Lift Contractions

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$G = (V, E)$ is finite, undirected graph, no loops, no multiple edges
We study graph containment relations.
These are defined by the type of graph operation permitted.
We consider a number of basic graph operations, such as the

- vertex deletion
- edge deletion
- edge contraction
- vertex dissolution

The edge contraction of an edge $e = uv$ removes $u$ and $v$ from $G$, and replaces them by a new vertex adjacent to precisely those vertices to which $u$ or $v$ were adjacent.

If one of the two vertices, say $u$, has exactly two neighbors which in addition are nonadjacent, then we call this operation the vertex dissolution of $u$. 
### Known Graph Containment Relations

<table>
<thead>
<tr>
<th>Containment Relation</th>
<th>VD</th>
<th>ED</th>
<th>EC</th>
<th>VDi</th>
</tr>
</thead>
<tbody>
<tr>
<td>minor</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>induced minor</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>topological minor</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>induced topological minor</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>contraction</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
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<td>no</td>
<td>no</td>
<td>yes</td>
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<tr>
<td>subgraph</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
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<tr>
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<td>no</td>
<td>no</td>
</tr>
<tr>
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<td>no</td>
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<tr>
<td>isomorphism</td>
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<td>no</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

VD = Vertex Deletion  
ED = Edge Deletion  
EC = Edge Contraction  
VDi = Vertex Dissolution

**Example.** A graph $G$ contains a graph $H$ as an induced minor if $G$ can be modified to $H$ by a sequence of graph operations which may only include vertex deletions and edge contractions (and vertex dissolutions) but no edge deletions.
Another Graph Operation

Let \( e = uv \) and \( e' = uw \) be two different edges incident with the same vertex \( u \) in a graph \( G \).

The lift (or splitting off) of \( e \) and \( e' \) removes \( e \) and \( e' \) from \( G \) and then adds the edge \( vw \) in case \( v \) and \( w \) are nonadjacent.

- A lift does not change the number of vertices of \( G \).
- A dissolution can be simulated by a lift and a vertex deletion.
Immersions are known.

Theorem (Robertson & Seymour, 2010)

*In every countable sequence of graphs, there exist two different graphs such that one contains the other one as an immersion.*

We introduce the other two containment relations.

In particular we are interested in lift contractions.

Our results also hold for lift minors as any lift contraction is a lift minor.
1. We present three useful lemmas on
   - complete graphs
   - grids
   - fans.

2. We give our main results and sketch their proofs using these lemmas.
Complete Graphs

Let $K_p$ denote the complete graph on $p$ vertices.

Lemma (Complete Lemma)

The graph $K_{2n}$ contains every $n$-vertex graph as a lift contraction.

Proof. Let $G$ be an $n$-vertex graph.

Each vertex $u$ in $G$ corresponds to two vertices $u$ and $u'$ in $K_{2n}$.

Then, for each non-edge $uv$ in $G$, we perform the following sequence of graph operations in $K_{2n}$. 
Afterwards contract any remaining edge $ww'$ in $K_{2n}$. This yields $G$. 
Let $M_{k,\ell}$ denote the $k \times \ell$ grid.

![Grid Diagram]

**Lemma (Grid Lemma)**

The grid $M_{2n,2n^2-2}$ contains every $n$-vertex graph as a lift contraction.

By the **Complete Lemma** we only have to show that $M_{2n,2n^2-2}$ contains $K_{2n}$ as a lift contraction.
$n = 2$
Let $F_k$ denote the $k$-vertex fan.

**Lemma (Fan Lemma)**

The fan $F_{4n^2-2n+2}$ contains every $n$-vertex graph as a lift contraction.

By the **Complete Lemma** we only have to show that $F_{4n^2-2n+2}$ contains $K_{2n}$ as a lift contraction.
$n = 2$
Our Main Results

1. Every connected graph of degeneracy $\geq 400n$ has every $n$-vertex graph as a lift contraction.

2. There is a constant $c$ such that every connected graph of treewidth $\geq c \cdot n^4$ has every $n$-vertex graph as a lift contraction.

3. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every 2-connected graph of pathwidth $\geq f(n)$ has every $n$-vertex graph as a lift contraction.

4. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every connected graph on $\geq f(n)$ vertices and minimum degree $\geq 3$ has every $n$-vertex graph as a lift contraction.

1-4 do not hold if lift contractions are replaced by contractions or lifts, respectively.

**Counter example:** a sufficiently large complete graph.
Theorem

*Every connected graph of degeneracy at least $400n$ contains every $n$-vertex graph as a lift contraction.*

DeVos, Dvorak, Fox, McDonald, Mohar and Scheide (2011) showed that such a graph contains $K_{2n}$ as an immersion.

Due to the **Complete Lemma**, we only have to show that the sequence $S$ of graph operations that yields $K_{2n}$ can be modified into a **sequence of edge contractions and lifts**.

By definition, $S$ consists of vertex deletions, edge deletions and lifts.

1. Remove all vertex deletions and edge deletions from $S$.
2. Replace every lift that makes the graph disconnected by a sequence of edge contractions and one vertex dissolution.
3. Contract all edges adjacent to a vertex not in the $K_{2n}$.
Theorem

There exists a constant $c$ such that every connected graph $G$ of treewidth at least $c \cdot n^4$ contains every $n$-vertex graph as a lift contraction.

Robertson and Seymour (1986) showed that a graph $G$ contains a large grid as a minor provided the treewidth of $G$ is large enough.

We can use this and the Grid Lemma to show a bound on the treewidth.

We can obtain the bound of the theorem as follows.

Birmelé, Bondy and Reed (2007) showed that any graph that does not contain the $k$-prism (i.e. the $C_k \times K_2$) as a minor has treewidth $O(k^2)$.

Note that $F_k$ is a minor of the $k$-prism.

We then show how to use the Fan Lemma.
Proof Sketch 3

**Theorem**

There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every 2-connected graph of pathwidth at least $f(n)$ contains every $n$-vertex graph as a lift contraction.

Thomas (1996) showed that for any two graphs $G$ and $H$ such that $G$ is an outerplanar graph and $H$ has a vertex whose removal leaves a tree, there is a constant $c_{G,H}$ such that every 2-connected graph of pathwidth at least $c_{G,H}$ contains $G$ or $H$ as a minor.

The fan is outerplanar and can be changed to a tree (path) after removing the dominating vertex.

Hence, we take both $G$ and $H$ to be a fan.

We then show how to apply the Fan Lemma.
Future Work

Determine the computational complexity of the problems that are to test whether

- a given graph $G$ contains a fixed graph $H$ as a lift minor;
- a given graph $G$ contains a fixed graph $H$ as a lift contraction.

- When both $G$ and $H$ are part of the input, these two problems are NP-complete. This can be observed from a corresponding result of Matoušek and Thomas (1992) for minors and contractions, respectively.

- Grohe, Kawarabayashi, Marx and Wollan (2011) showed that for any fixed graph $H$ the problem of testing if a given graph $G$ contains $H$ as an immersion can be solved in cubic time.