

Problem 1 (30 points). Set up a linear programming problem and solve it for the following situation. A farmer owns a farm which produces corn, soybeans, and oats. There are 10 acres of land. Each crop that planted has certain requirements for labor and capital and provides a certain profit. The following table represents for each crop required labor (in hours), capital (in \$\$), and net profit (in \$\$):

Corn (per acre)	9	200	17
Soybeans (per acre)	5	250	27
Oats (per acre)	10	100	29

The farmer has \$2000 and 90 hours available for working this crops. How much of each crop should be planted to maximize profit ?

Solution (brief)

Suppose the farmer plants  $x$  acres of corn,  $y$  acres of soy and  $z$  acres of oats.

The constraints are:

$$x + y + z \leq 10 \text{ (A)}$$

$$9x + 5y + 10z \leq 90 \text{ (B)}$$

$$200x + 250y + 100z \leq 2000 \text{ (C)}$$

$$x \geq 0 \text{ (D)}$$

$$y \geq 0 \text{ (E)}$$

$$z \geq 0 \text{ (F)}$$

Objective function to maximize:  $17x + 27y + 29z$ . Optimal solution  $(0, 2, 8)$ .

Problem 2 (55 points). For the previous problem construct the set  $S$  of all feasible solutions and find:

- all extreme points of  $S$ ;
- all degenerate extreme points of  $S$ ; How many hyperplanes intersect in each such point ?
- at least 3 basic non-feasible systems;
- a non-basic system, if possible.
- The objective function is defined by vector  $(17, 27, 29)$ . Consider objective functions defined by other vectors:  $(17, 29, 27)$ ,  $(15, 30, 30)$ ,  $(25, 25, 25)$ . In each case find all optimal solutions.
- For each vertex of  $S$  suggest an objective function such that this vertex is the unique optimum.
- Construct, if possible, an objective function such that the problem is unbounded.

Solution (brief)

$S$  is a pyramid with the apex  $(10, 0, 0)$  and the pentagon base with vertices  $(0, 0, 0)$ ,  $(0, 8, 0)$ ,  $(0, 20/3, 10/3)$ ,  $(0, 2, 8)$ ,  $(0, 0, 9)$ .

(a) See above.

(b) It is  $(10, 0, 0)$ . Five hyperplanes intersect here.

(c)  $[BCD] = (0, 5.5, 6.25)$ ,  $[CDE] = (0, 0, 20)$ ,  $[BDF] = (0, 18, 0)$ ,  $[ADE] = (0, 0, 10)$ ,  $[ADF] = (0, 10, 0)$ .

(d) All inequalities are linearly independent. So all systems are basic.

(e)  $(17, 29, 27)$ . Point  $(0, 20/3, 10/3)$ .

$(15, 30, 30)$ . Any point of segment  $[(0, 20/3, 10/3), (0, 2, 8)]$ .

$(25, 25, 25)$ . Any point of triangle with the vertices  $(10, 0, 0)$ ,  $(0, 20/3, 10/3)$ ,  $(0, 2, 8)$ .

(f)  $(10, 0, 0)$   $(1, 0, 0)$ ,  $(0, 0, 0)$   $(-1, -1, -1)$ ,  $(0, 8, 0)$   $(0, 1, 0)$ ,  $(0, 20/3, 10/3)$   $(17, 29, 27)$ ,  $(0, 2, 8)$   $(17, 27, 29)$ ,  $(0, 0, 9)$   $(0, 0, 1)$ .

(g) The set is bounded, so any linear function is bounded on it.

Problem 3 (30 points). A polyhedron  $P$  is the intersection of  $m$  half-spaces in  $\mathbb{R}^n$ . For (a)  $m = 4, n = 3$ , (b)  $m = 4, n = 4$ , (c)  $m = 3, n = 4$

how many vertices can be in  $P$  ?

how many degenerate vertices can be in  $P$  ?

how many non-basic  $n$ -systems can exist ?

how many basic non-feasible  $n$ -systems can exist ?

Provide a construction for each possible answer

Solution

We know how to find the total number of  $n$ -systems ( $m$  choose  $n$ ). A system can be basic or non-basic, a basic system can be feasible (vertex, maybe some vertices coincide) or non-feasible.

(a) The total number of systems is 4. We'll show that both number of vertices and number of non-feasible basic  $n$ -systems can be any 0 to 4.

The set of 4 half-spaces with 4 vertices:

$x \geq 0$ ,  $y \geq 0$ ,  $z \geq 0$ ,  $x + y + z \leq 1$  (You can take any other system). For any vertex three of these four inequalities become equalities and one is satisfied as inequality. Thus, if we change the comparison sign in inequality, one of the basic systems becomes non-feasible. We can change any number of comparison signs and get this number of non-feasible basic systems.

The number of non-basic systems can be 0, 1, 2 or 4. Examples.

0. Each three of four are linearly independent (see above).

1. Three are linearly dependent. The rest is independent with any two of these three:  $x \geq 0$ ,  $y \geq 0$ ,  $z \geq 0$ ,  $x + y \leq 1$

2. There are three independent. The fourth is parallel to one of them.  $x \geq 0$ ,  $y \geq 0$ ,  $z \geq 0$ ,  $x \leq 1$

4. All four are parallel.  $x \geq 0$ ,  $x \geq -1$ ,  $x \leq 1$ ,  $x \leq 2$

Proof that three of 4 systems cannot be non-basic. Suppose that  $A, B, C, D$  are vectors orthogonal to corresponding planes and  $A, B, C$  is the only independent set of 3.  $D$  cannot be parallel to one of these three vectors, otherwise one more system will be independent. Then, since  $A, B, D$  are dependent, we can express  $D$  through  $A$  and  $B$  and both coefficients will be non-zero.  $A, C$  and  $D$  are dependent, so some combination with all non-zero coefficients is zero.

Now we substitute the expression we got for  $D$ . We got a linear combination of  $A, B, C$ , which is zero and the coefficient of  $B$  is not zero. So, they are also dependent. Contradiction.

It can be only one degenerate vertex, if all four planes intersect in it (or none otherwise). Example  $x + y \leq z, -x + y \leq z, x - y \leq z, -x - y \leq z$

(b) There is only one  $n$ -system. It can be basic or non-basic. It cannot be basic non-feasible, as the vertex belong to all half-spaces. The vertex cannot be degenerate, as we have only  $n$  hyperplanes.

(c) There is no  $n$ -systems at all. So, the answer to all questions is zero.

Problem 4 (35 points). A network is given by the capacity matrix:

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094000
002030
020100
004025
030208
900000
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Let the source  $s$  and sink  $t$  be: (a) (1,6), (b) (1,4), (c) (4,6). For each case find the maximal flow from  $s$  to  $t$  and a minimal cut.

(a) The cut: (123-456). (34)+(25)=4. The flow of 4 is reached by (1256) = 3 and (1346) = 1.

(b) The cut: (12356-4). (34)+(54)=3. The flow of 3 is reached by (1254) = 2 and (134) = 1.

(c) The cut: (43-1256). (45)+(46)+(32)=9. The flow of 9 is reached by (46) = 5, (456) = 2 and (43256) = 2.

Any solution is good if:

the cut contains all vertices, the source at the left, the sink at the right;

the capacity of cut equals to the sum of capacities of all arcs going from the left part of cut to the right part;

the flow doesn't exceed the capacities of edges;

the volume of flow equals to the capacity of the cut.