Open problems and conjectures

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1 There is a unique friendship two-graph?

Definition 1 *A friendship graph is a graph in which every two distinct vertices have exactly one common neighbor.*

These graphs were characterized by Erdős, Rényi, and Sós [40] as follows: a friendship graph consists of triangles incident to a common vertex.

Kotzig generalized friendship graphs to graphs in which every pair of vertices is connected by λ paths of length k. He conjectures that, for $k \geq 3$, there is no finite graph in which every pair of vertices is connected by a unique path; see also Bondy [15] and Kostochka [60].

The concept of friendship graphs can be naturally extended as follows.

Definition 2 *A two-graph is an ordered pair $G = (G_0, G_1)$ of edge-disjoint graphs G_0 and G_1 on the same vertex-set $V(G_0) = V(G_1) = V$; in other words, the edges of G are colored with colors 0 and 1. In a* friendship two-graph*, every unordered pair of distinct vertices u, v is connected by a unique bicolored 2-path.*

Remark 1 *It is easily seen that the pairs of adjacency matrices of friendship two-graphs are solutions to the matrix equation $AB + BA = J - I$, where A and B are $n \times n$ symmetric $0-1$ matrices, J is an $n \times n$ matrix whose every entry is 1, and I is the identity $n \times n$ matrix.*

Interestingly, a somewhat similar matrix equation $AB = J - I$ characterizes so-called partitionable graphs; see [29] for the definition and more details.

A friendship two-graph F on seven vertices is given in Figure 1.

Conjecture 1 *There exist no other finite friendship two-graphs.*

This conjecture is shown in [25] for the two-graphs with a dominating vertex. (For example, 7 is a dominating vertex in F.) It is also shown in [25] that there is no finite friendship two-graph with the minimum vertex degree at most two. Yet, an uncountable family of *infinite* friendship two graphs on a countable vertex-set is constructed in [25].

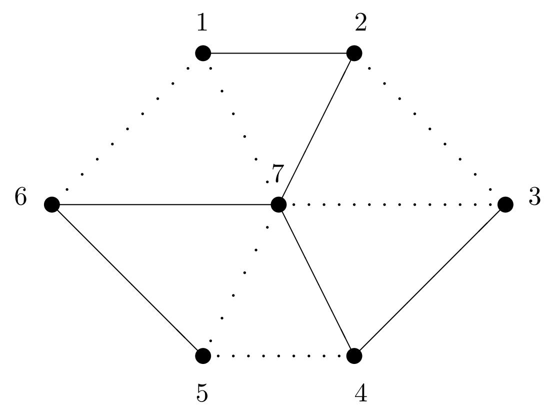

Figure 1: A (unique?) finite friendship two-graph F.

Remark 2 *Somewhat similar situation holds for the so-called biconnected graphs and complementary connected d-graphs; see Section 3 and also [22, 52, 57] for the definitions and more details.*

There is a unique finite biconnected graph P_4 and two finite complementary connected d-graphs Π and Δ; yet, there are infinite (although countable) families of infinite biconnected graphs and complementary connected d-graphs.

2 Set-difference graphs

Intersection and measured intersection graphs are common in the literature; see for example, surveys [46, 47], and [28], chapter 4. A similar concept of measured set-difference graphs was introduced in [24]. Given a hypergraph $\mathcal{H} = \{H_1, \ldots, H_m\}$, let us assign to it a graph $G = G(\mathcal{H})$ on the vertex-set $[m] = \{1, \ldots, m\}$ in which (i, j) is an edge if and only if the corresponding sets H_i and H_j are "sufficiently different". More precisely, for any integer positive threshold k, we introduce three graphs $G_{\mathcal{H}}^{avg,k}, G_{\mathcal{H}}^{\max,k}$, and $G_{\mathcal{H}}^{\min,k}$, in which (i, j) is an edge, respectively, if and only if:

(avg) $|H_i \setminus H_j| + |H_j \setminus H_i| \geq 2k$;

(max) $\max(|H_i \setminus H_j|, |H_j \setminus H_i|) \geq k$;

(min) $\min(|H_i \setminus H_j|, |H_j \setminus H_i|) \geq k$,

For example, let hypergraph $\mathcal{H} = \binom{[5]}{2}$ consists of all, $m = 10$, subsets of cardinality 2 from a ground set of cardinality 5. It is easy to verify that if $k = 2$ then all three graphs defined above are isomorphic to the Petersen graph.

Then, by definition, each of the above three graph classes is hereditary and monotone increasing with respect to k. It is shown in [24] that every graph G can be realized by a hypergraph in all three cases if k is sufficiently large.

In the first two cases (avg and max) $k = \Omega(\log m)$ is needed [24]. Yet, somewhat surprisingly, in the last case (min) we know no graph which could not be realized by a hypergraph with $k = 2$.

Conjecture 2 *For every graph G there is a hypergraph $\mathcal{H}$ such that $G = G_{\mathcal{H}}^{\min,2}$.*

Let us remark that $k = 1$ would not suffice. Indeed, $G = G_{\mathcal{H}}^{\min,1}$ for some $\mathcal{H}$ if and only if G is a co-comparability graph; see [28] for the definitions.

Given a graph $G = (V, E)$, let $k^\rho(G)$ denote the smallest k for which G is (ρ, k)-realizable, where $\rho \in \{avg, \max, \min\}$.

Proposition 1 *([24]) For every G and ρ we have $k^\rho(G) \leq |V| - 1$.*

Let $k^\rho(m)$ denote the maximum of $k^\rho(G)$ taken over all simple graphs with m vertices. It was shown in [24] that $k^\rho(m) = \Omega(m)$ for $\rho \in \{avg, \max\}$.

More precisely, let $\mathcal{H} = 2^X$ denote the family of all distinct subsets from a given set X of cardinality $|X| = 2k$. It is not difficult to see that $G_{\mathcal{H}}^{avg,k} = 2^{2k-1}K_2$ is a matching that consists of 2^{2k} vertices and 2^{2k-1} edges.

Similarly, let $\mathcal{H} = \binom{X}{k}$ denote the hypergraph of all subsets of cardinality k from a ground set X of cardinality $2k$. It is not difficult to see that $G_{\mathcal{H}}^{\max,k} = \frac{1}{2}\binom{2k}{k}K_2$ is a matching that consists of $\binom{2k}{k}$ vertices and $\frac{1}{2}\binom{2k}{k}$ edges. Another way to realize the same matching is by the hypergraph $\mathcal{H}$ that consists of all subsets of cardinality k or $k-1$ from a ground set of cardinality $2k-1$.

Conjecture 3 *All three above constructions are extremal: the first one for $\rho = avg$ and the last two for $\rho = \max$.*

Partial results in this direction are obtained in [24].

Theorem 1 *Let $G = tK_2$ be the matching that consists of t edges.*
If G is (avg, k)-realizable then $t = O(k^2 2^{2k})$.
If G is $(\max, k)$-realizable then $t = O(k^2\binom{3k}{k})$.

Both above results are based on an inequality proven by Füredi [41] and generalizing an earlier result by Bollobas [14].

3 CIS graphs and d-graphs

3.1 CIS-graphs

Definition 3 *We say that graph G is a CIS-graph, or that it has the CIS-property, if every maximal clique C and every maximal stable set S of G has a common vertex. (Abbreviation CIS stands for "Cliques intersect Stable Sets".)*

Problem 1 *Characterize CIS-graphs.*

By definition, G is a CIS-graph if and only if the complementary graph $\overline{G}$ is a CIS-graph. Furthermore, let us substitute a vertex v of a graph G' by a graph G'' and denote the obtained graph by G. It is easy to see that G is a CIS-graph if and only if both G' and G'' are CIS-graphs. In other words, the CIS-property is closed under complementation and exactly closed under substitution. Yet, it is not hereditary.

In 1980s Claude Berge noticed that in a CIS-graph every induced P_4

$(v_1, v_2), (v_2, v_3), (v_3, v_4)$

must be extendable to an induced bull-graph (sometimes called also A-graph)

$(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_0, v_2), (v_0, v_3)$.

Unlike the former, the latter is a CIS-graph. Thus, CIS-property is *not* hereditary. For this reason, CIS-graphs cannot be characterized in terms of forbidden induced subgraphs.

Proposition 2 *Any graph is an induced subgraph of a CIS-graph.*

Proof. Given a graph G_0, let us add an individual simplicial vertex to every its maximal clique. Obviously, the obtained graph G contains G_0 and it is easy to verify that G has the CIS-property. □

Let us notice, yet, that the size of G might be exponential in the size of G_0.

Problem 1 is difficult, perhaps, because the CIS-property is not hereditary. However, some necessary and some sufficient conditions are known.

Definition 4 *Given an integer $k \geq 2$, a* comb G_k *is a graph with $2k$ vertices k of which, $v_1, \ldots, v_k$, form a clique C, the remaining k, $v'_1, \ldots, v'_k$, form a stable set S, and (v_i, v'_i) is an edge for all $i \in [k] = \{1, \ldots, k\}$; furthermore, there are no other edges. The complementary graph $\overline{G}_k$ is called an* anti-comb.

Clearly, S and C switch in the complementary graphs.

Obviously, the combs and anti-combs are not CIS-graphs, since $C \cap S = \emptyset$.

Hence, if a CIS-graph G has an induced comb or anti-comb then it must be settled, that is, G must contain a vertex v_0 connected to all vertices of C and to no vertex of S. In particular, the 2-comb and 2-anti-combs are both isomorphic to P_4. Hence, they must be settled, as Claude Berge noticed.

However, all these conditions are only necessary but not sufficient.

The following graph was suggested by Ron Holzman in 1994. It has $\binom{5}{1} + \binom{5}{2}$ $= 5 + 10 = 15$ vertices, where subsets $S = \{v_1, \ldots, v_5\}$ and $C = \{v_{12}, \ldots, v_{45}\}$ induce a stable set and clique, respectively; $V = C \cup S$ (hence, G is a split graph); furthermore, every pair (v_i, v_{ij}), where $i, j = 1, \ldots, 5$ and $i \neq j$, is an edge, and there are no more edges. Let us denote this graph by $G(5, 1, 2)$

It is not difficult to verify that $G(5, 1, 2)$ contains no induced 5-combs and 4-anti-combs. Furthermore, all induced combs and anti-combs in $G(5, 1, 2)$ are settled. For example, the 4-comb induced by vertices $(v_{12}, v_{13}, v_{14}, v_{15}, v_2, v_3, v_4, v_5)$ is settled by v_1 and the 3-anti-comb induced by $(v_{12}, v_{13}, v_{23}, v_1, v_2, v_3)$ is settled by v_{45}, etc. However, $G(5, 1, 2)$ is not a CIS-graph, since $C \cap S = \emptyset$.

Conjecture 4 *If a graph G contains no induced $G(5, 1, 2)$ or its complement and every induced comb and anti-comb is settled in G then G is a CIS–graph.*

Sufficient for the CIS-property conditions are given by the next theorem.

Theorem 2 *If all 2-combs are settled in G and it has no 3-combs and 3-anti-combs then G is a CIS-graph.*

In early 1990s, this claim was suggested as a conjecture by Vasek Chvátal to his student from RUTCOR, Wenan Zang, who published first partial results in [73]. The problem was finally solved in [36, 37] and independently in [4].

One could try to relax the sufficient conditions of Theorem 2 as follows.

Let us say that graph G satisfies the property $\mathbf{comb}(i, j)$ if G contains no i-combs and j-anti-combs and all combs and anti-combs of G are settled.

Then, Theorem 2 claims that $\mathbf{comb}(3, 3)$ implies the CIS-property, while Holzman's example shows that $\mathbf{comb}(5, 4)$ does not.

Problem 2 *The cases of $\mathbf{comb}(4, 4)$ and $\mathbf{comb}(i, 3)$ for $i > 3$ remain open.*

3.2 Almost CIS-graphs

Definition 5 *An* almost CIS-graph *has a unique pair (C, S) of disjoint maximal clique C and maximal stable sets S. It is called the non-CIS pair.*

Proposition 3 *Every split graph has at most one non-CIS pair.*

Proof. Let (A, B) be a split partition of a split graph G, where A is a clique and B is a stable set. Obviously, a maximal clique C distinct from A consists of a proper subset of A and one vertex $u \in B$; respectively, a maximal stable set S distinct from B consists of a proper subset of B and one vertex $v \in A$. It is easy to see that $C \cap S = \{u\}$ if u and v are non-adjacent, and $C \cap S = \{v\}$ otherwise. $\Box$

In other words, every split graph is either CIS or almost CIS. The next claim shows when the first option takes place.

Proposition 4 *A split graph G has more than one split partition if and only if G is a CIS-graph.*

Proof. Let $A \cup B$ be a split partition of G. By Proposition 3, (A, B) is the only possible non-CIS-pair (C, S) in G. If, indeed, (A, B) is such a pair then G is an almost CIS-graph, by the definition. If not, then either clique A or stable set B is not maximal. In this case G is a CIS-graph. $\square$

Thus, every split graphs with a unique split partition is an almost CIS-graph.

The above definition and two simple propositions were given in [4], where it was also conjectured that the inverse claim holds too, that is, almost CIS-graphs are exactly split graphs that have a unique split partition.

Partial results were obtained in [26] and the conjecture was proved in [72].

3.3 CIS-d-graphs and Δ-conjecture

Definition 6 *A d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ is a complete graph whose edges are arbitrarily partitioned into d subsets (colored with d colors). Graph $G_i = (V, E_i)$ is called the ith chromatic component of $\mathcal{G}$, where $i \in [d] = \{1, \ldots, d\}$.*

In case $d = 2$ a d-graph is just a graph, or more precisely, a pair: a graph and its complement. Thus, d-graphs can be viewed as a generalization of graphs.

Choose a maximal independent set $S_i \subseteq V$ in every graph G_i and denote by $\mathcal{S} = \{S_i \mid i \in [d]\}$ the obtained set-family; furthermore, let $S = \bigcap_{i=1}^{d} S_i$. Obviously, $|S| \leq 1$ for every $\mathcal{S}$; indeed, if $v, v' \in S$ then $(v, v') \notin E_i$ for all $i \in [d]$, that is, this edge has no color.

Definition 7 *We say that $\mathcal{G}$ has the CIS property and call $\mathcal{G}$ a CIS d-graph if $S \neq \emptyset$ for every family $\mathcal{S} = \{S_i \mid i \in [d]\}$ of maximal independent sets.*

Δ-Conjecture No CIS-d-graph contains a triangle colored by three pairwise distinct colors.

This conjecture was suggested in 1978 [[51], p. 71, remark after Claim 17]. It is trivial for $d = 2$. In 1982 Andrei Gol'berg noticed that general case can be reduced to the case $d = 3$. This and other partial results are surveyed in [4].

Problem 3 *Characterize CIS-d-graphs.*

It is shown in [4] that, modulo Δ-conjecture, this problem is reduced to Problem 1. However, the latter is sufficiently difficult itself.

4 Partitionable graphs

In this section, we recall several conjectures from [3, 7, 23].

4.1 Equivalent definitions

The results of Lovász [64] and Padberg [69] yield certain properties of minimally imperfect graphs. Following the paper by Bland, Huang and Trotter [13], for integers $\alpha, \omega \geq 2$, we say that a graph $G = (V, E)$ is (α, ω)-*partitionable* if for every vertex $v \in V$ the induced subgraph $G[V \setminus \{v\}]$ admits a partition into α cliques of cardinality ω and also admits a partition into ω stable sets of cardinality α. It is easy to see that α must be equal to the maximum cardinality of a stable set in G, and similarly that ω is the maximum cardinality of a clique in G. According to [13] (α, ω)-partitionable graphs have the following properties:

(i) G has exactly $n = \alpha\omega + 1$ vertices;

(ii) G has exactly n maximum stable sets of cardinality α;

(iii) G has exactly n maximum cliques of cardinality ω;

(iv) Each maximum clique meets exactly $n - 1$ of the maximum stable sets (and misses exactly one);

(v) Each maximum stable set meets exactly $n - 1$ of the maximum cliques (and misses exactly one);

(vi) Each vertex belongs to exactly ω maximum cliques;

(vii) Each vertex belongs to exactly α maximum stable sets.

With this definition, results of Lovász and Padberg simply say that every minimally imperfect graph with maximum clique size ω and stability number α is (α, ω)-partitionable. The converse is not true: there exist infinitely many partitionable graphs that are not minimally imperfect.

It was shown in [23] that in fact all properties (i-vii) result from the following single axiom.

A set-family $\mathcal{C}$ on the vertex-set V is called *partitionable* if $|\mathcal{C}| \leq |V|$ and for every $v \in V$ the difference $V \setminus \{v\}$ is a union of some pairwise disjoint sets of $\mathcal{C}$.

Then $\mathcal{C}$ is the family of $n = \alpha\omega + 1$ maximum cliques of cardinality α. Furthermore, for every $C \in \mathcal{C}$, the unique vis-a-vis stable set $S = S(C)$ consists of all vertices $v \in V$ such that C participates in a (unique) partition of $V \setminus \{v\}$ by the sets of $\mathcal{C}$.

4.2 Indifferent pairs

It is important to notice that a given (α, ω)-partitionable graph G may admit a pair of vertices that lies neither in a maximum clique nor in a maximum stable set of G. We call such a pair *indifferent*. As observed in [35], adding to or removing from the edge set of G any choice of indifferent pairs yields another (α, ω)-partitionable graph having the same maximum cliques and maximum stable sets as G. We call any such graph a *variant* of G. Since the properties studied and used in this section will always be based not on mere adjacency but on the arrangement of maximum cliques and maximum stable sets in a partitionable graph, the corresponding results will hold true for any variant of the graph.

4.3 Small transversal

Let us say that a subset of vertices T is a *transversal* if T meets every maximum-size clique and every maximum-size stable set in G. Let us say that a (transversal) subset of vertices is *small* if its cardinality is at most $\alpha + \omega - 1$. Chvátal [34] proved:

Lemma 1 *A minimally imperfect graph contains no small transversal.* $\square$

We present a criterion which enables us to find a small transversal in many partitionable graphs. Let us say that a maximum clique C of G *covers* a vertex $x \in C$ if and every maximum clique C' containing x satisfies $|C' \cap C| \geq 2$. Similarly, a maximum stable set S covers a vertex $x \in S$ if every maximum stable set S' containing x satisfies $|S' \cap S| \geq 2$.

Lemma 2 *([7]). If a vertex of a partitionable (α, ω)-graph is covered by both a α-clique and ω-stable set then this graph has a small transversal.*

Proof Let G be a partitionable graph and x be a vertex of G covered by some C and S. Let S' be the unique maximum stable set disjoint from C and C' be the unique maximum clique disjoint from S. Obviously $C' \neq C$ and $S' \neq S$. So S' and K' have a common element y, or else S' would be disjoint from two maximum cliques C' and C. Now we claim that $T = S \cup C \cup \{y\} \setminus \{x\}$ is a small transversal. It is clear that T has cardinality $\alpha + \omega - 1$. Moreover, $S \setminus \{x\}$ meets every maximum clique of G except C' and the maximum cliques containing x. However, C' is met by y, and a maximum clique containing x is met by $C \setminus \{x\}$, since C is a mother of x. So, T meets every maximum clique of G and similarly every maximum stable set. $\square$

4.4 Partitionable circulants

Now let us recall a subfamily of partitionable graphs introduced in [35].

For any two sets of integers X, Y, let $X + Y$ denote the set $\{x + y \mid x \in X,\ y \in Y\}$. If $X = \{x\}$ we will often write $x + Y$ instead of $\{x\} + Y$.

Let α and ω be two integers greater than or equal to two. Given factorizations $\omega = m_1 m_3 \cdots m_{2k-1}$ and $\alpha = m_2 m_4 \cdots m_{2k}$, where each factor is a positive integer $m_i \geq 2$, we consider the Chvátal-Graham-Perold-Whitesides graph defined as follows. First write:

$$\begin{aligned}
\mu_i &= m_1 m_2 \cdots m_i \quad (\mu_0 = 1), \\
M_i &= \{0, \mu_{i-1}, 2\mu_{i-1}, \ldots, (m_i - 1)\mu_{i-1}\}, \\
C &= M_1 + M_3 + \cdots + M_{2k-1}, \\
S &= M_2 + M_4 + \cdots + M_{2k}, \\
n &= m_1 m_2 \cdots m_{2k} + 1.
\end{aligned}$$

Then let x_C (resp. x_S) be the n-dimensional characteristic vector of C (resp. S) with respect to the set $\{0, 1, 2, \ldots, n-1\}$. Let A_C (resp. A_S) be the matrix whose rows are the n possible circular permutations of x_C (resp. x_S). There exists an (α, ω)-partitionable graph whose ω-clique matrix is A_C and whose α-stable set matrix is A_S. Such a graph is obtained by taking as vertices the elements $0, \ldots, n-1$ of the cyclic group $\mathbb{Z}_n$ and adding an edge xy whenever the difference $x - y$ modulo n is equal to the difference of two elements in C. This graph will be denoted by $C[m_1, m_2, \ldots, m_{2k}]$. Such graph and their variants will be called CGPW graphs. The variant in which no indifferent pair is an edge is called the *normalized* variant.

Theorem 3 *No CGPW graph with $\alpha > 2$ and $\omega > 2$ is minimally imperfect.*

It was proven in [7]. For the normalized CGPW graphs the result was obtained earlier by Grinstead [45], who proved that the normalized variant of any CGPW graph contains an odd hole or an odd antihole. However, these arguments do not extend the general case, because of the indifferent pairs. The proof in [7] uses only the arrangements of maximum cliques and maximum stable sets and so is true for all variants, but it does not exhibit a minimally imperfect proper subgraph. Instead, it proves the existence of a small transversal.

In order to prove Theorem 3, it appears necessary to distinguish between three types of CGPW graphs. More precisely, let us consider two properties:

$$m_1 = m_3 = \cdots = m_{2k-1} = 2, \tag{1}$$

$$m_2 = m_4 = \ldots = m_{2k} = 2. \tag{2}$$

Let us say that a CGPW graph is of Type 1 if it satisfies none of (1) and (2), of Type 2 if it satisfies exactly one of them, and of Type 3 if it satisfies both.

In [7], Theorem 3 was derived from the next three propositions.

Proposition 1 ([59]) *If G is a CGPW graph of Type 1 with $\alpha \geq 3$ and $\omega \geq 3$ then G admits a small transversal.* $\square$

Proposition 2 *If G is a CGPW graph of Type 2 with $\alpha \geq 3$ and $\omega \geq 3$ then G admits a small transversal.*

Proposition 3 *If G is a CGPW graph of Type 3 then G has an induced C_5.*

Conjecture 5 *Every partitionable circulant is a CGPW graph.*

This Conjecture is implicit in the paper by N.G. de Bruijn (1956) [32] on near-factorizations of cyclic groups; see also [35] and [33]. In [7], it was proven for the case $\min(\alpha, \omega) \leq 5$.

It is easy to see that the CGPW graph $C[m_1, m_2, \ldots, m_{2k}]$ is an odd hole or odd anti-hole when $k = 1$ and $m_1 = 2$ or $m_2 = 2$ respectively, that is, $C[2, m] = C_{2m+1}$ is the $2m+1$-hole and $C[m, 2]$ is the corresponding anti-hole. Being minimal imperfect, they have no small transversals. It is also obvious that they have no indifferent pairs. Interestingly, graph $C[2, 2, 2, 2]$ shares these properties: it has neither small transversals [35, 59] nor indifferent pairs.

Conjecture 6 *([59, 7]) Any other partitionable graph (distinct from $C[2, m]$, $C[m, 2]$, and $C[2, 2, 2, 2]$) has both (an indifferent pair and a small transversal).*

Graph $C[2, 2, 2, 2]$ was discovered in [35] and rediscovered in [59]. It is a circulant (that is, has circular symmetry) with 17 vertices $V = \{0, 1, \ldots, 16\}$; pair (i, j) is an edge if and only if $i - j \ (mod\ 17) \in \{2, 6, 7, 8, 9, 10, 11, 15\}$, that is $C[2, 2, 2, 2] = C_{17}(2, 6, 7, 8)$. It is isomorphic to its complement $C_{17}(1, 3, 4, 5)$ and, hence, has no indifferent pairs. It is not difficult to verify that it has no small transversal either, yet, there are $2 \times 17 = 34$ five-holes.

4.5 Critical cliques

The following conjecture was suggested in [23].

An ω-clique C of a partitionable graph G meets at least $2\omega - 2$ other ω-cliques; C is called *critical* if equality holds.

It is shown in [23] that if G has a critical clique then it can be reduced to a smaller partitionable graph G'; in particular, G cannot be minimal imperfect. For example, $C[2, 2, 2, 2]$ is a partitionable $(4, 4)$-graph without critical cliques.

Conjecture 7 *([7]) Any $(\alpha, 3)$-graph has a critical clique.*

It was verified for $\alpha \leq 10$; see [23].

4.6 Even-hole-free circulants

Given non-negative integer k and m, let us introduce a graph $G(k; m) = (V; E)$ with circular symmetry as follows: $V = \mathbb{Z}_n = \{1, \ldots, n\}$, where $n = k(2m+1)$, and $(i; j) \in E$ iff $i - j + t(2m + 1) = 0$ or $+1$ or $-1 \mod n$ for some integer t. For convenience, the loops $i = j$ are included.

For exampple. if $k = 5, m = 2$ then $n = 25$ and $(i; j) \in E$ if and only if

$i - j (mod\ 25) \in \{4; 5; 6; 9; 10; 11; 14; 15; 16; 19; 20; 21; 24; 0; 1\}$.

It is not difficult to check that graph $G(k; m)$ has no even hole (in fact, it can only have holes of length $2m + 1$); furthermore,

$\omega(G(k; m)) = 2k;\ \ 2k + \lceil k/m \rceil \leq \chi(G(k; m)) \leq 2k + \lceil k/m \rceil + 1.$

Conjecture 8 *([3]). Each even-hole-free circulant is isomorphic to a $G(k; m)$.*

4.7 Perfect, partitionable, and kernel-solvable graphs

Given a graph $G = (V, E)$, let us assign to every its edge $e = (u, v)$ either the directed arc $[u, v)$, or $[v, u)$, or both. The obtained directed multi-graph $D = (V, A)$ is called an orientation of G.

A vertex-subset $K \subseteq V$ is called a *kernel* if K is

(i) *independent* and (ii) *absorbant*,

that is, for each u from $V \setminus K$ there is an arc $[u, v) \in A$ such that $v \in K$.

An orientation D is called clique-acyclic if every clique of G has a kernel in D. Orientation D is called kernel-less if it has no kernel. Graph G is called kernel-solvable if every its clique-acyclic orientation has a kernel. Berge and Duchet (1983) conjectured that:

(BD1) Perfect graphs are kernel-solvable and

(BD2) Kernel-solvable graphs are perfect.

BD1 was proved in [20]; see also [1]. BD2 follows from the Strong Perfect Graph Theorem but no independent proof is known.

An orientation D of a *partitionable* graph G is called *uniform* if

- D is kernel-less and clique acyclic;
- for each maximum stable set S there is a unique unabsorbed vertex $v(S)$;
- vertex $v(S)$ belongs to the vis-a-vis clique $C(S)$ of S;
- for each vertex v there exists a unique maximal stable set $S(v)$ which does not absorb v.

In 1998, Sebo proved that every kernel-less and clique-acyclic orientation of a minimal imperfect graph is uniform.

Conjecture 9 *Each partitionable graph has a uniform orientation.*

This, if true, implies BD2.

4.8 Edge-minimal and locally minimal kernel-less digraphs

Obviously, an even directed cycle (dicycle) has two kernels formed by the even and odd vertices, respectively, while an odd dicycle has no kernel. This simple observation can be generalized as follows. Let $k(D)$ denote the number of kernels in a digraph D.

Theorem 4

- *(i) $k(D) \leq 1$ when all dicycles in D are odd;*
- *(ii) $k(D) \geq 1$ when all dicycles in D are even;*
- *(iii) $k(D) = 1$ when D has no dicycles.*

Claim (i) is an easy exercise. Claim (ii) is the Richardson (1953) Theorem: an odd dicycle free digraph has a kernel, [70]. Claim (iii) is the von Neumann and Morgenstern (1944) Theorem: an acyclic digraph has a unique kernel. This claim easily implies that finite (acyclic) positional games with perfect information can be solved in pure positional uniformly optimal strategies [68]; see the next Section for more details. Obviously, claims (i) and (ii) imply (iii).

However, (iii) is simpler than (ii). The latter can be reformulated as follows: an arc-minimal kernel-less digraph is an odd dicycle plus ℓ isolated vertices, where $\ell \geq 0$.

In 1980, Pierre Duchet conjectured that not only all arc-minimal but also all locally arc-minimal digraphs have the same structure. In other words, any other kernel-less digraph (which is not an odd dicycle plus several isolated vertices, if an) contains an arc that can be deleted and the reduced digraph still remains kernel-less; [38], see also [9]. This statement, if true, would drastically strengthen the Richardson Theorem.

However, the directed circulant $G_{43}(1,7,8)$ is a counterexample [5].

In particular, it is shown in [5] that $G_n(1,7,8)$ has a kernel if and only if $n \equiv 0 \pmod 3$ or $(mod\, 29)$.

Problem 4 *Characterize the locally arc-minimal kernel-less directed circulants.*

The following radical relaxation of the Duchet conjecture is still open.

Conjecture 10 *No directeed circulant $G_n(i,j)$ can be a locally arc-minimal kernel-less digraph.*

5 Nash equilibria in pure positional strategies

5.1 Modeling positional games by directed graphs

Given a finite directed graph (digraph) $G = (V, E)$ in which loops and multiple arcs are allowed, a vertex $v \in V$ is a *position* and a directed edge (or arc) $e = (v, v') \in E$ is a *move* from v to v'. A position of out-degree 0 (that is, with no moves) is called a *terminal*. We denote by V_T the set of all terminals.

Let us also fix an *initial* position $v_0 \in V$. Furthermore, let us introduce a set of n players $I = \{1, \ldots, n\}$ and a partition $P : V = V_1 \cup \ldots \cup V_n \cup V_R \cup V_T$.

Each player $i \in I$ controls all positions in V_i, and V_R is the set of random positions, in which moves are not controlled by a players but by nature. For each $v \in V_R$ a probability distribution over the set of outgoing edges is fixed.

Let $C = C(G)$ denote the set of all simple directed cycles (dicycles) of G. For instance, a loop $c_v = (v, v)$ is a dicycle of length 1, and a pair of oppositely directed edges $e = (v, v')$ and $e' = (v', v)$ form a dicycle of length 2.

A directed path (dipath) p that begins in v_0 is called a *walk*. It is called a *play* if it ends in a terminal vertex $a \in V_T$, or it is infinite. Since the considered digraph G is finite, every infinite play contains infinitely repeated positions.

For example, it might consist of an initial part and a dicycle repeated infinitely. Finally, a walk is called a *debut* if it is a simple path, that is no vertex is repeated.

The interpretation of this model is standard. The game starts at $v = v_0$ and a walk is constructed as follows. The player who controls the endpoint v of the current walk can add to it a move $(v, w) \in E$. If $v \in V_R$ then a move $(v, w) \in E$ is chosen according to the given probability distribution. The walk can end in a terminal position or it can last infinitely. In both cases it results in a play.

5.2 Outcomes and terminal payoff

We will consider the **AIPFOOT games**, in which All Infinite Plays Form One Outcome (in addition to the Terminal outcomes), and will denote this special outcome by a_∞ or c. Thus, $A = V_T \cup \{a_\infty\}$ is the set of outcomes, while $V_T = \{a_1, \ldots, a_p\}$ is the set of terminal positions, or terminals, of G.

A *payoff or utility function* is a mapping $u : I \times A \to \mathbb{R}$ whose value $u(i, a)$ is standardly interpreted as a profit of player $i \in I$ in case of outcome $a \in A$.

A payoff is called *zero-sum* whenever $\sum_{i \in I} u(i, a) = 0$ for every $a \in A$.

The quadruple (G, P, v_0, u) will be called a *positional game*, and we call the triple (G, P, v_0) a positional game form.

Remark 3 *It is convenient to represent a game as a game form plus the payoffs. In fact, several structural properties of games, like existence of a NE, may hold for some families of game forms and all possible payoffs.*

Two-person zero-sum games are important. Chess and Backgammon are two well-known examples. In both, every infinite play is defined as a draw.

Another important special case is provided by the n-person games in which the infinite outcome a_∞ is the *worst* for all players $i \in I$. These games will be called the AIPFOOW games. They were introduced in [21] in a more general setting of additive payoffs, which is a generalization of the terminal case.

Example. Somebody from a family I should clean the house. Whenever $i \in I$ makes a terminal move, it means that (s)he has agreed to do the work. Although such a move is less attractive for i than for $I \setminus \{i\}$, yet, an infinite play means that the house will not be cleaned, which is unacceptable for everybody.

Remark 4 *In absence of random moves, the values $u_i = u(i, *)$ are irrelevant, only the corresponding pseudo-orders $\succ_i$ over A matter. Moreover, in this case, ties can be eliminated, without any loss of generality. In other words, we can assume that $\succ_i$ is a complete order over A and call it the* preference *of the player $i \in I$ over A. The set of n such preferences is called the* preference profile. *However, in presence of random moves, the values $u(i, a)$ matter, since their probabilistic combinations will be compared.*

5.3 Pure, Positional, and Stationary Strategies

A *pure strategy* x_i of a player $i \in I$ is a mapping assigning a move $e = (v, v') \in E$ to each walk that starts in v_0 and ends in v provided $v \in V_i$. In other words, it is a "general *plan*" of player i for the whole game.

A strategy x_i is called *stationary* if every time the walk ends at vertex $v \in V_i$, player i chooses the same moved. Finally, strategy x_i is called *positional* if for each $v \in V_i$ the chosen move depends only on this position v, not on the previous positions and/or moves of the walk. By definition, all positional strategies are stationary and all stationary strategies are pure.

Let us note also that when all players are restricted to their pure positional strategies, the resulting play will consist of an initial part (if any) and a simple dicycle repeated infinitely. This dicycle appears after a position is repeated.

We will restrict ourselves to pure positional strategies. Why to do so? This needs a motivation. The simplest answer is "why not?" or, to state it more politely, why to apply more sophisticated strategies in cases when positional positional strategies would suffice?

In 1950, Nash introduced his concept of equilibrium and proved that it exists, in mixed strategies, for every n-person game in normal form. Yet, finite positional games with perfect information can be always solved in pure strategies. For this reason, we restrict all players to their pure strategies and will not even mention the mixed ones.

However, restriction of all players to their pure positional strategies is by far less obvious. In some cases the existence of a Nash equilibrium (NE) in positional strategies fails; in some other it becomes an open problem; finally, in several important cases it holds, which, in our view justifies the restriction to positional strategies. To outline such cases is one of our goals.

There are also other arguments in favor of positional strategies; for example, "poor memory" can be a reason.

- In parlor games, not many individuals are able to remember the whole debut. Solving a Chess problem, you are typically asked to find an optimal move in a given position. No Chess composer will ever specify all preceding moves. Yet, why such an optimal move does not depend on the debut, in the presence of dicycles ? This needs a prove.
- In other, non-parlour, models, the decision can be made by automata without memory.
- The set of strategies is doubly exponential in the size of a digraph, while the set of positional strategies is "only" exponential.

Remark 5 *In [21], we used term "stationary" as a synonym to "positional". Yet, it is better to reserve the first one for the repeated games or positions.*

5.4 Normal form and Nash equilibria

Let X_i denote the set of all pure positional strategies of a player $i \in I$ and let $X = \prod_{i \in I} X_i$ be the set of all *strategy profiles or situations.*

In absence of random moves, given $x \in X$, a unique move is defined in each position $v \in V \setminus V_T = \cup_{i \in I} V_i$. Furthermore, these moves determine a *play* $p = p(x)$ that begins in the initial position v_0 and results in a terminal $a = a(x) \in V_T$ or in a dicycle $c = c(x) \in C(G)$, which will be repeated infinitely.

The obtained mapping $g : X \to A = \{c\} \cup V_T$ is called a *positional game form.* Given also a payoff $u : I \times A \to \mathbb{R}$, the pair (g, u) defines a *positional game in normal form.*

In general, random moves can exist. In this case, a Markov chain appears for every fixed $x \in X$. (Now, a play is a probabilistic realization of this chain.) One can efficiently compute the probabilities $q(x, a)$ to come to a terminal $a \in V_T$ and $q(x, c)$ of an infinite play; of course, $q(x, c) + \sum_{a \in V_T} q(x, a) = 1$ for every situation $x \in X$. Furthermore, $u(i, x) = u(i, c)q(i, c) + \sum_{a \in V_T} u(i, a)q(x, a)$ is the effective payoff of player $i \in I$ in situation $x \in X$.

Standardly, a situation $x \in X$ is called a *Nash equilibrium* (NE) if:
$u(i, x) \geq u(i, x')$ for every player $i \in I$ and for each strategy profile x' which might differ from x only in the ith coordinate, that is, $x'_j = x_j$ for all $j \in I \setminus \{i\}$.

In other words, x is a NE, if no player $i \in I$ can make a profit by replacing x_i by a new strategy x'_i, provided all other players $j \in I \setminus \{i\}$ keep their old strategies x_j. This definition is applicable in absence of random moves, as well.

A NE x, in a positional game (G, P, v_0, u) (with any type of payoff u) is called *subgame perfect or ergodic* if x remains a NE in game (G, P, v, u) for every initial position $v \in V \setminus V_T$.

Remark 6 *If G is an acyclic digraph, in which v_0 is a source (that is, each position $v' \in V$ can be reached from v_0 by a directed path) then the name "subgame perfect" is fully justified. Indeed, in this case any game (G, P, v, u) is a subgame of (G, P, v_0, u). Yet, in general, in presence of dicycles, terms* "ergodic" *or* "uniformly optimal" *would be more accurate.*

Let us call a game form (G, P, v_0) *Nash-solvable* if the corresponding game (G, P, v_0, u) has a NE for every possible utility function u.

5.5 Additive and terminal payoffs

Given a digraph $G = (V, E)$, a *local reward* is a mapping $r : I \times E \to \mathbb{R}$. Standardly, the value $r(i, e)$ is interpreted as the profit obtained by player $i \in I$ whenever the play passes $e \in E$.

Let us recall that, in absence of random moves, each situation $x \in X$ defines a unique play $p = p(x)$ that begins in the initial position v_0 and either terminates at $a(x) \in V_T$ or results in a simple dicycle $c = c(x)$. The *additive effective payoff* $u : I \times X \to \mathbb{R}$ is defined in the former case as the sum of all local rewards of the obtained play, $u(i, x) = \sum_{e \in p(x)} r(i, e)$, and the latter case it is $u(i, x) \equiv -\infty$ for all $i \in I$. In other words, all infinite plays are equivalent and ranked as the worst by all players, that is, we obtain a natural extension of AIPFOOW games. Let us note however that in the first case payoffs depend not only on the terminal position $a(x)$ but on the entire play $p(x)$.

The following two assumptions were considered in [21]:

(i) all local rewards are non-positive: $r(i, e) \leq 0$ for all $i \in I$ and $e \in E$;

(ii) all dicycles are non-positive: $\sum_{e \in c} r(i, e) \leq 0$ for all dicycles $c \in C(G)$.

Obviously, (i) implies (ii). Moreover, it was shown in 1958 by Gallai [43] that in fact these two assumptions are equivalent, since (i) can be enforced by a potential transformation whenever (ii) holds; see [43] and also [21] for definitions and more details.

Remark 7 *In [21], all players $i \in I$ minimize cost function $-u(i, x)$ instead of maximizing payoff $u(i, x)$. Hence, conditions (i) and (ii) turn into non-negativity conditions in [21].*

Standardly, we assume that all infinite plays form one outcome a_∞. Furthermore, in agreement with (i, ii), let us assume that this outcome is the worst one for each player, or in other words, the AIPFOOW property holds.

Conjecture 11 *n-Person AIPFOOW games with additive payoffs and without random moves have NE in pure positional strategies if conditions (i, ii) hold.*

In [21], it was demonstrated that conditions (i,ii) are essential.

Furthermore, it was shown in [21] that Conjecture 11 holds for the so-called *play-once* games, in which each player controls only one position. The proof is based on the observation that in a minimal counterexample every play (a directed path from the initial position v_0 to a terminal $a_i \in A$) and every dicycle $c \in C(G)$ must have a position in common.

It was also observed in [21] that the terminal AIPFOOW payoffs is a special case of additive ones. To see this, let us just set $r(i, e) \equiv 0$ unless e is a terminal move and notice also that no terminal move can belong to a dicycle. Hence, conditions (i) and (ii) hold automatically and the following conjecture is a relaxation of the previous one.

Conjecture 12 *Every n-person AIPFOOW game with terminal payoffs and without random moves has a NE in pure positional strategies.*

In [21], this conjecture was proven for two cases:

(a) at most two players $n \leq 2$ and (b) an most two terminals $p \leq 2$.

Recently, the latter result was strengthened to $p \leq 3$ in [27].

Yet, in the terminal case, it is not clear whether the AIPFOOW condition is essential at all. Gimbert and Sörensen (private communications; see more detail in [2]) assumed that the previous conjecture can be strengthened as follows:

Conjecture 13 *Every n-person AIPFOOT game with terminal payoffs and without random moves has a NE in pure positional strategies.*

It was shown in [2] that the last conjecture holds for the two-person games. The proof is based on old criteria of ash-solvability for two-person game forms.

5.6 Nash, zero-sum, and ± 1-solvability of two-person games

Let us recall basic definitions. Given a set of players $I = \{1, \ldots, n\}$ and outcomes $A = \{a_1, \ldots, a_p\}$, an n-person game form g is a map $g : X \to A$, where $X = \prod_{i \in I} X_i$ and X_i is a finite set of strategies of player $i \in I$.

Furthermore, a utility or payoff function is a mapping $u : I \times A \to \mathbb{R}$.

Standardly $u(i, a)$ is a profit of player $i \in I$ in case of outcome $a \in A$.

A payoff u is called *zero-sum* if $\sum_{i \in I} u(i, a) = 0$ for all $a \in A$.

The pair (g, u) is called a *game in normal form*. Given a game (g, u) a strategy profile $x \in X$ is a NE if $u(i, g(x)) \geq u(i, g(x'))$ for every $i \in I$ and every x' that differs from x only in coordinate i. A game form g is called *Nash-solvable* if for every utility function u the obtained game (g, u) has a NE.

Furthermore, a two-person game form g is called:

- *zero-sum-solvable* if for each zero-sum utility function u the obtained zero-sum game (g, u) has a NE, which is called a *saddle point* in this case;

- ± 1-*solvable* if solvability holds for each zero-sum u that takes only values $+1$ and -1.

Necessary and sufficient conditions for zero-sum solvability were obtained by Edmonds and Fulkerson [39] in 1970; see also [49]. Somewhat surprisingly, these conditions remain necessary and sufficient for Nash-solvability as well [50], see also [53] and [16]. Moreover, all three types of solvability are equivalent for the two-person game forms. Unfortunately, this useful property does not extend the case of n-person game forms already for $n = 3$; see examples in [50, 53, 16].

5.7 Proof of Conjecture 13 for the two-person case

We want to prove that every two-person AIPFOOT game without random moves has a NE in pure positional strategies.

Let $\mathcal{G} = (G, P, v_0, u)$ be such a game, in which $u : I \times A \to \{-1, +1\}$ is a zero-sum ± 1 utility function. As we just mentioned, it would suffice to prove solvability in this case [50].

Let $A_i \subseteq A$ denote the outcomes winning for player $i \in I = \{1, 2\}$.

Let us also recall that $V_i \subseteq V$ denotes the subset of positions controlled by player $i \in I = \{1, 2\}$. Without any loss of generality, we can assume that $c \in A_1$, that is, $u(1, c) = 1$, while $u(2, c) = -1$, or in other words, player 1 likes dicycles. Let $W^2 \subseteq V$ denote the set of positions in which player 2 can enforce (not necessarily in one move) a terminal from A_2, and let $W^1 = V \setminus W^2$.

By definition, player 2 wins whenever $v_0 \in W^2$. Let x_2 denote such a winning strategy; note that x_2 can be defined arbitrarily in $V_2 \cap W^1$.

We have to prove that player 1 wins whenever $v_0 \in W^1$. Indeed, for an arbitrary vertex v, if $v \in W^1 \cap V_2$ then player 2 cannot leave W^1, that is, $v' \in W^1$ for every move $(v, v') \in E$. Furthermore, if $v \in W^1 \cap V_1$ then player 1 can stay in W^1, that is, (s)he has a move $(v, v') \in E$ such that $v' \in W^1$. Let player 1 choose such a move for every position $v \in W^1 \cap V_1$ and arbitrary moves in all remaining positions, from $W^2 \cap V_1$. This rule defines a strategy x_1 of player 1. Let us show that x_1 wins whenever $v_0 \in W^1$. Indeed, in this case the play cannot enter W^2. Hence, it either will terminate in A_1 or result in a dicycle; in both cases player 1 wins. Thus, player 1 wins when $v_0 \in W^1$, while player 2 wins when $v_0 \in W^2$. $\square$

Remark 8 *We proved a little more than we planed to, namely, in case of* ± 1 *zero-sum payoffs the obtained strategies* x_1 *and* x_2 *are positional and uniformly optimal, or in other words, that situation* $x = (x_1, x_2)$ *is a subgame perfect saddle point. Moreover, this result is not difficult to extend to all (not only* ± 1 *zero-sum games [22]. However, it cannot be extended further, since a non-zero-sum two-person AIPFOOT game might have, in pure positional strategies, a unique NE, which is not subgame perfect; see Section 5.11 below.*

5.8 Acyclic games always have Nash equilibria

In the absence of dicycles, every finite n-person positional game (G, P, v_0, u) with perfect information has a subgame perfect NE in pure positional strategies. In 1950, this theorem was proved by Kuhn [61]; see also [62]. Strictly speaking, he considered only trees, yet, the suggested method, so-called *backward induction* can easily be extended to any acyclic digraphs; see, for example, [42].

The moves of a NE are computed recursively, position by position.
We start with the terminal positions and proceed eventually to the initial one. To every node and every player we shall associate a value, initialized by setting $u_i(v) = u(i, v)$ for all terminals $v \in V_T$. We proceed with a position $v \in V$ after all its immediate successors $w \in S(v)$ are done. If $v \in V_i$ then we set $u_i(v) = \max(u_i(w) \mid w \in S(v))$, and chose $w \in S(v)$ realizing this maximum, and set $u_j(v) = u_j(w)$ for all players $j \in I$. If $v \in V_R$ then we set

$$u_i(v) = mean\,(u_i(w) \mid w \in S(v)) \;\; = \sum_{w \in S(v)} p(v, w) u_i(w) \;\text{ for all }\; i \in I.$$

By construction, the obtained situation x is a subgame perfect NE. Thus, we will consider only games with dicycles, otherwise there is nothing to prove.

Yet, backward induction fails when the digraph G contains a dicycle.

5.9 For $n \geq 3$, in presence of directed cycles and moves of chance, Nash-solvability of AIPFOOW games fails

A simple example was suggested in [21]; see also [22].

Let $I = \{1, 2, 3\}$ and $A = \{a_1, a_2, a3, c\}$ and let s consider a dicycle c with three positions v_1, v_2, v_3 controlled by players $1, 2, 3$, respectively. Each player $i \in I$ has two options: either to proceed along c, or to terminate in a_i. The last option is the second in the preference list of i; it is better (worse) if the next (previous) player terminates, while the dicycle itself is the worst option for all. In other words, the preference profile is

$$u_1 : a_2 \succ a_1 \succ a_3 \succ c, \;\; u_2 : a_3 \succ a_2 \succ a_1 \succ c, \;\; u_3 : a_1 \succ a_3 \succ a_2 \succ c.$$

Finally, there is one position of chance v_0 ("in the middle of the cycle"), in which there are three moves leading to positions v_1, v_2, v_3 with *strictly positive* probabilities p_1, p_2, p_3, respectively.

Let s show that this game has no NE in pure positional strategies.

Its normal form is of size $2 \times 2 \times 2$, since each of three players has two positional strategies. Let us show that none of the eight situations is a NE. First, let us consider two situations: all three players terminate or all three move along the dicycle c. Obviously, each of these two situations can be improved by any one of the three player.

Now, let us show that none of the remaining six situations is a NE either. For example, consider the situation in which player 1 terminates, while 2 and 3 proceed. Then, player 2 is unhappy and can improve the situation by choosing termination. Yet, after this, player 1 can switch to move along c and improve again. Thus, we arrive to a situation in which player 2 terminates, while 3 and 1 proceed. Clearly, this situation is just the clockwise shift by $120°$ of the one we started with. Hence, after repeating the same procedure two more times, we get the so-called improvement cycle including all six considered situations.

However, the above game has a NE $x = (x_1, x_2, x_3)$ in pure stationary, but not positional, strategies. Such a strategy x_i, $i \in I = \{1, 2, 3\}$, requires to terminate in a_i whenever the play comes to v_i from v_0, and to proceed along c to v_{i+1} whenever the play comes to v_i from v_{i-1} (where standardly the indices are taken modulo 3). By definition, all these strategies x_i, $i \in I$, are pure and stationary but not positional. Let us show that the obtained situation x is a NE. Indeed, each player i could try to improve and get his best outcome a_{i+1} instead of a_i, which is his second best. Yet, to do so, this player i needs to proceed along c rather than terminate at a_i. Then, by definition of $x = (x_1, x_2, x_3)$, the other two players would also proceed along c. Thus, the play would result in c, which is the worst outcome for all.

Let us note that the above game has only one random position and one dicycle, which is the worst outcome for all players. Furthermore, the game is play-once, that is, each of the three players controls only one position.

Thus, this example leaves no hopes for Nash-solvability of n-person AIP-FOOW games, which havve both, dicycles and random moves, when $n \geq 3$.

Therefore, our main result (and hopes) are related to the two-person case; yet, even then one should not be too optimistic, as the following example shows.

5.10 On Nash-solvability of two-person games with both, dicyces and moves of chance

Let us reduce n from 3 to 2 in the previous example. The corresponding normal game form is of size 2×2. Each entry consists of two outcomes, which appear with probabilities p_1 and $p_2 = 1 - p_1$, respectively.

$$\begin{array}{cc} a_1a_2 & a_1a_1 \\ a_2a_2 & c\ \ c \end{array}$$

It is easily seen that the corresponding game has a NE in pure positional strategies unless all its four situations form an improvement cycle. This happens, indeed, if and only if the preference profile is

$u_1 : c \succ a_1 \succ a_2$ and $u_2 : a_1 \succ a_2 \succ c$ or their inverse permutations.

Yet, in both cases the obtained game is not zero-sum and not AIPFOOW.

Thus, Nash-solvability could hold only for:

(a) zero-sum case and/or (b) AIPFOOOW case.

In case (a) it holds; see, for example, [22], where the existence of a saddle point in pure positional uniformly optimal strategies can be derived from the basic results of the theory of stochastic games with perfect information developed in 1957 by Gillette [44]; see also [63].

Problem 5 *Whether these results hold for the two-person AIPFOOW games: (b') in general? (b'') in absence of moves of chance?*

5.11 On subgame perfect Nash-solvability

Let us delete the random position v_0 in two examples given in the last two sections. It is not difficult to verify that both reduced game forms become Nash-solvable; otherwise Conjecture 13 would be disproved. However, it is also easy to check that there exist no *subgame perfect* Nash equilibria.

In fact, this observation is general. Given a game $\mathcal{G}$ without moves of chance, add to it a vertex v_0 and a move from v_0 to each (non-terminal) position v of $\mathcal{G}$; furthermore, let us define a strictly positive probability $p(v_0, v)$ for each v.

Proposition 5 *The obtained game $\mathcal{G}'$ has a NE if and only if the original game $\mathcal{G}$ has a subgame perfect NE.*

Thus, we naturally arrive to the following questions.

Problem 6 *Whether a two-person AIPFOOW game has a subgame perfect NE: (b') in general? (b'') in absence of the moves of chance?*

Recently for both Problems 5 and 6, the general part b' was answered in negative, while b'' still remains open.

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