Generating k-Vertex Connected Spanning Subgraphs and k-Edge Connected Spanning Subgraphs

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Abstract

We show that k-vertex connected spanning subgraphs of a given graph can be generated in incremental polynomial time for any fixed k. We also show that generating k-edge connected spanning subgraphs, where k is part of the input, can be done in incremental polynomial time. These results are based on properties of minimally k-connected graphs which might be of independent interest.

1 Introduction

Vertex and edge connectivity are two of the most fundamental concepts in network reliability theory. While in the simplest case only the connectedness of an undirected graph, that is, the presence of a spanning tree, is required, in practical applications higher levels of connectivity are often desirable. Given the possibility that the edges of the network can randomly fail the reliability of the network is defined as the probability that the operating edges provide a certain level of connectivity. Most methods computing network reliability depend on the efficient generation of all minimal subsets of network edges which guarantee the required connectivity [Cou87, Val79].

In this paper we consider the problems of generating k-vertex connected spanning subgraphs and k-edge connected spanning subgraphs. An undirected graph G on at least $k+1$ vertices is *k-vertex connected* if every subgraph of G obtained by removing at most $k-1$ vertices is connected. Similarly, a graph G is *k-edge connected* if every subgraph of G obtained by removing at most $k-1$ edges is connected. A subgraph of a graph G is *spanning* if it has the same vertex set as G.

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For a fixed integer k we define the problem of generating minimal k-vertex connected spanning subgraphs as follows:

Minimal k-Vertex Connected Spanning Subgraphs Problem

Input: A k-vertex connected graph G

Output: The list of all minimal k-vertex connected spanning subgraphs of G

We define the problem of generating minimal k-edge connected spanning subgraphs as follows (note that k is a part of the input):

Minimal k-Edge Connected Spanning Subgraphs Problem

Input: A positive integer k and a k-edge connected graph G

Output: The list of all minimal k-edge connected spanning subgraphs of G

Note that in both generation problems the output may consist of exponentially many subgraphs in terms of the input size. Thus, the efficiency of generation algorithms is measured customarily in both the input and output size (see e.g., [JP88, LLK80, Val79]). An algorithm generating all elements of a family $\mathcal{F}$ runs in *incremental polynomial time* if generating N elements of $\mathcal{F}$ (or all if $\mathcal{F}$ has less than N elements) can be done in time polynomial in N and the size of the input, for an arbitrary integer N.

Our problems include as a special case the problem of generating spanning trees ($k = 1$), which can be solved efficiently [ASU97, RT75]. The problem of generating 2-vertex connected subgraphs and its generalization for matroids has been considered in [KBB$^+$].

The remainder of the paper is organized as follows. In Section 1.1 we state our main results and in Section 1.2 we recall a technique from [KBB$^+$05] used to prove the main results. The proofs of our theorems are in Sections 2 and 3.

1.1 Main Results

We show that both generation problems can be solved efficiently, i.e., in incremental polynomial time.

Theorem 1 *All minimal k-vertex connected spanning subgraphs of a given graph can be generated in incremental polynomial time.*

We remark that the running time of our algorithm depends exponentially on k. The complexity of the above problem when k is also part of the input remains an open question.

Theorem 2 *All minimal k-edge connected spanning subgraphs of a given graph can be generated in incremental polynomial time.*

1.2 The $X - e + Y$ method

In this section we recall a technique from [KBB$^+$05], which is a variant of the supergraph approach introduced by [SS02]. Let E be a finite set, and $\pi : 2^E \to \{0, 1\}$ be a monotone Boolean function, i.e., one for which $X \subseteq Y$ implies $\pi(X) \le \pi(Y)$. We assume that $\pi(\emptyset) = 0$ and $\pi(E) = 1$. We also assume that an efficient algorithm for evaluating $\pi(X)$ in polynomial time in the size of E is available for every $X \subseteq E$. Let

$$\mathcal{F} = \{X \mid X \subseteq E \text{ is a minimal set satisfying } \pi(X) = 1\}.$$

Our goal is to generate all sets belonging to $\mathcal{F}$.

Let us remark first that for every $X \subseteq E$ for which $\pi(X) = 1$ we can derive a subset $Y \subseteq X$ such that $Y \in \mathcal{F}$, by evaluating π exactly $|X|$ times. This can be accomplished by deleting one-by-one elements of X the removal of which does not change the value of π. To formalize this, we can fix an arbitrary linear order $\prec$ on elements of E, without any loss of generality, and define a mapping $\mu : \{X \subseteq E \mid \pi(X) = 1\} \to \mathcal{F}$ by $\mu(X) = X \setminus Z$, where Z is the lexicographically first subset of X, with respect to $\prec$, such that $\pi(X \setminus Z) = 1$ and $\pi(X \setminus (Z \cup e)) = 0$ for every $e \in X \setminus Z$. Clearly, by trying to delete elements of X in their $\prec$-order, we can compute $\mu(X)$, as we remarked above, by evaluating π exactly $|X|$ times.

We next introduce a directed graph $\mathcal{G} = (\mathcal{F}, \mathcal{E})$ on vertex set $\mathcal{F}$, where the neighborhood $N(X)$ of $X \in \mathcal{F}$ and the family $\mathcal{Y}_{X,e}$ are defined by

$$N(X) = \{\mu((X \setminus e) \cup Y) \mid e \in X, Y \in \mathcal{Y}_{X,e}\}, \quad \text{and}$$

$$\mathcal{Y}_{X,e} = \{Y \mid Y \subseteq E \setminus X \text{ is a minimal set satisfying } \pi((X \setminus e) \cup Y) = 1\}.$$

In other words, for every set $X \in \mathcal{F}$ and for every element $e \in X$ (since $X \in \mathcal{F}$, we have $\pi(X \setminus e) = 0$) we extend $X \setminus e$ in all possible ways to a set $X' = (X \setminus e) \cup Y$ for which $\pi(X') = 1$, and introduce each time a directed arc from X to $\mu(X')$. We call the obtained directed graph $\mathcal{G}$ a *supergraph* of our generation problem.

Proposition 1 ([KBB$^+$05]) *The supergraph $\mathcal{G} = (\mathcal{F}, \mathcal{E})$ is strongly connected.* $\square$

Since $\mathcal{G}$ is strongly connected by performing a breadth-first search in $\mathcal{G}$ we can generate all elements of $\mathcal{F}$ as follows:

$Traversal(\mathcal{G})$

Find an initial vertex $X^0 \leftarrow \mu(E)$, and initialize two queues $\mathcal{P} = \mathcal{Q} = \emptyset$.

Perform a breadth-first search of $\mathcal{G}$ starting from X^o:

output X^0 and insert it to the queue $\mathcal{P}$
while $\mathcal{P} \neq \emptyset$ **do**
  take the first vertex X out of the queue $\mathcal{P}$, and insert it to $\mathcal{Q}$
  for every $e \in X$ **do**
    for every $Y \in \mathcal{Y}_{X,e}$ **do**
      compute the neighbor $X' \leftarrow \mu((X \setminus e) \cup Y)$
      if X' is not in $\mathcal{P} \cup \mathcal{Q}$ **then**
          output X' and insert it to $\mathcal{P}$
    endfor
  endfor
endwhile

Proposition 2 ([KBB+05]) *If the sets of $\mathcal{Y}_{X,e}$ can be generated in incremental polynomial time for every $X \in \mathcal{F}$ and $e \in X$, then $Traversal(\mathcal{G})$ generates all elements of $\mathcal{F}$ in incremental polynomial time.* □

2 Proof of Theorem 1

In this section we apply the $X - e + Y$ method to the generation of all minimal k-vertex connected spanning subgraphs.

For a given k-vertex connected graph (V, E) we define a Boolean function π as follows: for a subset $X \subseteq E$ let

$$\pi(X) = \begin{cases} 1, & \text{if (V,X) is } k\text{-vertex connected;} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly π is monotone, $\pi(\emptyset) = 0$, $\pi(E) = 1$, and $\pi(X)$ can be evaluated in time polynomial in the number of edges [ET75]. Then $\mathcal{F} = \{X \mid X \subseteq E$ is a minimal set satisfying $\pi(X) = 1\}$ is the family of edge sets of all minimal k-vertex connected spanning subgraphs of (V, E). For $X \in \mathcal{F}$ and $e \in X$ we define

$$\mathcal{Y}_{X,e} = \{Y \mid Y \subseteq E \setminus X \text{ is a minimal set satisfying } \pi((X \setminus e) \cup Y) = 1\}.$$

By Proposition 2 we only need to prove that we can generate all elements of $\mathcal{Y}_{X,e}$ in incremental polynomial time. We devote the rest of this section to the proof of the following proposition.

Proposition 3 *All elements of $\mathcal{Y}_{X,e}$ can be generated in incremental polynomial time for every $X \in \mathcal{F}$ and $e \in X$.*

In Section 2.1 we introduce a poset describing the structure of the graph $(V, X \setminus e)$. In Section 2.3 we show that a family of sublattices is 2-Helly. Then in Section 2.4 we characterize $\mathcal{Y}_{X,e}$ and, combining our previous results, we prove Proposition 3.

In Section 2.2 we show that the poset introduced in Section 2.1 has bounded width. This observation is not necessary to show that generating all minimal k-vertex connected subgraphs can be done in incremental polynomial time.

2.1 $(k-1)$-separators of $(V, X \setminus e)$

A *k-separator* of a graph is a set of k vertices whose removal (simultaneously removing all edges adjacent to those vertices) makes the graph no longer connected. Note that a k-vertex connected graph has no k'-separators for $k' < k$.

Let $G = (V, X)$ be a minimal k-vertex connected spanning subgraph of a k-vertex connected graph (V, E) (see Figure 1).

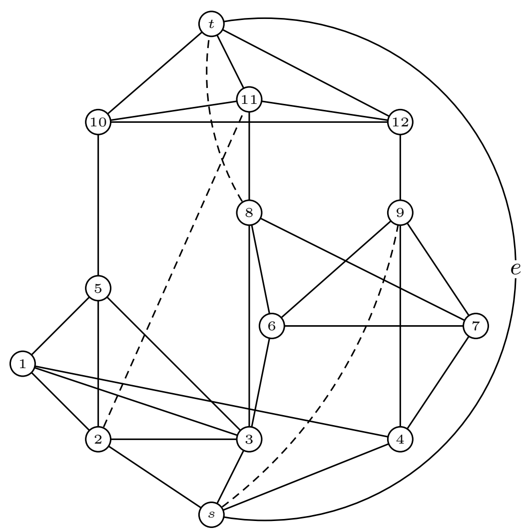

Figure 1: 4-vertex connected graph (V, E) and its minimal 4-vertex connected subgraph $G = (V, X)$. Solid lines are edges in X.

Let $e = st$ be an arbitrary edge of G and let W be a $(k-1)$-separator of $G_e = (V, X \setminus e)$. Note that W contains neither s nor t, since otherwise W would also be a $(k-1)$-separator of G. We denote by S_W and T_W the vertex sets of the components (i.e., maximal connected subgraphs) of $G_e[V \setminus W]$ containing s and t, respectively.

In the remainder of this section we denote by $N(\cdot)$ a neighborhood in the graph G_e.

Claim 1 *$G_e[V \smallsetminus W]$ consists of two components, $G_e[S_W]$ and $G_e[T_W]$ (see Figure 2).*

Proof: Suppose that there is a component of $G_e[V \smallsetminus W]$ not incident to e. Then that component is also a component of $G[V \smallsetminus W]$, a contradiction with $G[V \smallsetminus W]$ being connected. Hence $G_e[V \smallsetminus W]$ has two components, one containing s and the other containing t. $\square$

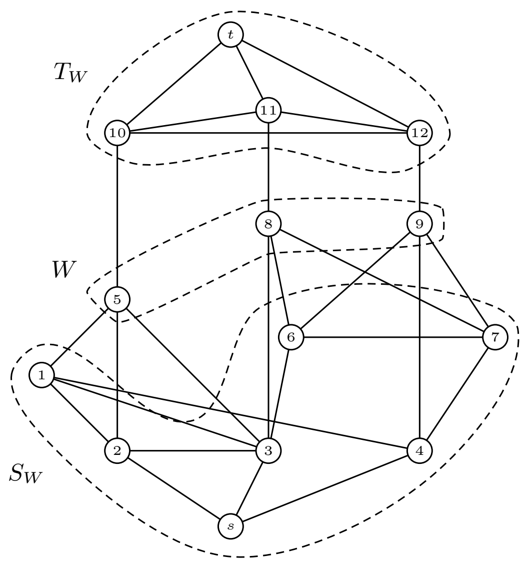

Figure 2: 3-seperator $W = \{5, 8, 9\}$ and the corresponding 3-source $S_W = \{s, 1, 2, 3, 4, 6, 7\}$.

Let $\mathcal{W}$ be the set of all $(k-1)$-separators of $G_e = (V, X \smallsetminus e)$ and let $\mathcal{S} = \{S \subseteq V \mid |N(S)| = k-1, s \in S, t \notin S \cup N(S)\}$. We call an element of $\mathcal{S}$ a $(k-1)$-*source*. Note that the mapping $W \longmapsto S_W$ is a bijection between $\mathcal{W}$ and $\mathcal{S}$ whose inverse is $S \longmapsto N(S)$.

Consider the poset $L = (\mathcal{S}, \subseteq)$ of the $(k-1)$-sources ordered by inclusion (see Figure 3).

Proposition 4 *The poset L with operations $\cap$ and $\cup$ is a lattice.*

Proof: Let S, S' be $(k-1)$-sources of G_e. We show that $S \cup S'$ and $S \cap S'$ are also $(k-1)$-sources. Clearly $s \in S \cup S'$, $s \in S \cap S'$, $t \notin S \cup S' \cup N(S \cup S')$ and $t \notin (S \cap S') \cup N(S \cap S')$.

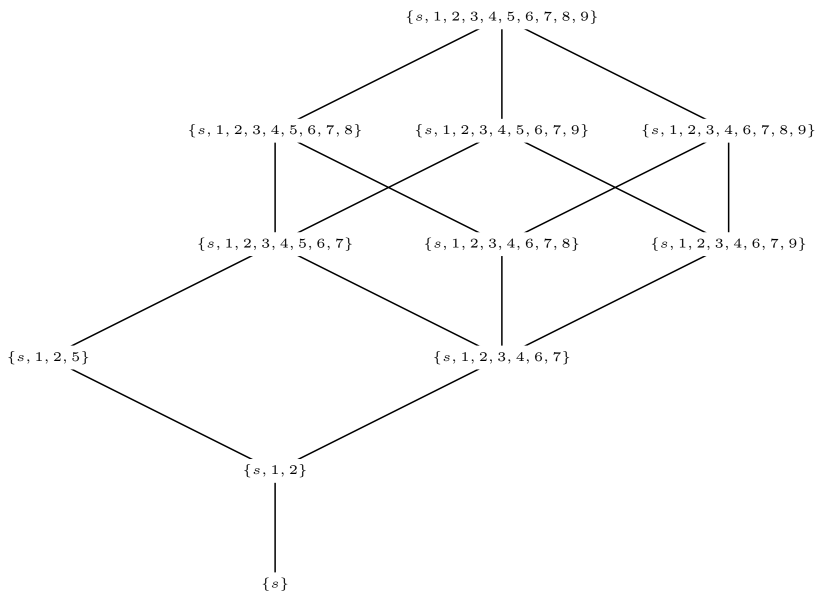

Figure 3: Poset of 3-sources of G_e.

By the submodularity of the neighborhood function we have $|N(S \cap S')| + |N(S \cup S')| \leq |N(S)| + |N(S')|$. Since $|N(S)| = |N(S')| = k-1$, the right hand side of the above inequality equals $2k-2$. On the other hand by since G_e is $(k-1)$-vertex connected and the removal of $N(S \cup S')$ or $N(S \cap S')$ disconnects G_e, we obtain $|N(S \cup S')| \geq k-1$ and $|N(S \cap S')| \geq k-1$. Thus $|N(S \cup S')| = |N(S \cap S')| = k-1$. □

We will show that the ordering of $(k-1)$-sources in L has a natural interpretation for the corresponding $(k-1)$-separators.

Since the graph G_e is $(k-1)$-vertex connected, by Menger's Theorem it contains $k-1$ internally vertex disjoint s-t paths. Let $P_1 = sv_1^1 \ldots v_{l_1}^1 t$, $P_2 = sv_1^2 \ldots v_{l_2}^2 t$, $\ldots$, $P_{k-1} = sv_1^{k-1} \ldots v_{l_{k-1}}^{k-1} t$ denote such a collection of paths (see Figure 4). We denote by V_P the set of all vertices belonging to the paths $P_1, \ldots, P_{k-1}$. Note that not all vertices in V necessarily belong to V_P.

Consider a $(k-1)$-separator W. Since the removal of W disconnects G_e, W contains at least one internal vertex from each path P_i, $i = 1, \ldots, k-1$. As W has $k-1$ vertices, $W = \{v^1_{\alpha(W,1)}, \ldots, v^{k-1}_{\alpha(W,k-1)}\}$, where $\alpha(W,i)$ is the index of the vertex of P_i belonging to W.

Claim 2 *Let $W, U \in \mathcal{W}$. $S_W \subseteq S_U$ if and only if $\alpha(W,i) \leq \alpha(U,i)$ for all $i = 1, \ldots, k-1$.*

Proof: First let $S_W \subseteq S_U$ and suppose on the contrary that $\alpha(W,i) > \alpha(U,i)$ for some i. Note that $v^i_{\alpha(W,i)}$ is the only vertex of the path P_i belonging to W. Vertices $v^i_1, \ldots, v^i_{\alpha(W,i)-1}$ of P_i, including $v^i_{\alpha(U,i)}$, must therefore belong to

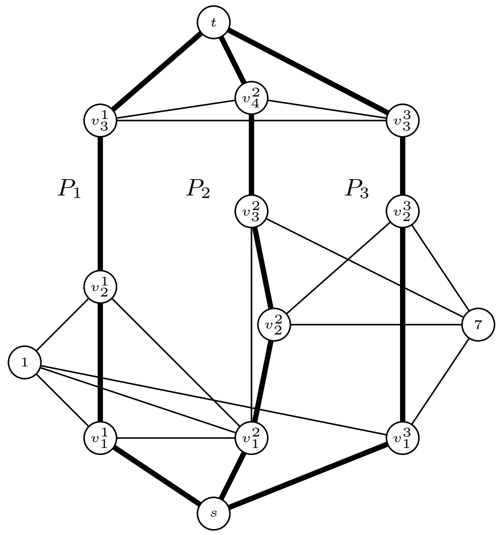

Figure 4: Internally vertex disjoint paths P_1, P_2, P_3 of G_e represented by thick edges.

S_W, since they are reachable from s in $G_e[V \smallsetminus W]$ using edges of P_i. Also note that $v^i_{\alpha(U,i)} \notin S_U$, since $v^i_{\alpha(U,i)} \in U = N(S_U)$. Hence $v^i_{\alpha(U,i)} \in S_W \smallsetminus S_U$, a contradiction.

Conversely, assume $\alpha(W,i) \leq \alpha(U,i)$ holds for all $i = 1, \ldots, k-1$. Observe that vertices in $S_W \cap V_P$ belong to S_U, since they are reachable from s in $G_e[V \smallsetminus U]$ using edges of P_i. Since $G_e[S_W]$ is connected, every component of $G_e[S_W \smallsetminus V_P]$ has a neighbor in $S_W \cap V_P$. As U does not contain vertices of $S_W \smallsetminus V_P$ and $S_W \cap V_P \subseteq S_U$, the vertices in $S_W \smallsetminus V_P$ are reachable from s in $G_e[V \smallsetminus U]$. Thus $S_W \subseteq S_U$. □

Lemma 1 *Let S_W, S_U be $(k-1)$-sources of G_e. Either $S_W \cap T_U = \emptyset$ or $T_W \cap S_U = \emptyset$.*

Proof: We partition $\{1, \ldots, k-1\}$ into sets I, J and K as follows:

$$I = \{i \mid \alpha(W,i) > \alpha(U,i)\},$$

$$J = \{i \mid \alpha(W,i) = \alpha(U,i)\},$$

$$K = \{i \mid \alpha(W,i) < \alpha(U,i)\}.$$

Let $C = \{v^i_{\alpha(U,i)} \mid i \in I\} \cup \{v^i_{\alpha(W,i)} \mid i \in I\} \cup \{v^i_{\alpha(W,i)} \mid i \in J\}$. Observe that $|C| = 2|I| + |J|$.

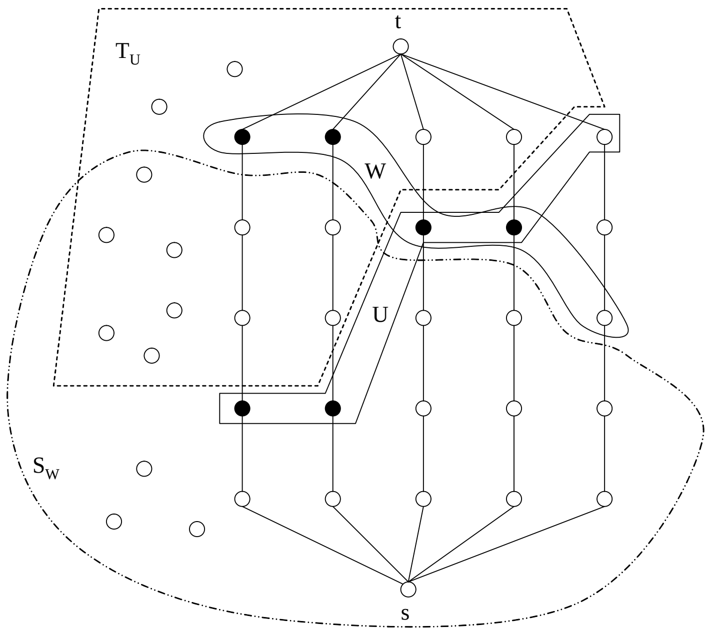

Figure 5: $(k-1)$-separators W and U. Black nodes are vertices of C.

We show that $N(S_W \cap T_U) \subseteq C$. Note that $V \setminus ((S_W \cap T_U) \cup C) = T_W \cup S_U$ (see Figure 5). Since W and U are $(k-1)$-separators of G_e, there is no edge between $S_W \cap S_U$ and $T_W \cup S_U$, thus $N(S_W \cap T_U) \subseteq C$.

Let $D = \{v^i_{\alpha(U,i)} \mid i \in K\} \cup \{v^i_{\alpha(W,i)} \mid i \in K\} \cup \{v^i_{\alpha(W,i)} \mid i \in J\}$. Similarly, we obtain that $N(T_W \cap S_U) \subseteq D$.

Suppose for contradiction that $S_W \cap T_U \neq \emptyset$ and $T_W \cap S_U \neq \emptyset$. Since $S_W \cap T_U$ contains neither s nor t, the removal of $N(S_W \cap T_U)$ disconnects G. As G is k-vertex connected, we obtain $k \leq |N(S_W \cap T_U)| \leq |C|$, thus

$$2|I| + |J| \geq k.$$

Similarly, we have

$$2|K| + |J| \geq k.$$

Recall that I, J and K partition $\{1, \ldots, k-1\}$, thus

$$k - 1 = |I| + |J| + |K|.$$

Combining this with the above inequalities we obtain $2((k-1) + |I| + |J| + |K|) \geq 2(k + |I| + |J| + |K|)$, a contradiction. $\square$

2.2 The Width of L

In this section we show that the lattice L has bounded width. This observation is not necessary to show that generating all minimal k-vertex connected subgraphs can be done in incremental polynomial time.

Corollary 1 *If S_W and S_U are incomparable in L then there exists some $i \in \{1, \dots, k-1\}$ such that $|\alpha(W,i) - \alpha(U,i)| = 1$, i.e., the vertices $v^i_{\alpha(W,i)}$ and $v^i_{\alpha(U,i)}$ are adjacent on the path P_i.*

Proof: Suppose on the contrary that $|\alpha(W,i)-\alpha(U,i)| > 1$ for all $i = 1, \dots, k-1$. Then since S_W and S_U are incomparable, there exist $j, l \in \{1, \dots, k-1\}$ such that $\alpha(U,j)+1 < \alpha(W,j)$ and $\alpha(W,l)+1 < \alpha(U,l)$. Then

$$v^j_{\alpha(U,j)+1} \in S_W \cap T_U,$$

$$v^l_{\alpha(W,l)+1} \in T_W \cap S_U$$

contradicting Lemma 1. □

The *width* of a poset is the size of its largest antichain. We show that the width of L is bounded.

Proposition 5 *The width of L is at most 2^{k-1}.*

Proof: We associate to every $(k-1)$-separator W a 0-1 vector

$$\pi(W) = (\alpha(W,1) \mod 2,\ \dots\ , \alpha(W,k-1) \mod 2).$$

By Corollary 1, if two $(k-1)$-separators W, U are incomparable, there exists some $i \in \{1, \dots, k-1\}$ such that $|\alpha(W,i)-\alpha(U,i)| = 1$, implying $\pi(W) \neq \pi(U)$.

Since the number of different 0-1 vectors of length $k-1$ is 2^{k-1}, every antichain in P has size at most 2^{k-1}. □

The bound in Proposition 5 cannot be significantly improved upon, as the following example shows.

Example 2.1 *Consider a minimal k-vertex connected graph G on a vertex set $\{s, t, x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}\}$ whose edges set is defined as follows (see Figure 6):*

- *we have the edges $x_1y_1, x_2y_2, \dots, x_{k-1}y_{k-1}$ and st,*
- *the vertices $s, x_1, \dots, x_{k-1}$ form a clique,*
- *the vertices $t, y_1, \dots, y_{k-1}$ form a clique.*

Let I be a subset of $\{1, \dots, k-1\}$. Then $W_I = \{y_i \mid i \in I\} \cup \{x_i \mid i \in \{1, \dots, k-1\} \setminus I\}$ is the $(k-1)$-separator of G_e and $S_I = \{s\} \cup \{x_i \mid i \in I\}$ is the corresponding $(k-1)$-source of G_e. Thus the poset of $(k-1)$-sources of G_e contains the poset B_{k-1} of all subsets of $\{1, \dots, k-1\}$ ordered by inclusion. The width of B_{k-1} is $\binom{k-1}{\lfloor \frac{k-1}{2} \rfloor} \approx \frac{2^{k-1}}{\sqrt{k-1}}$.

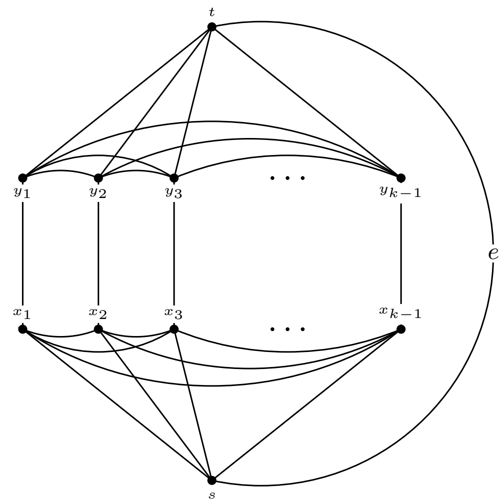

Figure 6: Minimal k-vertex connected graph G.

2.3 Helly Property for a Family of Sublattices

Let $(P, \preceq)$ be a lattice with operations $\wedge, \vee$ and let $\mathcal{H}$ be a family of sublattices of P. As stated in [Ber89, Example 2 on page 21], the hypergraph $\mathcal{H}$ is 2-Helly. For the sake of completeness we present the proof below.

Lemma 2 *Let $\mathcal{A} \subseteq \mathcal{H}$. If for every $Q', Q'' \in \mathcal{A}$ we have $Q' \cap Q'' \neq \emptyset$ then $\bigcap_{Q \in \mathcal{A}} Q \neq \emptyset$.*

Proof: The proof is by induction on $|\mathcal{A}|$. For $|\mathcal{A}| = 2$ the claim clearly holds.

Suppose $|\mathcal{A}| \geq 3$. Let R, R' and R'' be three distinct elements of $\mathcal{A}$. By the induction hypothesis, there exist $x \in \bigcap_{Q \in \mathcal{A} \smallsetminus \{R\}} Q$, $y \in \bigcap_{Q \in \mathcal{A} \smallsetminus \{R'\}} Q$ and $z \in \bigcap_{Q \in \mathcal{A} \smallsetminus \{R''\}} Q$. Let

$$x^* = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

We show that $x^* \in \bigcap_{Q \in \mathcal{A}} Q$. Since for every $Q \in \mathcal{A} \smallsetminus \{R, R', R''\}$ we have $x, y, z \in Q$ we obtain that for every $Q \in \mathcal{A} \smallsetminus \{R, R', R''\}$ we have $x^* \in Q$.

As the formula for x^* is symmetric, it suffices to show that $x^* \in R$. Let $\hat{1}$ and $\hat{0}$ denote the maximum and minimum elements of R, respectively. Note that $y, z \in R$. Thus $\hat{0} \preceq y, z \preceq \hat{1}$. Since $\hat{0} \preceq y \wedge z$ and $x \wedge y, x \wedge z, y \wedge z \preceq x^*$, we obtain $\hat{0} \preceq x^*$. As $x \wedge y, x \wedge z, y \wedge z \preceq \hat{1}$ and x^* is the minimum upper bound of $x \wedge y, x \wedge z, y \wedge z$, we have $x^* \preceq \hat{1}$. $\square$

2.4 Proof of Proposition 3

For two vertices $u, v \in V$ let $D_{u,v} = \{S_W \in \mathcal{S} \mid u \in S_W, v \in T_W\}$.

Claim 3 *Either $D_{u,v} = \emptyset$ or $D_{v,u} = \emptyset$ for all $u, v \in V$.*

Proof: Suppose on the contrary that we have $S_W \in D_{u,v}$ and $S_U \in D_{v,u}$. Then $u \in S_W \cap T_U$ and $v \in T_W \cap S_U$, contradicting Lemma 1. $\square$

For an edge $f = uv$ let $D_f = D_{u,v} \cup D_{v,u}$ (see Figure 7).

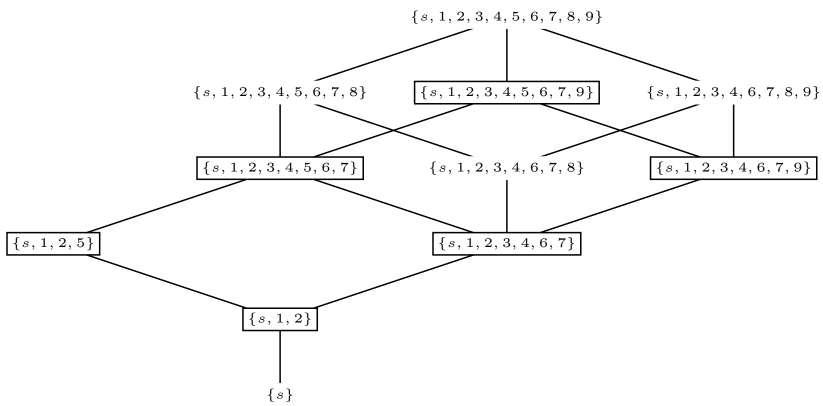

Figure 7: Elements of $D_{2,11}$ are in black rectangles. Note that $D_{11,2} = \emptyset$.

Claim 4 *D_f is a sublattice of L.*

Proof: Let $f = uv$. Without loss of generality we can assume that $D_f = D_{u,v}$. Let $S, S'' \in D_{u,v}$. Then $u \in S \cap S'$ and $v \notin (S \cap S') \cup N(S \cap S')$. Similarly, $u \in S \cup S'$ and $v \notin S \cup S' \cup N(S \cap S')$. Thus $S \cap S', S \cup S' \in D_f$. $\square$

We now consider the hypergraph $\mathcal{H}$ on vertex set $\mathcal{S}$ with edge set $\mathcal{E} = \{D_f \mid f \in E \setminus X\}$. We call a set of hyperedges whose union contains every vertex a *hyperedge cover*. We show that the generation of all elements of $\mathcal{Y}_{X,e}$ is equivalent to the generation of the minimal hyperedge covers of $\mathcal{H}$.

Claim 5 *Let $Y \subseteq E \setminus X$. The graph $(V, X \setminus e \cup Y)$ is k-vertex connected if and only if $\bigcup_{f \in Y} D_f = \mathcal{S}$.*

Proof: First suppose that $(V, X \setminus e \cup Y)$ is k-vertex connected and consider a $(k-1)$-source $S_W \in \mathcal{S}$. Since $(V, X \setminus e \cup Y)$ is k-vertex connected there is an edge $f \in Y$ such that one of its endpoints belongs to S_W and the other to T_W, implying $S_W \in D_f$.

Conversely, suppose that suppose that $(V, X \setminus e \cup Y)$ is not k-vertex connected, i.e., it has a $(k-1)$-separator W. Since W is also a $(k-1)$ separator of G_e, we have $S_W \in \mathcal{S}$. As no edge in Y connects $G_e[S_W]$ and $G_e[T_W]$, $S_W \notin \bigcup_{f \in Y} D_f$. $\square$

Claim 6 *$\mathcal{H} = (\mathcal{S}, \mathcal{E})$ can be constructed in time polynomial in the size of the graph G.*

Proof: Recall that $(k-1)$-sources are in one to one correspondence with $(k-1)$-separators. We can check if after removing a given set of $k-1$ vertices the graph G is still connected in linear time using, e.g., depth first search. Thus we can find all $(k-1)$-separators by repeating the above procedure for every $(k-1)$-element subset of V. Since k is fixed, the number of such subsets, $\binom{|V|}{k-1}$, is a polynomial of $|V|$.

To construct $\mathcal{H}$ we need to check for every $f \in E \setminus X$ and every $(k-1)$-separator W if S_W belongs to D_f, which can done in linear time for each pair. □

Claim 7 *All hyperedge covers of $\mathcal{H}$ can be generated in incremental polynomial time.*

Proof: We reduce the problem of generating all hyperedge covers of $\mathcal{H}$ to the problem of generating of maximal independent sets of a 2-conformal hypergraph, which can be done in incremental polynomial time [BEGK03].

First we observe that $\mathcal{C}$ is a minimal hyperedge cover of $\mathcal{H}$ if and only if $\mathcal{E} \setminus \mathcal{C}$ is a maximal independent set of $\mathcal{H}^T$, where $\mathcal{H}^T$ is the hypergraph defined by the transposed incidence matrix of $\mathcal{H}$.

As shown in [Ber89], a hypergraph is δ-Helly if and only if its transpose is δ-conformal. Since the edges of $\mathcal{H}$ is are sublattices of L, by Lemma 2 $\mathcal{H}$ is 2-Helly. Thus $\mathcal{H}^T$ is 2-conformal, which proves the claim. □

By Claim 7 we can generate N hyperedge covers of $\mathcal{H}$ in time polynomial in N and the size of $\mathcal{H}$. Thus by Claim 6 we can also generate N elements of $\mathcal{Y}_{X,e}$ in time polynomial in N and the size of G. Proposition 3 immediately follows.

3 Proof of Theorem 2

We apply the $X-e+Y$ method to the generation of all minimal k-edge connected spanning subgraphs.

For a graph $G = (V, E)$ we define a Boolean function π as follows: for a subset $X \subseteq E$ let

$$\pi(X) = \begin{cases} 1, & \text{if (V,X) is } k\text{-edge connected;} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{X \mid X \subseteq E \text{ is a minimal set satisfying } \pi(X) = 1\}$ is a family of edge sets of all minimal k-edge connected spanning subgraphs of G. For $X \in \mathcal{F}$ and $e \in X$ we define

$$\mathcal{Y}_{X,e} = \{Y \mid Y \subseteq E \setminus X \text{ is a minimal set satisfying } \pi((X \setminus e) \cup Y) = 1\}.$$

Proposition 6 below implies that we can generate all elements of $\mathcal{F}$ in incremental polynomial time.

Proposition 6 *All elements of $\mathcal{Y}_{X,e}$ can be generated in incremental polynomial time for every $X \in \mathcal{F}$ and $e \in X$ (see Figure 8).*

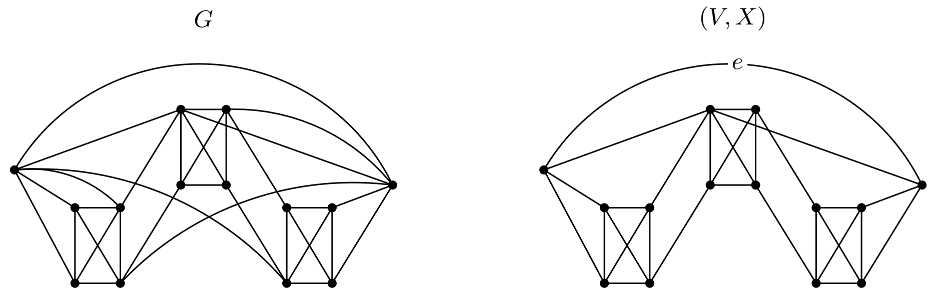

Figure 8: 4 -edge connected graph $G = (V, E)$ and a minimal 4-edge connected spanning subgraph (V, X).

Proof: We define the equivalence relation $\sim$ on V by

$$s \sim t \iff \text{there are k-edge disjoint paths from } s \text{ to } t \text{ in } (V, X \setminus e).$$

We call a set $\delta(S) = \{vw \in X \setminus e \mid v \in S, w \in V \setminus S\}$ for some $S \subseteq V$, a *cut*. An *s-t cut* is a cut for which $s \in S$, $t \in V \setminus S$.

To see that $\sim$ is transitive, suppose that $s \sim t$, $t \sim u$ and $s \nsim u$. By Menger's Theorem the minimum s-u cut $\delta(S)$ has less than k edges. Note that t belongs either to S or $V \setminus S$. Hence $\delta(S)$ is also s-t or t-u cut of size less than k, a contradiction.

Let H be a weighted graph obtained from $(V, X \setminus e)$ by identifying vertices in the equivalence classes of $\sim$ and replacing edges between two vertices by one, whose weight is equal to the number of these edges (see Figure 9). Let T be the

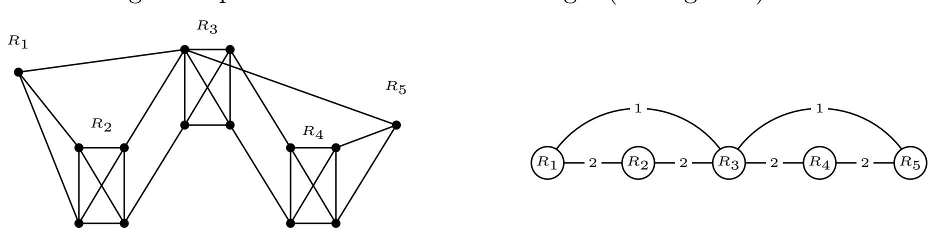

Figure 9: $(V, X \setminus e)$ and the weighted graph H.

Gomory-Hu cut-tree of H (see e.g., [CCPS98, page 78]).

Claim 8 *T is a path such that both ends contain an endpoint of e.*

Proof: Let s, t be vertices of H and let $\delta(S)$ be the minimum s-t cut. We denote the capacity of the cut $\delta(S)$ by $w(\delta(S))$. We first show that $w(\delta(S)) = k - 1$.

Suppose that $w(\delta(S)) \leq k-2$. Then removing all $k-1$ edges of $\delta(S) \cup e$ disconnects the graph (V,X), a contradiction with (V,X) being k-connected. Suppose that $w(\delta(S)) \geq k$. This implies that there are k edge disjoint paths between s and t, thus $s \sim t$, a contradiction with the construction of H.

We then show that every leaf of T contains an endpoint of e. Suppose that T has a leaf r that does not contain an endpoint of e. Since each vertex of T corresponds to exactly one vertex of H, r corresponds to the equivalence class R. Observe that $|\delta(R)| = k-1$ and removing edges of $\delta(R)$ from (V,X) disconnects it, a contradiction. Thus T has only two leaves, each containing an endpoint of e. $\square$

We denote by $r_1, \ldots, r_l$ the vertices of the path T and by $R_1, \ldots, R_l \subseteq V$ the corresponding equivalence classes. We next construct a directed multigraph D on vertex set $r_1, \ldots, r_l$ whose edge set is defined as follows:

- for each $i = 1, \ldots, l-1$, we add an arc $r_{i+1}r_i$,
- for each edge $f \in E \setminus X$, such the one endpoint belongs to R_i and the other to R_j, where $i < j$, we add an arc r_ir_j (see Figure 10).

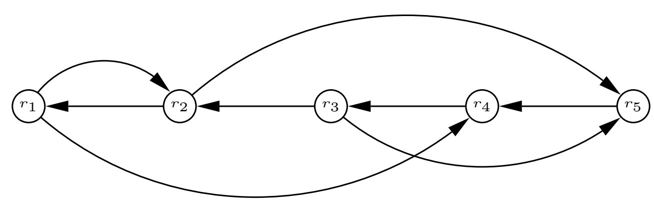

Figure 10: Directed multigraph D.

We then show that generating elements of $\mathcal{Y}_{X,e}$ is equivalent to the generation of minimal directed r_1-r_l paths in D.

For $f \in E$ we define $\alpha(f) = i$, $\beta(f) = j$ if one endpoint of f belongs to R_i and the other to R_j, where $i < j$.

Observe that for every pair of equivalence classes R_i, R_j, $i \neq j$, there is an edge $f \in Y$ such that $\alpha(f) \leq i$,$\beta(f) \geq j$. We conclude that $Y = \{f_1, \ldots, f_s\}$, such that

$$1 = \alpha(f_1) < \alpha(f_2) \leq \beta(f_1) < \alpha(f_3) \leq \ldots < \alpha(f_s) \leq \beta(f_{s-1}) < \beta(f_s) = l.$$

Thus Y corresponds to a directed path $r_{\alpha(f_1)}\ r_{\beta(f_1)}\ r_{\beta(f_1)-1} \ldots r_{\alpha(f_2)+1}\ r_{\alpha(f_2)}\ r_{\beta(f_2)}\ r_{\beta(f_2)-1} \ldots r_{\alpha(f_3)+1}\ r_{\alpha(f_3)} r_{\beta(f_3)} \ldots r_{\beta(f_s)}$.

Since all minimal directed paths between two vertices can be generated via backtracking with polynomial delay [RT75], Proposition 6 follows. $\square$

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