HOMEWORK 1, 
FEB. 12, DUE FEB. 28, 2008

Problem 1 (35 points)
Local distances between six vertices \( V = \{1, 2, 3, 4, 5, 6\} \) are given by the following vertex-vertex incidence matrix:

\[
\begin{array}{cccccc}
0 & 1 & x & 7 & x & x \\
x & 0 & 2 & x & 8 & x \\
x & x & 0 & 3 & x & 9 \\
8 & x & x & 0 & 4 & x \\
x & 7 & x & x & 0 & 3 \\
2 & x & 6 & x & x & 0 \\
\end{array}
\]

where symbol \( x \) stands for \( +\infty \).

a) Find the vertex-arc incidence matrix and set up an LP model to find the distance from 1 to 6 and from 6 to 1. Do not solve the obtained LP problem.

b) By the Dijkstra algorithm find the tree of the shortest paths and distance \( D_{i,j} \) from vertex \( i \) to all other vertices; do for \( i = 1, 3, 5 \).

c) Prove that \( D_{a,b} + D_{b,c} \geq D_{a,c} \geq 0 \) for every triplet \( a, b, c \in V \).

d) Is it true that \( D_{a,b} = D_{b,a} \) for every pair \( a, b \in V \)?

If yes, prove it; if not give an example.

Problem 2 (35 points) Local distances between vertices \( V = \{1, 2, 3, 4, 5\} \) are

\[
\begin{align*}
&d_{1,2} = 1, \quad d_{1,3} = 3, \quad d_{2,3} = 1, \quad d_{2,4} = 4, \quad d_{3,2} = 2, \\
&d_{3,4} = 1, \quad d_{4,2} = -3, \quad d_{4,3} = 2, \quad d_{5,2} = -1, \quad d_{5,4} = 3,
\end{align*}
\]

a) Find the vertex-vertex incidence matrix \( A_{VV} \).

b) Find the vertex-arc incidence matrix \( A_{VE} \) and set up LP models to find the distance from 1 to 3 and from 3 to 5. Do not solve the obtained LP problems.

c) Find the distance \( D_{i,j} \) for all pairs \( i, j \in V \).

Problem 3 (40 points) Given a pipeline with six nodes \( V = \{1, 2, 3, 4, 5, 6\} \) and 10 arcs:

\[
(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (3, 5), (4, 3), (4, 5), (4, 6), \text{ and } (5, 6).
\]

a) Write the vertex-vertex and vertex-arc incidence matrices of this graph and set up an LP model to find a maximum flow from 1 to 6 for a given capacity vector \( c \). Do not solve the obtained LP problem.

b) Consider the following four capacity vectors:

\[
(1, 5, 4, 3, 2, 4, 3, 2, 1, 5); \quad (5, 4, 3, 2, 1, 1, 2, 3, 4, 5); \\
(9, 1, 3, 5, 7, 8, 6, 4, 2, 10); \quad (9, 7, 5, 3, 1, 2, 4, 6, 8, 10).
\]

In each case find a minimum cut and maximum flow between nodes 1 and 6.
Problem 4 (30 points). For problem 2 on page 57 from the book:
a) set up a LP model and solve the obtained LP problem.
b) Find the feasible set and all extreme points (vertices), feasible and infeasible.
c) Find the value of objective function in each extreme point.

Problem 5 (10 points). For problem 11 on page 60 from the book, set up a LP model
(but do not solve the obtained LP problem) and consider the following two modifications:
a) There are upper limits: the manufacturer can sell at most 110 kg of Chewy, 70 kg of Crunchy, and 100 kg of nutty.
b) Nutty must contain exactly 5 % of raisins.