Lift Contractions

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Abstract

We introduce and study a new containment relation in graphs – lift contractions. \( H \) is a lift contraction of \( G \) if \( H \) can be obtained from \( G \) by a sequence of edge lifts and edge contractions. We show that a graph contains every \( n \)-vertex graph as a lift contraction, if (1) its treewidth is large enough, or (2) its pathwidth is large enough and it is 2-connected, or (3) its order is large enough and its minimum degree is at least 3.

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1 Introduction

All graphs in this paper are undirected, loopless, and without multiple edges (unless mentioned otherwise). $V(G)$ and $E(G)$ denote the vertex and edge set of a graph $G$, respectively. The degree of a vertex $v \in V(G)$ is the number of edges incident with it. Given an edge $e$ of a graph $G$, the result of the contraction of $e$ in $G$ is the graph obtained by removing $e$ from $G$ and then identifying its endpoints to a single vertex $v_e$. Given two edges $e_1 = \{x, x_1\}$ and $e_2 = \{x, x_2\}$ of $G$, incident with the same vertex $x$, and such that $x_1 \neq x_2$, we define the lift of $e_1$ and $e_2$ in $G$ as the graph obtained by removing $e_1$ and $e_2$ from $G$ and then add the edge $\{x_1, x_2\}$. If a contraction or lift creates multiple edges, we reduce their multiplicity to one and keep the graph simple.

The study of graph containment relations is one of the basic research avenues in graph theory. One of the most comprehensive studies of a containment relation is the theory of graph minors by Robertson and Seymour [5]. A graph $H$ is a minor of another graph $G$ if $H$ can be obtained from $G$ by a sequence of vertex deletions, edge removals, and edge contractions. Some more restricted graph containment relations than graph minors, like induced minors [3] or contractions [2], were also studied.

Graph immersions is another containment relation that has been consider in the literature [1]. A graph $H$ is an immersion of $G$ if $H$ can be obtained from $G$ by a sequence of vertex deletions, edge removals, and lifts. The last operation was introduced by Lovász under the name of splitting off as a reduction method to maintain edge connectivity.

In this paper, we introduce and study lift contractions. We say that a graph $H$ is a lift contraction of a graph $G$ if $H$ can be obtained from $G$ by a sequence of lifts and contractions. We also define lift minors. We say that a graph $H$ is a lift minor of a graph $G$ if $H$ can be obtained from $G$ by a sequence of vertex and edge deletions, lifts and contractions.

Being a lift contraction (lift minor) is a partial relation between graphs and we denote it by $H \leq_{lc} G$ ($H \leq_{lm} G$). If a graph $H$ can be obtained from $G$ by a sequence of contractions, we say that $H$ is a contraction of $G$ and we denote this by $H \leq_c G$. Clearly, $H \leq_c G \Rightarrow H \leq_{lc} G \Rightarrow H \leq_{lm} G$.

We identify three conditions on a graph $G$ that force any $n$-vertex graph as a lift contraction of $G$.

Theorem 1.1 There exists a constant $c$ such that every graph $G$ of treewidth at least $c \cdot n^4$ contains every $n$-vertex graph as a lift contraction.

Theorem 1.2 There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every 2-connected
graph of pathwidth at least $f(n)$ contains every $n$-vertex graph as a lift contraction.

**Theorem 1.3** There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph with at least $f(n)$ vertices and minimum degree at least 3 contains every $n$-vertex graph of as a lift contraction.

We note that none of the three conditions above forces all $n$-vertex graphs as a lift or as a contraction alone. Lifts do not change the number of vertices of the graph, so none of the conditions of Theorems 1.1, 1.2, and 1.3 forces an $n$-vertex graph as a lift. There are planar graphs satisfying the conditions of Theorems 1.1, 1.2, and 1.3 but none of them contain a non-planar graph as a contraction.

We also note that a consequence of [1] is that every graph of degeneracy at least 200 contains every $n$-vertex graph as a lift contraction.

In the next section, we prove two auxiliary lemmas and provide a sketch of the proof of Theorem 1.3. The proofs of Theorem 1.1 and 1.2 are omitted due to space restrictions and will appear in the full version of the paper.

2 Proof of Theorem 1.3

**Lemma 2.1** For every $n$-vertex graph $H$, $H \leq_{lc} K_{2n}$.

**Proof** We prove that every graph $n$-vertex graph $H$ is a lift contraction of $K_{2n}$. Let $H^+ = K_2 \times H$. First we prove that $H^+$ is a lift of $K_{2n}$. Let $V(H) = \{v_1, \ldots, v_n\}$ and $V(H^+) = \{v'_1, \ldots, v'_n, v''_1, \ldots, v''_n\}$. Let us assume that $V(K_{2n}) = V(H^+)$ and observe that $H^+$ is a spanning subgraph of $K_{2n}$. Let $R$ be the set of non-edges of $H$. Notice that each $\{v_i, v_j\} \in R$ corresponds to the vertices $v'_i, v'_j, v''_i, v''_j \in V(H^+)$ such that the edges $\{v'_i, v'_j\}, \{v''_i, v'_j\}, \{v'_i, v''_j\}, \{v''_i, v''_j\}$ are present in $K_{2n}$ but not in $H^+$. We use lifts to remove those edges. For every $\{v_i, v_j\} \in R$, we lift the pairs of edges $\{v'_i, v'_j\}, \{v''_i, v'_j\}$ and $\{v''_i, v''_j\}, \{v'_i, v'_j\}$. The result is $H^+$. Now we contract edges $\{v'_i, v''_i\}$ for all $i = 1, \ldots, n$ and obtain $H$ as claimed. \qed

The following observation can be easily proved by induction on $r$.

**Observation 1** For every $r \geq 2$, the complete $r$-partite graph, where each of its parts has $r - 1$ vertices, has a perfect matching $M$ such that for every two of its parts there is an edge in $M$ intersecting both of them.

For an integer $k > 1$, the $k$-fan is the graph obtained from the path $P_k$ on $k$ vertices by adding a dominating vertex $v_c$. We denote the $k$-fan by $F_k$ and say that $P_k$ is its spine and $v_c$ is its center.
Lemma 2.2 If $F_{n(n-1)} \leq \text{im} G$, then $K_n \leq_{lc} G$.

Proof If $F_{n(n-1)} \leq \text{im} G$, then there exists a graph $G' \leq_{lc} G$ such that $G'$ contains $F_{n(n-1)}$ as a spanning subgraph. Let the spine of this $n(n-1)$-fan in $G'$ be a path $P$ with $V_P = \{v_1^1, \ldots, v_{n-1}^1, v_1^2, \ldots, v_{n-2}^2, \ldots, v_1^n, \ldots, v_{n-1}^n\}$. Let $J$ be an $n$-partite graph with $V_P$ as its vertex set and let $M$ be a perfect matching of $J$ as in Observation 1. For each edge $\{v_i^j, v_{j'}^{j''}\} \in M$, where $j \neq j''$, we lift the pair of edges $\{v_i^j, v_e\}$ and $\{v_{j'}^{j''}, v_e\}$ in $G'$. In the resulting graph, we contract, for each $i \in \{1, \ldots, n\}$, all the edges in $\{\{v_j^i, v_{j+1}^i\} | j \in \{1, \ldots, n-2\}\}$ to a single vertex $u^i$. Observe that the resulting graph is a complete graph with vertex set $\{u^1, \ldots, u^n\}$. Hence, $K_n \leq_{lc} G' \leq_{lc} G$ as claimed. \hfill \Box

Let $W_k$ be the graph obtained from $F_k$ by adding an edge between its two vertices of degree 2. Let $K_{3,k}$ be the complete bipartite graph whose parts have exactly 3 and $k$ vertices. Given a graph $G$ and a set $S \subseteq V(G)$ we denote by $\text{cl}(G,S)$ the graph obtained from $G$ by adding all the edges between vertices of $S$ that are not already in $G$. Let $K_{4}^-$ be the graph obtained from $K_4$ by removing an edge.

Sketch of the Proof of Theorem 1.3. We set $k = 2n(n-1)$ and assume that $G$ does not contain $F_k$ as a lift minor. From Lemmas 2.1 and 2.2 it is enough to prove that $|V(G)|$ cannot be bigger than $f(k)$ where $f$ is a function that will be determined later in the proof.

We begin with a result from [4]: there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that every graph excluding $W_k$ and $K_{3,k}$ as a minor has a tree-decomposition of width at most $g(k)$ and such that the maximum size of the intersection of two different bags in it is at most 2. Notice that $W_k$ and $K_{3,k}$ both contain $F_k$ as a lift minor. Hence, $G$ has such a tree-decomposition; let this tree-decomposition be $\mathcal{X} = \{X_i | i \in V(T)\}$. We also assume that no bag of $\mathcal{X}$ is a subset of another bag. Among all such decompositions we take $\mathcal{X}$ to be one that maximizes the number of leaves and – subject to this – it maximizes the number of bags. Let $\mathcal{L} \subseteq \mathcal{X}$ be the set of bags corresponding to the leaves of $T$.

Claim. If $F_k \not\leq_{lc} G$, then $|\mathcal{L}| < k + k^{k+2}$.

Let us assume that $|\mathcal{L}| \geq k + k^{k+2}$ and consider the graphs $L_X = G[X]$ for each $X \in \mathcal{L}$. There are at most 2 vertices in each $L_X$ that have neighbors outside of $X$. Let $S_X$ be the set of such vertices for each $X \in \mathcal{L}$. Also, let $L_X = \text{cl}(G,S)$. As $\mathcal{X}$ has the maximum number of bags, $L_X$ is 3-connected for each $X \in \mathcal{L}$. If $|S_X| = 1$, then $K_4$ is a contraction of $L_X = L_X$. If $|S_X| = 2$, notice that the 3-connectivity of $L_X$ implies that $L_X$ can be contracted to $K_4^-$.
in such a way that the two vertices of degree 2 are the vertices of $S_\mathcal{X}$ (we call them base vertices of $K_4^-$).

We now construct an auxiliary graph $J$ by first adding in $G^- = G[\bigcup_{X \in \mathcal{X} \setminus \mathcal{L}} X]$ edges between all pairs of vertices that belong to some $S_L, L \in \mathcal{L}$, where $|S_L| = 2$. Let us call these edges additional. We assign weights to the vertices and edges of $J$: each vertex $v \in V(J)$ receives weight $|\{L \in \mathcal{L} \mid \{v\} = S_L\}|$ and each edge $e \in E(J)$ receives weight $|\{L \in \mathcal{L} \mid e = S_L\}|$. Observe that the additional edges are exactly those with positive weights. Clearly, the sum of the weights of the edges and vertices of $J$ is $\geq k + k + 2$. Notice that the sum of the vertex-weights of $J$ is $< k$; otherwise, contracting in $G$ all edges on $J$ gives a graph containing as a lift contraction at least $k$ copies of $K_4$ attached to the same vertex and this would imply $F_k$ as a lift contraction. So far we proved that the sum of edges weights of $J$ is at least $k + k + 2$. We now consider the graph $J^*$ obtained from $J$ by repetitively removing or contracting non-additional edges while maintaining 2-connectivity of the resulting graph. If during such a contraction two edges become one, the weight of the new edge is the sum of the weights of the two edges. Notice that the total edge-weight of $J^*$ is the same as in $J$, that is $\geq k + k + 2$. Also, at most two edges of zero weight may survive in $J^*$ and this may happen only when $J^*$ is a triangle where two or one of its edges have positive weights. Notice that none of the edges in $J^*$ may have weight $\geq k$ as, then, the same sequence of edge contractions and removals in $G$ would create a graph that contains as a lift minor $k$ copies of $K_4$ with their base vertices identified and this graph contains $F_k$ as a lift minor. We obtain that the number of edges in $J^*$ is at least $k + k + 1 > 2$. Thus all edges of $J^*$ have positive weight.

Our next step is to prove that the maximum degree of $J^*$ is less than $k$. Suppose for contradiction that some vertex $y$ of $J^*$ is incident with $\geq k$ edges. Recall that $J^*$ is 2-connected and thus $J^* \setminus y$ is connected. Therefore, if we contract in $J^*$ all edges of $J^*$ that are not incident to $y$, we create a single edge with total weight $\geq k$. This implies that $G$ has $F_k$ as a lift minor because, as before, $G$ contains as a lift minor $k$ copies of $K_4^-$ with their base vertices identified. So far, we proved that $J^*$ has $> k$ vertices all of degree $< k$. This means that $J^*$ has a path of length $\geq k$. This path in $G$ corresponds to $k$ copies of $K_4^-$ joined in a sequential way (the “right” base vertex of the one is the “left” of the next). This construction contains $F_k$ as a lift minor and the claim follows.

Notice that fact that each bag of $\mathcal{X}$ has at most $g(n(n - 1))$ vertices implies that $\mathcal{X}$ has at least $f(n)/g(k)$ bags. Therefore, the tree $T$ has $f(n)/g(k)$ vertices and from the above claim, less than $k + k + 2$ of them are leaves (recall that $k = n(n - 1)$). But then we can choose the function $f$ such that $T$ contains
a path $P$ of length $4k^3 + 4$ with all its vertices of degree 2. By the fact that
the minimum degree of $G$ is at least 3, we obtain that at most the half of the
graphs induced by the bags corresponding to the vertices of $P$ are bridges. By
contracting all these bridges, we may assume that this path has length $2k^3 + 2$
and no bridges at all. We assume that the vertices of $P$ are consecutive integers
from the set $I = \{0, \ldots, 2k^3 + 1\}$. Notice that for each $i \in I \setminus \{0\}$, the set
$S_i = X_{i-1} \cap X_i$ has cardinality one or two. We also set $G_i = G[X_i]$ and we
set $\overline{G}_i = \text{cl}(\text{cl}(G_i, S_i), S_{i+1})$ where $i \in I \setminus \{0, 2k^3 + 1\}$. Moreover, from the
fact that $\mathcal{X}$ was chosen so to have the maximum number of leaves, there is
no other bag of $\mathcal{X}$ containing all the vertices of $S_i$. Also, by the fact that the
minimum degree of $G$ is at least 3, we obtain that at most the half of the $G_i$s are bridges. Therefore, by contracting all these bridges, we may assume that
each $\overline{G}_i$ is either 3-connected or a triangle.

Observe also that if $S_i = \{x\}$ and $S_j = \{y\}$, $j > i$ and $G_{i,j} = \cup_{h \in \{i, \ldots, j\}} G_i$ is
2-connected, then $G_{i,j}$ contains $K^{1}_{i}$ as a lift minor with base vertices the vertices
in $x$ and $y$. But this means that no more than $2k$ $S_i$s may have cardinality 1 as
this would imply the existence of a $F_k$ as a lift minor of $G$. As a consequence of
this, there is a subsequence of $|S_1|, \ldots, |S_{k^2}|$ that consists of $2k^2$ consecutive 2's.
This, in turn, means that either there are $2k$ consecutive $\overline{G}_i$s where $|S_i \cap S_{i+1}| = 1$
or there are $k$ consecutive $\overline{G}_i$s where $|S_i \cap S_{i+1}| = 0$. In both cases $F_k$ is a lift
minor of $G$ and the result follows. \hfill $\square$

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