knew it. Yet felt that I did more than they do during the 8th year of the high-school computing "infinitely" many examples of the series-parallel networks. But will it be enough to impress the examiner? I was not completely sure and decided to check the triangle inequality for the corresponding resistances:

$$\Lambda_{i,k}^{-1} + \Lambda_{k,j}^{-1} \geq \Lambda_{i,j}^{-1} \ \ \forall \ i, j, k. \tag{2}$$

To my pleasant surprise, (2) indeed followed from (1), but not that easily. It took several hours and pages to derive it.

Next morning I reported, with pride, all this to an examiner. But, to my unpleasant surprise, he was not happy. Not at all.
   - What devices are in your network?
   - Only passive resistances.
   - And this is all?! No capacitors, launching coils, or generators?!
   - No, nothing like that...
   - How dare you to come with this?! We taught you Maxwell equations... And here... Resistances!

I shrugged off this question. He was pretty loud, so another examiner came to us and saved me from almost certain failure. I vaguely recall that his name was Kaloshnikov and he was teaching PhisLabs.
   - What is all this noise about?
   My examiner explained.
   - So what? I find all this curious. Let me finish the exam with this student...

In fact, they demonstrated two possible reactions to resistance distances that are still typical for physicists and electrical engineers.

I should confess that I did not give much importance to this result either and did not publish it until 1987. Thenadays, I worked for the Schmidt Institute of Physics of the Earth of the Soviet Academy of Sciences, in the Laboratory of Math. Methods of Earthquakes Prediction directed by Alexei Gvishiani. He was graduated in 1975 from Mech.-Mat. MSU, where studied functional analysis advised by Israel Gelfand and Alexander Kirillov. (Also, he is the grandson of the Soviet prime minister, Alexej Kosygin, 1904–1980). Although network resistances are pretty far from earthquakes, yet, not farer than the functional analysis. So Gvishiani and I decided to generalize and publish the result. We replaced the classical Ohm law $y_e^* = y_e/\mu_e$ by the monomial conductivity law

$$y_e^* = y_e^r/\mu_e^s, \tag{3}$$

where e is a conductor of the network, y_e is the tension (or the potential drop) on e and y_e^* is the current through e, furthermore, μ_e is the resistance of e, while r and s are real strictly positive parameters that do not depend on e.

For example, the quadratic law, $r = 1/2$, is typical for hydraulics or gas dynamics, where y_e is the pressure drop and y_e^* is the flow through the pipe e.

It appears that the effective resistance $\mu_{i,j}$ is well defined for any two nodes i, j, and for every three nodes i, j, k the following metric inequality holds:

$$\mu_{i,k}^{s/r} + \mu_{k,j}^{s/r} \geq \mu_{i,j}^{s/r} \ \ \forall \, i, j, k. \tag{4}$$

Interestingly, there are several proofs for the linear conductivity law, $r = 1$. Yet, I know only one general proof, but it is certainly the simplest one.

Let us fix potentials x_i and x_j in the nodes i and j. Without any loss of generality, assume that $x_i \geq x_j$. Then, some current $y_{i,j}^*$ will flow from i to j and some potential x_k in the node k will appear such that $x_i \geq x_k \geq x_j$.

Now, let us similarly fix x_i and x_k and determine the flow $y_{i,k}^*$. All we have to prove is the inequality $y_{i,j}^* \geq y_{i,k}^*$. And this is not difficult. Then, by symmetry, we have $y_{i,j}^* \geq y_{j,k}^*$. Finally, any good high-school student could derive the triangle inequality from the last two as follows.

$$y_{i,j}^* = \frac{(x_i - x_j)^r}{\mu_{i,j}^s} \geq \frac{(x_i - x_k)^r}{\mu_{i,k}^s} = y_{i,k}^*; \quad y_{i,j}^* = \frac{(x_i - x_j)^r}{\mu_{i,j}^s} \geq \frac{(x_j - x_k)^r}{\mu_{j,k}^s} = y_{j,k}^*, \tag{5}$$

which can be obviously rewritten as

$$\left(\frac{\mu_{i,k}}{\mu_{i,j}}\right)^{s/r} \geq \frac{x_i - x_k}{x_i - x_j}; \quad \left(\frac{\mu_{k,j}}{\mu_{i,j}}\right)^{s/r} \geq \frac{x_k - x_j}{x_i - x_j}. \tag{6}$$

Summing up these two inequalities one gets exactly (4).

It is also easy to show that (4) holds with equality iff $y_{i,j}^* = y_{i,k}^* = y_{k,j}^*$, which happens iff k belongs to every path between i and j.

We chose the journal of "Functional Analysis and its Applications", founded and led by Israel Gelfand. I still believe that the metric inequality for resistances does belong to the functional analysis. Moreover, as far as applications are concerned, our paper was just fine, unlike most of the papers in this journal.

However, Gelfand was not too happy. I can only guess about his reasons. Perhaps, in contrast to my physics-examiner, who decided that there are no serious "devices" and (2) is too "mathematical", Gelfand decided that there are too many "devices" and (4) is not enough "functionally analytic". Anyway, he suggested to withdraw the paper from "Functional Analysis" and resubmit it to the "Communications of the Moscow Math. Soc. in Russian Math. Surveys." This was a good journal and a reasonable compromise.

It would be natural for Gelfand to call Gvishiani, his former student, and tell him the verdict. Yet, he chose a more sophisticated way. I got a call from Andrei Zelevinsky, my former classmate and another Gelfand's student (perhaps, the best one; now he is a famous mathematician, the inventor of Cluster Algebras). I asked Andrei what is wrong with submitting to the "Functional Analysis" and got a pretty curious answer. He said that it is one a few journals that have its peculiar style and submitting our paper there is like coming in a ski suit (although in a very good one) to a banquet, where all other wear fracs.

Anyway, the offer was decent and it was accepted. The only disadvantage (but quite a serious one) was that I had to squeeze the eight pages long paper to just two pages, because the "Communications of the Moscow Math. Soc." were (and still are) strictly limited to this size.

In any case, the paper was never cited before I republished the more detailed (10 pages) version of it in the "Discrete Applied Mathematics", in 2010.

It was recently discovered by Pavel Chebotarev that the metric inequality, for $r = s = 1$, was published much earlier, in 1967, by Gerald E. Subak-Sharpe, a Professor of Electrical Engineering from New York City. Till 2012 his work was not cited either.