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Programming Under Probabilistic Constraint with Discrete Random Variable

András Prékopa[a] Béla Vizvári[b] Tamás Badics[c]

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RUTCOR • Rutgers Center for Operations Research • Rutgers University • P.O. Box 5062 • New Brunswick New Jersey • 08903-5062
Telephone: 908-445-3804
Telefax: 908-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/~rrr

[a]RUTCOR, Rutgers Center for Operations Research, Rutgers University, New Brunswick, NJ, 08903. prekopa@rutcor.rutgers.edu
[b]Dept. of Operations Research, Eötvös Loránd University, H-1088 Budapest, Múzeum krt. 6-8. vizvari@cs.elte.hu
[c]RUTCOR, Rutgers Center for Operations Research, Rutgers University, New Brunswick, NJ, 08903. badics@rutcor.rutgers.edu

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András Prékopa   Béla Vizvári   Tamás Badics

Abstract. The most important static stochastic programming models, that can be formulated in connection with a linear programming problem, where some of the right-hand side values are random variables, are: the simple recourse model, the probabilistic constrained model and the combination of the two. In this paper we present algorithmic solution to the second and third models under the assumption that the random variables have a discrete joint distribution. The solution method uses the concept of a p-level efficient point (pLEP) intoduced by the first author (1990) and works in such a way that first all pLEP's are enumerated, then a cutting plane method does the rest of the job.

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1 Introduction

Stochastic programming problems are formulated in such a way that we start from an underlying problem or base problem, in which we observe that some of the parameters are random variables; then, we reformulate the problem, by the use of some decision principle, and obtain a stochastic programming formulation. Assume that the base problem is:

$$\begin{array}{l} \text{Min}\, \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ \mathbf{Ax} = \mathbf{b} \\ \mathbf{Tx} \geq \xi \\ \mathbf{x} \geq \mathbf{0}, \end{array} \tag{1}$$

where $\mathbf{A}$ is an $m \times n$, $\mathbf{T}$ is an $r \times n$ matrix, and $\xi = (\xi_1, ..., \xi_r)^T$ is a random vector. Starting from problem (1), usually we formulate a recourse, or a probabilistic constrained stochastic programming problem, or a problem which combines the two formulations. In the present paper we look at the latter two problems under the assumption that ξ has a discrete distribution with a finite number of possible values, and propose algorithmic solutions for them.

Let $q_1, ..., q_r$ be nonnegative constants and designate by $T_1, ..., T_r$ the rows of the matrix $\mathbf{T}$. Then, our stochastic programming problem is the following:

$$\begin{array}{l} \text{Min}\left(\mathbf{c}^T\mathbf{x} + \sum_{i=1}^r q_i E[\xi_i - T^i\mathbf{x}]_+\right) \\ \text{subject to} \\ \begin{array}{rcl} \mathbf{Ax} & = & \mathbf{b} \\ P(\mathbf{Tx} \geq \xi) & \geq & p \\ \mathbf{x} & \geq & \mathbf{0}, \end{array} \end{array} \tag{2}$$

where p is a given probability $(0 < p < 1)$.

If $F_i(z)$ designates the probability distribution function of ξ_i, i.e., $F_i(z) = P(\xi_i \leq z)$, $z \in \mathbb{R}$, $i = 1, ..., r$, then problem (2) can be written in the form (see, e.g., Prékopa (1990)):

$$\begin{array}{l} \text{Min}\left(\mathbf{c}^T\mathbf{x} + \sum_{i=1}^r q_i \left(\mu_i - T^i\mathbf{x} + \int_{-\infty}^{T_i\mathbf{x}} F_i(z)dz\right)\right) \\ \text{subject to} \\ \begin{array}{rcl} \mathbf{Ax} & = & \mathbf{b} \\ P(\mathbf{Tx} \geq \xi) & \geq & p \\ \mathbf{x} & \geq & \mathbf{0}. \end{array} \end{array} \tag{3}$$

Let $z_{j1} < ... < z_{jk_j}$ be the possible values of ξ_j, $j = 1, ..., r$. We supplement each set of possible values by two further values, which are z_{j0} and z_{j,k_j+1} in case of ξ_j, and assume that they satisfy the following relations: $z_{j0} < z_{j1}$, $z_{jk_j} < z_{j,k_j+1}$. These values should be chosen in such a way that if $\mathbf{x}$ satisfies $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, then it should also satisfy the inequalities

$$z_{j0} \leq T_j\mathbf{x} \leq z_{jk_j+1},\ \ j = 1, ..., r.$$

The set $\mathcal{Z} = \{(z_{1j_1}, ..., z_{rj_r}) \mid 0 \le j_i \le k_i + 1,\ i = 1, ..., r\}$ clearly contains all possible values of the random vector ξ. Due to stochastic dependency of the components of ξ, some elements of $\mathcal{Z}$ may have 0 probability, but for notational convinience we take $\mathcal{Z}$ as the set of possible values of ξ. Let $F(\mathbf{z})$ designate the probability distribution function of ξ, i.e., $F(\mathbf{z}) = P(\xi \le \mathbf{z}),\ \mathbf{z} \in \mathbb{R}^r$.

A point $\mathbf{z} \in \mathcal{Z}$ is called a p-level efficient point (pLEP) of the probability distribution of ξ, if $F(\mathbf{z}) \ge p$ and there is no $\mathbf{y} \in \mathcal{Z}$ satisfying $\mathbf{y} \le \mathbf{z},\ \mathbf{y} \ne \mathbf{z},\ F(\mathbf{y}) \ge p$. Let $\mathcal{E} = \{\mathbf{z}^{(1)}, ..., \mathbf{z}^{(N)}\}$ be the set of all pLEP's.

If we choose $q_1 = ... = q_r = 0$ in problem (3), then an equivalent form of the problem is the following:

$$\text{Min}\,\mathbf{c}^T\mathbf{x} \tag{4a}$$

subject to

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{4b}$$

$$\mathbf{x} \ge \mathbf{0}, \tag{4c}$$

$$\mathbf{T}\mathbf{x} \ge \mathbf{z}^{(i)}, \text{ for at least one } i \in \{1, ..., N\}. \tag{4d}$$

In the general case we intruduce the notations:

$$c_{ij} = -q_i z_{ij} + q_i \int_{z_{i0}}^{z_{ij}} F_i(z)dz,\ j = 0, ..., k_i + 1;\ i = 1, ..., r, \tag{5}$$

and apply the λ-representation for the piecewise linear convex functions

$$f_i(y_i) = -q_i y_i + q_i \int_{z_{i0}}^{z_{ij}} F_i(z)dz,\ z_{i0} \le y_i \le z_{ik_i+1},\ i = 1, ..., r. \tag{6}$$

This means that for each $i = 1, ..., r$ we have:

$$\begin{aligned} f_i(y_i) = \quad & \text{Min} \sum_{j=0}^{k_i+1} f_i(z_{ij})\lambda_{ij} \\ & \text{subject to} \\ & \sum_{j=0}^{k_i+1} z_{ij}\lambda_{ij} \quad = \quad y_i \\ & \quad \sum_{j=0}^{k_i+1} \lambda_{ij} \quad = \quad 1 \\ & \qquad \lambda_{ij} \quad \ge \quad 0, \quad j = 0, ..., k_i + 1. \end{aligned} \tag{7}$$

Using these and the pLEP's, problem (3) can be written in the form (the constant term $\sum_{i=1}^r q_i\mu_i$ is left out from the objective function):

$$\text{Min}\ \left(\mathbf{c}^T\mathbf{x} + \sum_{i=1}^{r}\sum_{j=0}^{k_i+1} c_{ij}\lambda_{ij}\right) \tag{8a}$$

subject to

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{8b}$$

$$\mathbf{T}_i\mathbf{x} - \sum_{j=0}^{k_i+1} z_{ij}\lambda_{ij} = 0 \tag{8c}$$

$$\sum_{j=0}^{k_i+1} \lambda_{ij} = 1 \tag{8d}$$

$$\mathbf{x} \geq \mathbf{0} \tag{8e}$$

$$\lambda_{ij} \geq 0, j = 0, ..., k_i + 1;\ i = 1, ..., r \tag{8f}$$

$$\mathbf{T}\mathbf{x} \geq \mathbf{z}^{(i)} \text{ for at least one } i \in \{1, ..., N\}. \tag{8g}$$

We present algorithmic solutions to problems (4), and (8). To this end, we have to study the set $\mathcal{E}$ from two points of view: how to enumerate them algorithmically and what are their geometric properties.

2 Algorithmic Enumeration of the p-Level Efficient Points

The algorithm is presented in a recursive form. When we enumerate the pLEP's in $\mathbb{R}^r$, we assume that an enumeration technique in $\mathbb{R}^{r-1}$ is available for functions which are not necessarily probability distribution functions in the sense that the sum of all probabilities we are dealing with may be smaller than 1. Thus, it is convenient to define the function $F(z_1, ..., z_r)$, the pLEP's of which are to be determined, right at the outset in a somewhat more general manner.

We assume that to each element u in $\mathcal{Z} = \mathbb{Z}_1 \times ... \times \mathbb{Z}_r$ a probability p_u is assigned such that $\sum_{\mathbf{u}\in\mathcal{Z}} p_{\mathbf{u}} \leq 1$, and then define

$$F(\mathbf{z}) = \sum_{\substack{\mathbf{u}\in\mathcal{Z}\\ \mathbf{u}\leq\mathbf{z}}} p_{\mathbf{u}},\ \ \mathbf{z} \in \mathbb{R}^n. \tag{9}$$

For the function (9) the set $\mathcal{E}$ of all pLEP's can similarly be defined as we have done it for the probability distribution functions in Section 1. In the present case, however, the set $\mathcal{E}$ may be empty for some p values. In case of $r = 1$, the pLEP is that element h of $\mathbb{Z}_1$, which satisfies

$$h = \text{argmin}\{l \mid F(z_{1l}) \geq p\}, \tag{10}$$

provided that such an h exists. In the general case the steps of the algorithm are as presented below. We assume that $F(z_{1,k_1+1}, ..., z_{r,k_r+1}) \geq p$.

Step 0. Initialize $k \leftarrow 0$. Go to **Step 1.**

Step 1. Let

$$\begin{aligned} z_{1,j_1} &= \operatorname{argmin}\{y \mid F(y, z_{2,k_2+1}, \ldots, z_{r,k_r+1}) \geq p\} \\ z_{2,j_2} &= \operatorname{argmin}\{y \mid F(z_{1,j_1}, y, z_{3,k_3+1}, \ldots, z_{r,k_r+1}) \geq p\} \\ &\cdot \\ &\cdot \\ &\cdot \\ z_{r,j_r} &= \operatorname{argmin}\{y \mid F(z_{1,j_1}, \ldots, z_{r-1,j_{r-1}}, y) \geq p\}. \end{aligned}$$

Go to **Step 2.**

Step 2. Let $\mathcal{E} \leftarrow \{z_{1,j_1}, \ldots, z_{r,j_r}\}$. Go to **Step 3.**

Step 3. Let $k \leftarrow k+1$. If $j_1 + k > k_1 + 1$, then go to **Step 5.** If $j_1 + k \leq k_1 + 1$, then go to **Step 4.**

Step 4. Enumerate all pLEP's of the function $F(z_{1,j_1+k}, \mathbf{y})$ of the variable $\mathbf{y} \in \mathbb{R}^{r-1}$, and eliminate those which dominate at least one element in $\mathcal{E}$ ($\mathbf{y}$ dominates $\mathbf{z}$, if $\mathbf{y} \geq \mathbf{z}$, and $\mathbf{y} \neq \mathbf{z}$). If $\mathcal{H}$ is the set of the remaining pLEP's, which may be empty, then let $\mathcal{E} \leftarrow \mathcal{E} \cup \mathcal{H}$. Go to **Step 3.**

Step 5. Stop, $\mathcal{E}$ is the set of all pLEP's of the function $F(\mathbf{z})$, $\mathbf{z} \in \mathcal{Z}$.

Example 1. Let $\mathcal{Z}_1 = \mathcal{Z}_2 = \{0,1,2,3,4,5,6,7,8,9\}$, $p_{ih} = 0.019$, if $0 \leq i \leq 4$, $0 \leq h \leq 5$, or $h = 8, 9$; $p_{ih} = 0.038$, if $0 \leq i \leq 4$, $h = 6$; $p_{ih} = 0$, if $0 \leq i \leq 4$, $h = 7$; $p_{ih} = 0.001$ if $5 \leq i \leq 9$, $0 \leq h \leq 9$, and $p = 0.6$.

In **Step 1** we determine

$$\begin{aligned} 3 &= \operatorname{argmin}\{y \mid F(y, 9) \geq 0.6\} \\ 6 &= \operatorname{argmin}\{y \mid F(3, y) \geq 0.6\}. \end{aligned}$$

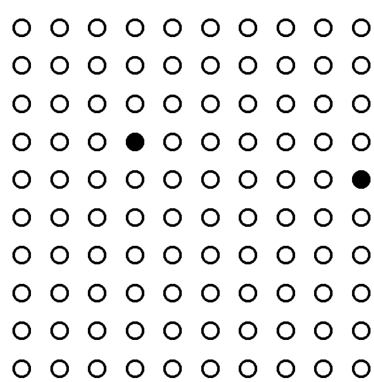

Figure 1: The lattice points are assigned to the set of integers $\{(i,h) \mid 0 \leq i,\ h \leq 9\}$. The corresponding probabilities are those mentioned in Example 1 of Section 2. The marked points are the 0.6-level efficient points.

Thus, $(z_{1,j_1}, z_{2,j_2}) = (3,6)$, and at the end of **Step 2** we have $k = 0$, $\mathcal{E} = \{(3,6)\}$.

In **Step 3** we take $k = 1$, and go to **Step 4**, where we obtain $\mathcal{H} = \{(4,6)\}$. Then we eliminate $(4,6)$, and define $\mathcal{E} = \{(3,6)\}$.

Then we go to **Step 3** but $\mathcal{H}$ will be empty for $k = 2,3,4,5$. In case of $k = 6$ we obtain $\mathcal{H} = \{(9,5)\}$, and the procedure terminates. The set of all pLEP's is $\mathcal{E} = \{(3,6),(9,5)\}$ (see Figure 1).

3 Properties of the p-Level Efficient Points

First we remark that the notion of a pLEP can be defined for arbitrary probability distributions. In addition to discrete distributions with finite supports we will discuss discrete distributions, where the support is the set $\mathbb{Z}^r_+$ of all lattice points of the nonnegative orthant of $\mathbb{R}^r$ or a subset of it.

A theorem of Vizvári (1977, Thm 4.5, p.28) asserts that any subset $\mathcal{H}$ of $\mathbb{Z}^r_+$ which has the property that if $\mathbf{x}, \mathbf{y} \in \mathcal{H}$, $\mathbf{x} \neq \mathbf{y}$, and neither $\mathbf{x} \leq \mathbf{y}$ nor $\mathbf{y} \leq \mathbf{x}$ holds, is necessarily finite. This implies that for any $0 < p < 1$, and for any probability distribution with support in $\mathbb{Z}^r_+$ the set of all pLEP's is finite. The question that what are the sets that may be pLEP sets is answered by the following theorem.

Theorem 1 *Let $\mathcal{H}$ be any nonempty subset of $\mathbb{Z}$ (or $\mathbb{Z}^r_+$) such that if $\mathbf{x}, \mathbf{y} \in \mathcal{H}$, $\mathbf{x} \neq \mathbf{y}$, then neither $\mathbf{x} \leq \mathbf{y}$ nor $\mathbf{y} \leq \mathbf{x}$ holds. Then, for any $0 < p < 1$ there exists a probability distribution with support $\mathbb{Z}$ (or $\mathbb{Z}^r_+$), the set of all pLEP's of which is $\mathcal{H}$.*

Proof. Let N be the number of elements of $\mathcal{H}$. If $N = 1$ then the probability p is assigned to the single point $\mathbf{x}$ of $\mathcal{H}$ and the probability $1-p$ is assigned to another point $\mathbf{z}$ of $\mathbb{Z}$ (or $\mathbb{Z}^r_+$) satisfying $\mathbf{x} \neq \mathbf{z}$, $\mathbf{x} \leq \mathbf{z}$. Assume that $N \geq 2$. At first let us suppose that $\frac{1}{N} \leq p < 1$. Then

$$\varepsilon = \frac{1-p}{N-1} \leq p.$$

Let $\mathbf{z} \in \mathbb{Z}$ (or $\mathbf{z} \in \mathbb{Z}^r_+$) be a point in $\mathbb{R}^r$ that satisfies $\mathbf{z} \leq \mathbf{x}$, for every $\mathbf{x} \in \mathcal{H}$. We define the probability distribution by assigning probability $p - \varepsilon$ to $\mathbf{z}$, and probability ε to each element of $\mathcal{H}$. Assume that $0 < p < \frac{1}{N}$. Let $\mathbf{z} \in \mathbb{Z}$ (or $\mathbf{z} \in \mathbb{Z}^r_+$) be a point in $\mathbb{R}^r$ that satisfies $\mathbf{z} \geq \mathbf{x}$, for every $\mathbf{x} \in \mathcal{H}$. Then we define the probability distribution by assigning probability $1 - Np$ to $\mathbf{z}$, and probability p to each element of $\mathcal{H}$.

□

Definition 1 *A set $\{z^{(i)}, i = 1, ..., N\}$ is said to be a discrete convex set if $z^j \notin riconv\{z^{(i)}, i = 1, ..., N\}$, $j = 1, ..., N$.*

In most cases we have encountered so far the pLEP sets turned out to be convex.

Below we present three examples to show that the set of pLEP's can be nonconvex, even in case of some well-known discrete probability distributions.

Example 1. Let $r = 2$ and define the birariate probability distribution on $\mathbb{Z}_+^2$ as follows

$$p_{ik} := \frac{1}{(i+1)(i+2)} \frac{1}{(k+1)(k+2)}, \quad i \geq 0, \ \ k \geq 0. \tag{11}$$

The probability distribution function of this distribution is $F(z_1, z_2) = F(z_1)F(z_2)$ if z_1, z_2 are nonegative integers, where $F(z) = \frac{z}{z+1}, \ z \in \mathbb{Z}_+^1$.

pLEP's (i,k)	Probability levels $F(i,k)$
(99,10)	0.9
(32,13)	0.9004329
(27,14)	0.9
(24,15)	0.9
(22,16)	0.9002557545
(21,17)	0.9015151515
(19,18)	0.9
(18,19)	0.9
(17,21)	0.9015151515
(16,22)	0.9002557545
(15,24)	0.9
(14,27)	0.9
(13,32)	0.9004329
(10,99)	0.9

Table 1: The set of 0.9-level efficients points and the corresponding probabilities in case of the discrete distribution given by (11).

Let $p = 0.9$. The set of pLEP's together with the corresponding probability distribution function values are presented in Table 1. The set $\mathcal{E}$ is nonconvex, because the points $(21, 17)$, and $(17, 21)$ are interior points of the convex hull of the pLEP's. By symmetry it is enough to show it for one of these points. It can easily be checked that $(21, 17) = \lambda_1(22, 16) + \lambda_2(19, 18) + \lambda_3(10, 99)$ where

$$\lambda_1 = \frac{153}{225}, \quad \lambda_2 = \frac{71}{225}, \quad \lambda_3 = \frac{1}{225}.$$

Example 2. Bivariate geometric distribution. Let again $r = 2$, and define the bivariate probability distribution as follows:

$$p_{ik} := (1-q)^2 q^{i+k}, \ \ i, k \geq 0. \tag{12}$$

Thus, each marginal distribution is geometric, with probabilities $(1-q)p^i, \ i = 0, 1, \ldots.$

If we choose $q = 0.95$, and $p = 0.9$, then there are 27 pLEP's. We present them, together with the corresponding probability levels (values of the probability distribution function), in

Table 2. The set $\mathcal{E}$ is nonconvex, because the points $(55,60)$ and $(60,55)$ are interior points of the convex hull of $\mathcal{E}$. By symmetry it is enough to show it for one of these points. We can easily check that $(55,60) = \lambda_1(54,61) + \lambda_2(56,58) + \lambda_3(143,44)$, where

$$\lambda_1 = \frac{2720}{3961}, \quad \lambda_2 = \frac{1224}{3961}, \quad \lambda_3 = \frac{17}{3961}.$$

pLEP's (i,k)	Probability levels $F(i,k)$
(44,143)	0.9000016434
(45,99)	0.9001705286
(46,87)	0.9002818328
(47,80)	0.9003904620
(48,75)	0.9003710351
(49,71)	0.9000762339
(50,69)	0.9013348881
(51,66)	0.9006190340
(52,64)	0.9007330886
(53,62)	0.9003042365
(54,61)	0.9013589450
(55,60)	0.9021475573
(56,58)	0.9003777116
(57,57)	0.9005120454
(58,56)	
(60,55)	
(61,54)	
(62,53)	
(64,52)	
(66,51)	
(69,50)	$F(k,i) = F(i,k)$
(71,49)	
(75,48)	
(80,47)	
(87,46)	
(99,45)	
(143,443)	

Table 2: The set of 0.9-level efficient points and the corresponding probabilities, in case of the discrete distribution given by (12).

Example 3. Bivariate Poisson distribution. Let $r = 2$, and define

$$p_{ik} := \frac{\lambda^i}{i!} e^{-\lambda} \frac{\lambda^k}{k!} e^{-\lambda}, \; i,k \geq 0. \tag{13}$$

Each marginal distribution is a Poisson distribution, with parameter λ.

For the case of $\lambda = 100$, and $p = 0.01$ the set of pLEP's, and the corresponding probability distribution function values are tabulated in Table 3. The points $(84, 91)$, $(91, 84)$ are interior points of the convex hull of $\mathcal{E}$. In fact, the point $(84, 91)$ is in the interior of the triangle determined by the points $(83, 92)$, $(85, 89)$, $(133, 70)$. The assertion for $(91, 84)$ follows by symmetry.

pLEP's (i, k)	Probability levels $F(i, k)$
(70,133)	0.01000114075
(75,107)	0.01030997036
(79,102)	0.01055318148
(80,98)	0.01012056273
(81,96)	0.01071687362
(82,94)	0.01088975998
(83,92)	0.01059854598
(84,91)	0.01144575315
(85,89)	0.01035405859
(86,88)	0.01066081614
(87,87)	0.01076508548
(88,86)	
(89,85)	
(91,84)	
(92,83)	
(94,82)	
(96,81)	$F(k, i) = F(i, k)$
(98,80)	
(102,79)	
(107,75)	
(133,70)	

Table 3: The set of 0.01-level efficient points and the corresponding probabilities, in case of the discrete distribution given by (13).

In all three examples there exist continuous, and logconcave probability distribution functions such that at the nonnegative lattice points the discrete and the continunous distribution functions coincide. Note that if $F(z)$ is a logconcave probability distribution function, then it is also quasi-concave, and thus, the set $\{z \mid F(z) \geq p\}$ is convex for any $0 < p < 1$. For facts concerning logconcavity see Prékopa (1995).

In example 1 the continuous distribution function is:

$$G(z_1, z_2) = \frac{z_1}{z_1 + 1} \frac{z_2}{z_2 + 1}. \tag{14}$$

In example 2 the continuous distribution function is:

$$G(z_1, z_2) = (1 - e^{-0.0513z_1})(1 - e^{-0.0513z_2}). \tag{15}$$

Finally, in example 3 the continuous distribution function is (see Prékopa (1995)):

$$\begin{aligned} G(z_1, z_2) &= \int_\lambda^\infty \frac{x^{z_1}}{\Gamma(z_1+1)} e^{-x} dx \int_\lambda^\infty \frac{y^{z_2}}{\Gamma(z_2+1)} e^{-y} dy \\ &= \frac{\int_\lambda^\infty x^{z_1} e^{-x} dx}{\int_0^\infty x^{z_1} e^{-x} dx} \frac{\int_\lambda^\infty y^{z_2} e^{-y} dy}{\int_0^\infty y^{z_2} e^{-y} dy}, \quad z_1 \geq -1, \ z_2 \geq -1. \end{aligned} \tag{16}$$

The logconcavity of the function (14) is trivial. Note that the probability density function of $G(z) = \frac{z}{z+1}$ is not logconcave, but logconvex.

The function (15) is also trivially logconcave. The probability density function of the exponential distribution function $(1 - e^{-\lambda z})' = \lambda e^{-\lambda z}$ is both logconcave, and logconvex: it is loglinear.

For the logconcavity of the function (16) see Prékopa (1995). There is no information about the logconcavity of the probability density function

$$\begin{aligned} \frac{d}{dz} G(z) &= \frac{d}{dz} \frac{\int_\lambda^\infty x^z e^{-x} dx}{\int_0^\infty x^z e^{-x} dx} \\ &= G(z) \left(\frac{\int_\lambda^\infty x^z \ln x e^{-x} dx}{\int_\lambda^\infty x^z e^{-x} dx} - \frac{\int_0^\infty x^z \ln x e^{-x} dx}{\int_\lambda^\infty x^z e^{-x} dx} \right). \end{aligned}$$

We know, however, that the elements of the Poisson distribution: $p_i = \frac{\lambda^i}{i!} e^{-\lambda}, \ i = 0, 1, 2, \ldots$ form a logconcave sequence.

Thus, in all of the three examples the bivariate p-quantile, defined by,

$$\{(z_1, z_2) \mid G(z_1, z_2) = p\}$$

is a convex curve and

$$\{(z_1, z_2) \mid G(z_1, z_2) \geq p\}$$

is a convex set. This suggests that if there exists a continous and logconcave probability distribution function that coincides with the discrete probability distribution function at the lattice points, then the set of pLEP's may not be far away from convexity.

4 Cutting Plane Method for the Solution of Relaxations of Problems (4) and (8)

First we consider problem (4). A relaxation of it is obtained by replacing the constraints

$$\begin{array}{rcl}\mathbf{Tx} & \geq & \sum_{i=1}^{N}\mathbf{z}^{(i)}\mu_i \\ & & \sum_{i=1}^{N}\mu_i = 1 \\ & & \mu_i \geq 0, i \in \{1,...,N\}\end{array} \tag{17}$$

for the constraint (4d). If in (17) we introduce surplus variables to make the first constraint equality and use (17) in that form, then the relaxation of problem (4) is the following:

$$\text{Min}\,\mathbf{c}^T\mathbf{x} \tag{18a}$$

subject to

$$\mathbf{Ax} = \mathbf{b} \tag{18b}$$

$$\mathbf{Tx} - \mathbf{u} - \sum_{i=1}^{N}\mathbf{z}^{(i)}\mu_i = \mathbf{0} \tag{18c}$$

$$\sum_{i=1}^{N}\mu_i = 1 \tag{18d}$$

$$\mathbf{x} \geq \mathbf{0},\ \mathbf{u} \geq \mathbf{0},\ \mu \geq \mathbf{0}, \tag{18e}$$

where $\mu = (\mu_1, ..., \mu_N)^T$. The matrix of the problem is presented in Figure 2.

$\mathbf{A}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{T}$	$-\mathbf{I}$	$-\mathbf{z}^{(1)} ... - \mathbf{z}^{(N)}$
$\mathbf{0}$	$\mathbf{0}$	1 ... 1

Figure 2: The matrix of the equality constraints in problem (18a-d).

Clearly, we have the relation:

$$\{\mathbf{x} \mid \mathbf{Ax} = b;\ \mathbf{x} \geq \mathbf{0};\ \exists i \in \{1, ..., N\}, \mathbf{Tx} \geq \mathbf{z}^i\} \subset \{(\mathbf{x} \mid (\mathbf{x}, \mathbf{u}, \mu) \text{ is feasible in (18a-e)}\} \tag{19}$$

which shows that problem (18a-e) is in fact a relaxation of problem (4).

We enumerate the pLEP's and then solve problem (18a-d) by a cutting plane method. We omit the constraints (18c-d), and subsequently generate cuts towards the constraints (17).

Let $\overline{\mathbf{x}}$ be a relative interior point of conv$\{\mathbf{z}^{(i)}, i = 1, ..., N\}$. We can take, e.g.,

$$\overline{\mathbf{z}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{z}^{(i)}.$$

Since the set of pLEP's can be concentrated on an affine manifold with dimension smaller than r, first we look at the system of linear equations (for the components of $\mathbf{w}$):

$$\mathbf{w}^T(\mathbf{z}^{(i)} - \overline{\mathbf{z}}) = 0, \quad i = 1, ..., N. \tag{20}$$

If $\mathbf{w}_1, ..., \mathbf{w}_k$ is a maximum number of linearly independent solutions of (20), then we append the contraints

$$\mathbf{w}_l^T(\mathbf{Tx} - \mathbf{u}) = 0, \quad l = 1, ..., k \tag{21}$$

to the constraint $\mathbf{Ax} = \mathbf{b}$, and keep them together throughout the procedure. For the sake of simplicity of the discussion, right at the outset we assume that the constraint $\mathbf{Ax} = \mathbf{b}$ already contains the constraints (21).

To create a cutting plane in the kth iteration, we formulate the auxiliary problem

$$\begin{aligned} \text{Min}\, \mathbf{e}^T\mu &= \alpha \\ \text{subject to} & \\ \textstyle\sum_{i=1}^{N}(\mathbf{z}^{(i)} - \overline{\mathbf{z}})\mu_i &= \mathbf{Tx}^k - \mathbf{u}^k - \overline{\mathbf{z}} \\ \mu \geq \mathbf{0}, & \end{aligned} \tag{22}$$

where $\mathbf{e} = (1, 1, ..., 1)^T$, $(\mathbf{x}^k, u^k)$ is the current optimal solution, and $\mu = (\mu_1, ..., \mu_N)$ is the decision vector. If problem (18a-d) has a feasible solution, then so does problem (22).

If $\alpha \leq 1$, then we terminate the procedure. The current optimal solution $(\mathbf{x}^k, u^k)$ is an optimal solution to problem (18a-d). If $\alpha > 1$, then we proceed in the following way.

Let $\mathbf{z}^{(i_1)} - \overline{\mathbf{z}}, ..., \mathbf{z}^{(i_{r-k})} - \overline{\mathbf{z}}$ be an optimal basis to problem (22). Then, find a $\mathbf{w} \neq \mathbf{0}$ such that

$$\begin{aligned} \mathbf{w}^T\mathbf{w}_i &= 0, \quad i = 1, ..., h \\ \mathbf{w}^T\left(\mathbf{z}^{(i_j)} - \mathbf{z}^{(i_1)}\right) &= 0, \quad j = 2, ..., r - h. \end{aligned} \tag{23}$$

The $r - 1$ equations (23) determine $\mathbf{w}$ up to a constant factor. Assume that $\mathbf{w}$ is determined in such a way that

$$\mathbf{w}^T(\mathbf{Tx}^k - \mathbf{u}^k - \overline{\mathbf{z}}) < 0. \tag{24}$$

Then, define $\mathbf{w}^{k+1} = \mathbf{w}$, and the cut:

$$(\mathbf{w}^{k+1})^T(\mathbf{Tx} - \mathbf{u} - \overline{\mathbf{z}}) \geq 0. \tag{25}$$

The algorithm that we designate by A1 can be summerized as follows.

Algorithm A1

STEP 1. Enumerate all pLEP's. Initialize $k \leftarrow 0$, and go to STEP 2.

STEP 2. Solve the following LP:

$$\begin{array}{c} \text{Min } \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ \mathbf{A}\mathbf{x} = \mathbf{b} \\ (\mathbf{w}^i)^T(\mathbf{T}\mathbf{x} - \mathbf{u} - \overline{\mathbf{z}}) \ \geq \ 0, \quad i = 1, ..., k \\ \mathbf{x} \geq \mathbf{0}, \ \mathbf{u} \geq \mathbf{0}. \end{array} \tag{26}$$

If $k = 0$, then ignore the constraints involving the cuts. Let $(\mathbf{x}^k, \mathbf{u}^k)$ be an optimal solution to problem (26).

STEP 3. Solve the LP (22). If $\alpha \leq 1$, then STOP, the current $(\mathbf{x}^k, \mathbf{u}^k)$ is an optimal solution to problem (18a-d). If $\alpha > 1$, then go to STEP 4.

STEP 4. Create the cut (25), set $k \leftarrow k + 1$, and go to STEP 2.

If the set of pLEP's is a convex discrete set, then all cuts generate facets of the convex polytope in the affine manifold given by

$$\left\{ \sum_{i=1}^{h} (\mathbf{w}_i - \overline{\mathbf{z}})\beta_i \mid \sum_{i=1}^{h} \beta_i = 1 \right\}.$$

Since the number of facets is finite, it follows that Algorithm A1 terminates in a finite number of steps, and provides us with an optimal solution.

The execution of the above cutting plane method can be done in a similar fashion as we do it in case of Gomory's method for the solution of the integer variable linear programming problem. This means that each time a new cut is created, a surplus variable is assigned to it (x_{n+k} in case of the kth cut) with nonnegativity and integrality restrictions and then, the problem is reoptimized by the use of the (lexicographic) dual method. In addition, once a variable x_{n+k} becomes basic at a later iteration, then we delete x_{n+k} together with its constraint from the problem.

The above remark also shows that Algorithm A1 can be combined with Gomory's algorithm to solve problem (18a-d) with the additional restriction that all components of $\mathbf{x}$ and $\mathbf{u}$ be integers. In this case we are dealing with cuts of the type (25) as well as Gomory cuts. Each time a Gomory cut is created, we reoptimize a problem of the type (18a-d).

Now we turn our attention to problem (8). The relaxation, after the introduction of the surplus vector $\mathbf{u}$, takes the form:

$$
\begin{array}{llllll}
\text{Min } \left(\mathbf{c}^T\mathbf{x} \right. & + & \left.\sum_{i=1}^{r}\sum_{j=0}^{k_i+1} c_{ij}\lambda_{ij}\right) & & & \text{(27a)}\\
\text{subject to} & & & & & \\
\quad \mathbf{A}\mathbf{x} & & & & = \mathbf{b} & \text{(27b)}\\
\quad \mathbf{T}_i\mathbf{x} & - & \sum_{j=0}^{k_i+1} z_{ij}\lambda_{ij} & & = \mathbf{0} & \text{(27c)}\\
 & & \sum_{j=0}^{k_i+1} \lambda_{ij} & & = 1 & \text{(27d)}\\
\quad \mathbf{T}\mathbf{x} & & & -\mathbf{u} \ -\sum_{j=1}^{N} \mathbf{z}^{(j)}\mu_j & = \mathbf{0} & \text{(27e)}\\
 & & & \sum_{j=1}^{N} \mu_j & = 1 & \text{(27f)}\\
\multicolumn{5}{l}{\quad \mathbf{x} \geq \mathbf{0};\ \lambda_{ij} \geq 0,\ j = 0,\dots,k_i+1,\ i = 1,\dots,r;\ \mathbf{u} \geq \mathbf{0};\ \mu \geq \mathbf{0}.} & \text{(27g)}
\end{array}
$$

The matrix of the equality constraints is illustrated in Figure 3.

$$
\left[\begin{array}{c|c|c|c}
\mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\hline
\mathbf{T} & \begin{array}{ccccccc} -z_{10} & \dots & -z_{1,k_1+1} & & & & \\ & & & \ddots & & & \\ & & & & -z_{r0} & \dots & -z_{r,k_r+1}\end{array} & \mathbf{0} & \mathbf{0}\\
\hline
\mathbf{0} & \begin{array}{ccccccc} 1 & \dots & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & \dots & 1\end{array} & \mathbf{0} & \mathbf{0}\\
\hline
\mathbf{T} & \mathbf{0} & -\mathbf{I} & \mathbf{z}^{(1)}\dots\mathbf{z}^{(N)}\\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & 1\ \dots\ 1
\end{array}\right]
$$

Figure 3: The matrix of the equality constraints in problem (27a-f).

Algorithm A1 solves problem (27a-f), too. If we compare Figures 2 and 3, we see that replacing the matrix

$$
\left[\begin{array}{c|c}
\mathbf{A} & \mathbf{0}\\
\hline
\mathbf{T} & \begin{array}{ccccccc} -z_{10} & \dots & -z_{1,k_1+1} & & & & \\ & & & \ddots & & & \\ & & & & -z_{r0} & \dots & -z_{r,k_r+1}\end{array}\\
\hline
\mathbf{0} & \begin{array}{ccccccc} 1 & \dots & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & \dots & 1\end{array}
\end{array}\right]
$$

taken from Figure 3, for the matrix **A** in Figure 2, we obtain the matrix of the equality constraints of problem (27a-g). Thus, a code which is suitable to solve problem (18a-e), also solves problem (27a-f) (Of course, the objective function coefficients and the right-hand side values need replacement as well). However, problem (27a-f) can be solved more efficiently if its special structure is exploited in full details. The dual type method of Prékopa (1990) and the improved dual type method of Findler, Prékopa and Fábián (1995) offer promising possibilities in this respect.

5 An Exact Solution of Problem (8)

The method is outlined in Prékopa (1990), and is based on the dual method presented in that paper. That dual method solves any problem of the type (8a-e), i.e., problem (8) without the restriction that $\mathbf{Tx}$ should dominate at least one pLEP.

Problem (8) is nonlinear. If, however, constraint (8g) is replaced by the single constraint that $\mathbf{Tx} \geq \mathbf{z}^{(i)}$ for a given i, then the problem becomes an LP. We may append the new constraint to the constraint $\mathbf{Ax} = \mathbf{b}$, but there is a more practical way to take it into account which decreases (rather than increases) the size of the problem: we simply ignore all elements in $\mathbf{z} \in \mathcal{Z}$ for which $\mathbf{z} \not\geq \mathbf{z}^{(i)}$. This means that we delete all terms from (8a), (8c-d) and all inequalities from (8e) which correspond to deleted elements of $\mathcal{Z}$. The obtained problem is of the type (8a-e) and can be solved by the dual method. This is because $\{\mathbf{z} \in \mathcal{Z} \mid \mathbf{z} \geq \mathbf{z}^{(i)}\}$ is a rectangular set, i.e., the Cartesian product of sets in $\mathbb{R}$, as is $\mathcal{Z}$ itself.

If we subdivide the set

$$\bigcup_{i=1}^{N} \left\{\mathbf{z} \mid \mathbf{z} \geq \mathbf{z}^{(i)}\right\} \tag{28}$$

into a finite number of disjoint rectangular sets, then, in view of the above remark, we can solve problem (8a-f) by the application of the dual method as many times as the number of the subdividing sets. In fact, each application of the dual method produces an optimal solution and an optimum value. That optimal solution which corresponds to the smallest optimum value, is the optimal solution of problem (8a-f). Note that a nondisjoint subdivision of the set (28) is inefficient because in that case $\mathbf{Tx}$ is allowed to visit parts of the set (28) more than once.

In view of the above remarks, we mainly have to be concerned with the subdivision of the set (28) and below we present a method to do it.

Method to subdivide the set (28) into pairwise disjoint rectangular sets.

In connection with a set $C \subset \mathbb{R}^r$ we will use the notation:

$$\mathbf{z} + C \ = \ C + \mathbf{z} \ = \ \{\mathbf{z} + \mathbf{y} \mid y \in C\}.$$

Let $H_1, ..., H_M$ be arbitrary subsets of $\mathbb{R}^r$. Then, the following formula holds true:

$$H_1 \cup ... \cup H_M \ = \ H_1 \cup \left(\bigcup_{i=2}^{M} \left(H_i \cap \overline{H}_1 \cap ... \cap \overline{H}_{i-1} \right) \right), \tag{29}$$

where H_1, $H_i \cap \overline{H}_1 \cap ... \cap \overline{H}_{i-1}$, $i = 2, ..., M$ are pairwise disjoint sets.

Let us define the sets

$$\begin{aligned} C_0(\mathbf{z}) &= \{\mathbf{y} \in \mathbb{R}^r \mid \mathbf{y} \geq \mathbf{z}\} \\ C_i(\mathbf{z}) &= \{\mathbf{y} \in \mathbb{R}^r \mid y_i < z_i\}, \quad i = 1, ..., r. \end{aligned} \tag{30}$$

If we apply formula (29) for the case $M = r$, $H_i = C_i(\mathbf{z})$, $i = 1, ..., r$, then we obtain

$$\bigcup_{i=1}^{r} C_i(\mathbf{z}) = C_1(\mathbf{z}) \cup \left(\bigcup_{i=2}^{r} \left(C_i(\mathbf{z}) \cap \overline{C_1(\mathbf{z})} \cap ... \cap \overline{C_{i-1}(\mathbf{z})} \right) \right). \tag{31}$$

The sets

$$\begin{aligned} D_1(\mathbf{z}) &= C_1(\mathbf{z}) \\ D_i(\mathbf{z}) &= C_i(\mathbf{z}) \cap \overline{C_1(\mathbf{z})} \cap ... \cap \overline{C_{i-1}(\mathbf{z})} \end{aligned} \tag{32}$$

are paiwise disjoint, and rectangular, i.e., each of them is the Cartesian product of one-dimensional intervals. We remark that

$$\bigcup_{i=1}^{r} C_i(\mathbf{z}) = \bigcup_{i=1}^{r} D_i(\mathbf{z}) = \overline{C_0(\mathbf{z})}. \tag{33}$$

Now, we use formula (29) for $M = N$, and $H_i = C_0(\mathbf{x}^{(i)})$, $i = 1, ..., N$. We obtain

$$\bigcup_{i=1}^{N} C_0(\mathbf{z}^{(i)}) = C_0(\mathbf{z}^{(1)}) \cup \left(\bigcup_{i=2}^{N} \left(C_0(\mathbf{z}^{(i)}) \cap \overline{C_0(\mathbf{z}^{(1)})} \cap ... \cap \overline{C_0(\mathbf{z}^{(i-1)})} \right) \right). \tag{34}$$

Taking into account (31)-(34), we derive our final formula:

$$\bigcup_{i=1}^{N} C_0(\mathbf{z}^{(i)}) = C_0(\mathbf{z}^{(1)}) \cup \left(\bigcup_{i=2}^{N} \left(C_0(\mathbf{z}^{(i)}) \cap \bigcap_{j=1}^{i-1} \left(\bigcup_{l=1}^{r} D_l(\mathbf{z}^{(j)}) \right) \right) \right). \tag{35}$$

Formula (35) suggests the following algorithm to subdivide the set (28).

Algorithm A2

STEP 0. Initialize $i \leftarrow 1$, $k \leftarrow 1$, $F_1^{(1)} = C_0(\mathbf{z}^{(1)})$.

STEP 1. If the current sets are $F_1^i, ..., F_k^i$, then form the sets $F_j^{(i)} \cap C_h(\mathbf{z}^{(i)})$, $j = 1, ..., k$; $h = 0, ..., r$. Let l be the number of nonempty sets among them.

STEP 2. Set $i \leftarrow i + 1$, $k \leftarrow l$. If $i = N$, then STOP, $F_1 = F_1^{(N)}, ..., F_k = F_k^{(N)}$ are the subdividing rectangular sets. Otherwise go to STEP 1.

Having created the above subdivision, problems (4) and (8) can be solved in such a way that we replace the constraint

$$\mathbf{Tx} \geq \mathbf{z}^{(i)} \text{ for at least one } i \in \{1, ..., N\}$$

by the constraint

$$\mathbf{Tx} \in F_h \text{ for at least one } h \in \{1, ..., k\}. \tag{36}$$

For each h we solve the LP (4a-c), $\mathbf{Tx} \in F_h$, or (8a-e), $\mathbf{Tx} \in F_h$ and take that optimal solution which produces the smallest objective function value. This will be the optimal solution to problem (4) or (8).

If we solve problem (8a-e), $\mathbf{Tx} \in F_h$, then we ignore those z_{ij}, λ_{ij} in the problem, for which $z_{ij} \notin F_h$. In other words, the set $\mathcal{Z}$ is now defined as the intersection of the $\mathcal{Z}$, defined in the Introduction, with the set F_h. For each $h = 1, ..., k$ the problem can be solved by the dual type method of Prékopa (1990), or the improved dual type method of Fiedler, Prékopa and Fábián (1995).

6 Numerical Example

The following numerical example is based on the paper by Prékopa and Boros (1991) about transportation network reliability calculation. The problem presented in this section is a transportation network capacity design problem, where the network reliability (i.e., the probability of the existence of a feasible flow=1 - loss of load Probability) is prescribed to be at least $p = 0.9975$.

Let (N, A) designate the network, where N is the set of nodes and A is the set of arcs. Our capacity design problem is of the following type:

$$\begin{array}{c} \text{Min} \left\{ \sum_{i \in N} c_i x_i \; + \; \sum_{(i,j) \in A} c_{ij} y_{ij} \right\} \\ \text{subject to} \\ X_i^{(l)} \leq x_i \leq X_i^{(u)} \\ Y_{ij}^{(l)} \leq y_{ij} \leq Y_{ij}^{(u)} \\ P(\text{a feasible flow exists}) \geq p, \end{array} \tag{37}$$

where x_i, y_{ij} are node and arc capacities, respectively, for which lower and upper bounds are prescribed. The c_i, c_{ij} constants represent capacity unit prices.

We take Example 3 of the paper by Prékopa and Boros as a basis to formulate the design problem.

The random event that a feasible flow exists can be formulated in terms of "cuts" which are presented in Table VII of that paper. Each cut is an inequality involving random demand values. The first, second and seventh cuts (cut 15, cut 11,12 and cut 1,2) represent inequalities that hold if and only if the random variables involved assume their largest possible values. Recall that the system reliability is prescribed to be at least $p = 0.9975$. This implies that these three cut inequalities can be separated from the others and can be handled as determinitic constraints in the problem. The other ten cut inequalities should hold by probability p.

We enumerated the set of all pLEP's and then formulated and solved the problem of the type (18a-e).

The original problem

Minimize
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15}$
$+y_{1,2} + y_{2,3} + y_{3,4} + y_{3,5} + y_{3,13} + y_{4,5} + y_{5,6} + y_{5,8} + y_{5,13} + y_{5,14} + y_{6,7} + y_{7,8}$
$+y_{7,10} + y_{8,9} + y_{8,11} + y_{9,10} + y_{9,11} + y_{9,15} + y_{10,11} + y_{11,12} + y_{13,14} + y_{14,15}$
Subject to
cut 15:
$x_{15} + y_{9,15} + y_{14,15} >= 4400$
cut 11,12:
$x_{11} + x_{12} + y_{8,11} + y_{9,11} + y_{10,11} >= 5395$
cut 1,2:
$x_1 + x_2 + y_{2,3} >= 3945$
Bounds
$0 <= x_1 <= 3125$
$0 <= x_2 <= 3030$
$0 <= x_3 <= 3020$
$0 <= x_4 <= 3200$
$0 <= x_5 <= 3770$
$0 <= x_6 <= 2500$
$0 <= x_7 <= 3800$
$0 <= x_8 <= 3900$
$0 <= x_9 <= 3415$
$0 <= x_{10} <= 1500$
$0 <= x_{11} <= 2880$
$0 <= x_{12} <= 1525$
$0 <= x_{13} <= 2800$
$0 <= x_{14} <= 3500$
$0 <= x_{15} <= 2500$

$2000 <= y_{1,2} <= 3000$
$1500 <= y_{5,6} <= 2250$
$1500 <= y_{2,3} <= 2250$
$1800 <= y_{3,4} <= 2700$
$1800 <= y_{3,5} <= 2700$
$1800 <= y_{3,13} <= 2700$
$1800 <= y_{4,5} <= 2700$
$2000 <= y_{5,8} <= 3000$
$1800 <= y_{5,13} <= 2700$
$1450 <= y_{5,14} <= 2175$
$2000 <= y_{6,7} <= 3000$
$2000 <= y_{7,8} <= 3000$
$1200 <= y_{7,10} <= 1800$
$3000 <= y_{8,9} <= 4500$
$1500 <= y_{8,11} <= 2250$
$1225 <= y_{9,10} <= 1835$
$1500 <= y_{9,11} <= 2250$
$1500 <= y_{9,15} <= 2250$
$2450 <= y_{10,11} <= 3675$
$1675 <= y_{11,12} <= 2500$
$1200 <= y_{13,14} <= 1800$
$2000 <= y_{14,15} <= 3000$

The 10×37 matrix T.

0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0 0 0
0 0 0 0 0 0 1 1 1 1 1 1 0 0 1 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 1
0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 0 1 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0
0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1 1 0 0 0 0 1 1 0 0 0 0 0 0 0 1 0 0 0 0
1 1 1 1 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1 0
1 1 1 1 1 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1
1 1 1 1 1 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0
1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 1 0
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

The set of reduced pLEP's. (400×reduced pLEP = pLEP.)

79	10								
33	52	66	103	71	67	73	83	103	115
28	52	56	93	61	61	67	73	88	110
28	52	56	93	61	61	67	73	93	105
28	52	56	93	66	56	67	78	93	105
28	52	56	93	66	61	67	73	88	105
28	52	56	93	71	66	62	83	98	115
28	52	56	98	61	56	67	78	93	110
28	52	56	98	61	56	67	78	103	105
28	52	56	98	61	61	67	73	88	105
28	52	56	98	66	56	67	73	93	105
28	52	56	98	66	61	72	78	93	100
28	52	61	88	56	61	67	78	93	110
28	52	61	88	56	61	67	78	98	105
28	52	61	88	56	61	72	78	93	105
28	52	61	88	56	66	67	78	93	105
28	52	61	88	61	56	62	73	88	105
28	52	61	88	61	56	67	73	88	100
28	52	61	88	61	61	62	73	88	100
28	52	61	88	61	61	72	68	93	105
28	52	61	93	56	56	67	73	88	105
28	52	61	93	56	61	62	73	88	105
28	52	61	93	56	61	67	73	88	100
28	52	61	93	61	56	62	73	88	100
28	52	61	93	61	56	67	68	88	100
28	52	61	93	61	56	67	73	83	105
28	52	61	98	56	56	72	83	98	100
28	52	61	98	56	57	62	73	88	110
28	52	61	98	56	57	62	73	93	105
28	52	61	103	56	56	72	78	93	100
28	52	61	103	57	56	62	73	88	110
28	52	61	103	57	56	62	73	93	105
28	52	65	88	66	61	67	68	93	110
28	52	65	88	66	61	67	78	83	105
28	52	66	88	56	61	67	73	88	110
28	52	66	88	56	61	67	73	93	105
28	52	66	88	56	61	67	78	88	105
28	52	66	88	61	56	62	73	88	100

28	52	66	88	61	61	67	68	88	105
28	52	66	89	61	61	67	73	83	105
28	52	66	93	56	56	62	73	88	105
28	52	66	93	56	56	67	78	93	100
28	52	66	93	56	56	67	79	88	100
28	52	66	93	61	56	62	68	88	105
28	52	66	94	61	56	62	78	83	100
28	52	66	98	56	56	67	73	88	100
28	52	66	98	56	66	72	68	93	105
28	52	66	98	56	67	62	78	93	100
28	52	66	98	61	61	67	78	93	95
28	52	66	98	61	61	67	79	83	100
28	52	66	98	61	61	72	73	93	95
28	52	66	98	61	61	72	74	83	100
28	52	66	98	61	61	72	78	88	95
28	52	66	98	61	61	73	73	83	100
28	52	66	98	61	62	62	68	88	100
28	52	66	98	61	62	62	73	83	100
28	52	66	98	61	66	67	73	93	95
28	52	66	98	66	56	62	68	88	100
28	52	66	98	66	56	62	73	83	100
28	52	66	98	66	56	72	78	93	95
33	52	56	83	56	61	67	78	93	105
33	52	56	83	61	56	62	73	88	100
33	52	56	83	61	61	67	68	88	100
33	52	56	83	61	61	67	78	83	105
33	52	56	88	56	56	62	68	83	100
33	52	56	88	56	56	62	68	88	95
33	52	56	88	61	51	62	73	83	100
33	52	56	88	61	56	57	73	88	100
33	52	56	88	61	56	62	68	83	95
33	52	56	93	56	51	62	73	88	100
33	52	61	83	56	56	62	73	88	100
33	52	61	83	56	56	67	68	88	100
33	52	61	83	56	56	67	78	83	105
33	52	61	83	61	56	62	68	83	100
33	52	61	83	61	56	67	73	88	95
33	52	61	84	56	61	67	73	83	100
33	52	61	88	56	51	62	68	83	100
33	52	61	88	56	56	57	68	88	100
33	52	61	88	56	56	62	68	83	95
33	52	61	88	56	60	57	73	83	105

The set of pLEP's.

79 10

13200	20800	26400	41200	28400	26800	29200	33200	41200	46000
11200	20800	22400	37200	24400	24400	26800	29200	35200	44000
11200	20800	22400	37200	24400	24400	26800	29200	37200	42000
11200	20800	22400	37200	26400	22400	26800	31200	37200	42000
11200	20800	22400	37200	26400	24400	26800	29200	35200	42000
11200	20800	22400	37200	28400	26400	24800	33200	39200	46000
11200	20800	22400	39200	24400	22400	26800	31200	37200	44000
11200	20800	22400	39200	24400	22400	26800	31200	41200	42000
11200	20800	22400	39200	24400	24400	26800	29200	35200	42000
11200	20800	22400	39200	26400	22400	26800	29200	37200	42000
11200	20800	22400	39200	26400	24400	28800	31200	37200	40000
11200	20800	24400	35200	22400	24400	26800	31200	37200	44000
11200	20800	24400	35200	22400	24400	26800	31200	39200	42000
11200	20800	24400	35200	22400	24400	28800	31200	37200	42000
11200	20800	24400	35200	22400	26400	26800	31200	37200	42000
11200	20800	24400	35200	24400	22400	24800	29200	35200	42000
11200	20800	24400	35200	24400	22400	26800	29200	35200	40000
11200	20800	24400	35200	24400	24400	24800	29200	35200	40000
11200	20800	24400	35200	24400	24400	28800	27200	37200	42000
11200	20800	24400	37200	22400	22400	26800	29200	35200	42000
11200	20800	24400	37200	22400	24400	24800	29200	35200	42000
11200	20800	24400	37200	22400	24400	26800	29200	35200	40000
11200	20800	24400	37200	24400	22400	24800	29200	35200	40000
11200	20800	24400	37200	24400	22400	26800	27200	35200	40000
11200	20800	24400	37200	24400	22400	26800	29200	33200	42000
11200	20800	24400	39200	22400	22400	28800	33200	39200	40000
11200	20800	24400	39200	22400	22800	24800	29200	35200	44000
11200	20800	24400	39200	22400	22800	24800	29200	37200	42000
11200	20800	24400	41200	22400	22400	28800	31200	37200	40000
11200	20800	24400	41200	22800	22400	24800	29200	35200	44000
11200	20800	24400	41200	22800	22400	24800	29200	37200	42000
11200	20800	26000	35200	26400	24400	26800	27200	37200	44000
11200	20800	26000	35200	26400	24400	26800	31200	33200	42000
11200	20800	26400	35200	22400	24400	26800	29200	35200	44000
11200	20800	26400	35200	22400	24400	26800	29200	37200	42000
11200	20800	26400	35200	22400	24400	26800	31200	35200	42000
11200	20800	26400	35200	24400	22400	24800	29200	35200	40000
11200	20800	26400	35200	24400	24400	26800	27200	35200	42000
11200	20800	26400	35600	24400	24400	26800	29200	33200	42000

11200	20800	26400	37200	22400	22400	24800	29200	35200	42000
11200	20800	26400	37200	22400	22400	26800	31200	37200	40000
11200	20800	26400	37200	22400	22400	26800	31600	35200	40000
11200	20800	26400	37200	24400	22400	24800	27200	35200	42000
11200	20800	26400	37600	24400	22400	24800	31200	33200	42000
11200	20800	26400	39200	22400	22400	26800	29200	35200	40000
11200	20800	26400	39200	22400	26400	28800	27200	37200	42000
11200	20800	26400	39200	22400	26800	24800	31200	37200	40000
11200	20800	26400	39200	24400	24400	26800	31200	37200	38000
11200	20800	26400	39200	24400	24400	26800	31600	33200	40000
11200	20800	26400	39200	24400	24400	28800	29200	37200	38000
11200	20800	26400	39200	24400	24400	28800	29600	33200	40000
11200	20800	26400	39200	24400	24400	28800	31200	35200	38000
11200	20800	26400	39200	24400	24400	29200	29200	33200	40000
11200	20800	26400	39200	24400	24800	24800	27200	35200	40000
11200	20800	26400	39200	24400	24800	24800	29200	33200	40000
11200	20800	26400	39200	24400	26400	26800	29200	37200	38000
11200	20800	26400	39200	26400	22400	24800	27200	35200	40000
11200	20800	26400	39200	26400	22400	24800	29200	33200	40000
11200	20800	26400	39200	26400	22400	28800	31200	37200	38000
13200	20800	22400	33200	22400	24400	26800	31200	37200	42000
13200	20800	22400	33200	24400	22400	24800	29200	35200	40000
13200	20800	22400	33200	24400	24400	26800	27200	35200	40000
13200	20800	22400	33200	24400	24400	26800	31200	33200	42000
13200	20800	22400	35200	22400	22400	24800	27200	33200	40000
13200	20800	22400	35200	22400	22400	24800	27200	35200	38000
13200	20800	22400	35200	24400	20400	24800	29200	33200	40000
13200	20800	22400	35200	24400	22400	22800	29200	35200	40000
13200	20800	22400	35200	24400	22400	24800	27200	33200	38000
13200	20800	22400	37200	22400	20400	24800	29200	35200	40000
13200	20800	24400	33200	22400	22400	24800	29200	35200	40000
13200	20800	24400	33200	22400	22400	26800	27200	35200	40000
13200	20800	24400	33200	22400	22400	26800	31200	33200	42000
13200	20800	24400	33200	24400	22400	24800	27200	33200	40000
13200	20800	24400	33200	24400	22400	26800	29200	35200	38000
13200	20800	24400	33600	22400	24400	26800	29200	33200	40000
13200	20800	24400	35200	22400	20400	24800	27200	33200	40000
13200	20800	24400	35200	22400	22400	22800	27200	35200	40000
13200	20800	24400	35200	22400	22400	24800	27200	33200	38000
13200	20800	24400	35200	22400	24000	22800	29200	33200	42000

The problem of the type (18a-e).

Minimize
$(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15}$
$+y_{1,2} + y_{2,3} + y_{3,4} + y_{3,5} + y_{3,13} + y_{4,5} + y_{5,6} + y_{5,8} + y_{5,13} + y_{5,14} + y_{6,7} + y_{7,8}$
$+y_{7,10} + y_{8,9} + y_{8,11} + y_{9,10} + y_{9,11} + y_{9,15} + y_{10,11} + y_{11,12} + y_{13,14} + y_{14,15})$
Subject to
$cut\ 15:$
$x_{15} + y_{9,15} + y_{14,15} >= 4400$
$cut\ 11,12:$
$x_{11} + x_{12} + y_{8,11} + y_{9,11} + y_{10,11} >= 5395$
$cut\ 1,2:$
$x_1 + x_2 + y_{2,3} >= 3945$
$constraint\ 1:$
$x_{16} + x_{17} + x_{18} + x_{19} + x_{20} + x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} + x_{28} + x_{29} + x_{30}$
$+x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} + x_{37} + x_{38} + x_{39} + x_{40} + x_{41} + x_{42} + x_{43} + x_{44} + x_{45}$
$+x_{46} + x_{47} + x_{48} + x_{49} + x_{50} + x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} + x_{57} + x_{58} + x_{59} + x_{60}$
$+x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{66} + x_{67} + x_{68} + x_{69} + x_{70} + x_{71} + x_{72} + x_{73} + x_{74} + x_{75}$
$+x_{76} + x_{77} + x_{78} + x_{79} + x_{80} + x_{81} + x_{82} + x_{83} + x_{84} + x_{85} + x_{86} + x_{87} + x_{88} + x_{89} + x_{90}$
$+x_{91} + x_{92} + x_{93} + x_{94} + x_{95} + x_{96} + x_{97} + x_{98} + x_{99} + x_{100} + x_{101} + x_{102} + x_{103} + x_{104} = 1$
$constraint\ 2:$
$x_{10} + x_{11} + x_{12} + y_{7,10} + y_{8,11} + y_{9,10} + y_{9,11}$
$-14040x_{16} - 11200x_{17} - 14040x_{18} - 14040x_{19} - 14040x_{20}$
$-14040x_{21} - 14040x_{22} - 14040x_{23} - 14040x_{24} - 14040x_{25}$
$-14040x_{26} - 11200x_{27} - 11200x_{28} - 11200x_{29} - 11200x_{30}$
$-11200x_{31} - 11200x_{32} - 11200x_{33} - 11200x_{34} - 11200x_{35}$
$-11200x_{36} - 11200x_{37} - 11200x_{38} - 11200x_{39} - 11200x_{40}$
$-11200x_{41} - 11200x_{42} - 11200x_{43} - 11200x_{44} - 11200x_{45}$
$-11200x_{46} - 11200x_{47} - 11200x_{48} - 11200x_{49} - 11200x_{50}$
$-11200x_{51} - 11200x_{52} - 11200x_{53} - 11200x_{54} - 11200x_{55}$
$-11200x_{56} - 11200x_{57} - 11200x_{58} - 11200x_{59} - 11200x_{60}$
$-11200x_{61} - 11200x_{62} - 11200x_{63} - 11200x_{64} - 11200x_{65}$
$-11200x_{66} - 11200x_{67} - 11200x_{68} - 11200x_{69} - 11200x_{70}$
$-11200x_{71} - 11200x_{72} - 11200x_{73} - 11200x_{74} - 11200x_{75}$
$-11200x_{76} - 11200x_{77} - 11200x_{78} - 11200x_{79} - 11200x_{80}$
$-11200x_{81} - 11200x_{82} - 11200x_{83} - 11200x_{84} - 13200x_{85}$
$-13200x_{86} - 13200x_{87} - 13200x_{88} - 13200x_{89} - 13200x_{90}$
$-13200x_{91} - 13200x_{92} - 13200x_{93} - 13200x_{94} - 13200x_{95}$
$-13200x_{96} - 13200x_{97} - 13200x_{98} - 13200x_{99} - 13200x_{100}$
$-13200x_{101} - 13200x_{102} - 13200x_{103} - 13200x_{104} = 0$

$$
\begin{aligned}
&constraint\ 3:\\
&x_7+x_8+x_9+x_{10}+x_{11}+x_{12}+x_{15}+y_{5,8}+y_{6,7}+y_{14,15}\\
&-28520x_{16}-28520x_{17}-20800x_{18}-28520x_{19}-28520x_{20}\\
&-28520x_{21}-28520x_{22}-28520x_{23}-28520x_{24}-28520x_{25}\\
&-28520x_{26}-20800x_{27}-20800x_{28}-20800x_{29}-20800x_{30}\\
&-20800x_{31}-20800x_{32}-20800x_{33}-20800x_{34}-20800x_{35}\\
&-20800x_{36}-20800x_{37}-20800x_{38}-20800x_{39}-20800x_{40}\\
&-20800x_{41}-20800x_{42}-20800x_{43}-20800x_{44}-20800x_{45}\\
&-20800x_{46}-20800x_{47}-20800x_{48}-20800x_{49}-20800x_{50}\\
&-20800x_{51}-20800x_{52}-20800x_{53}-20800x_{54}-20800x_{55}\\
&-20800x_{56}-20800x_{57}-20800x_{58}-20800x_{59}-20800x_{60}\\
&-20800x_{61}-20800x_{62}-20800x_{63}-20800x_{64}-20800x_{65}\\
&-20800x_{66}-20800x_{67}-20800x_{68}-20800x_{69}-20800x_{70}\\
&-20800x_{71}-20800x_{72}-20800x_{73}-20800x_{74}-20800x_{75}\\
&-20800x_{76}-20800x_{77}-20800x_{78}-20800x_{79}-20800x_{80}\\
&-20800x_{81}-20800x_{82}-20800x_{83}-20800x_{84}-20800x_{85}\\
&-20800x_{86}-20800x_{87}-20800x_{88}-20800x_{89}-20800x_{90}\\
&-20800x_{91}-20800x_{92}-20800x_{93}-20800x_{94}-20800x_{95}\\
&-20800x_{96}-20800x_{97}-20800x_{98}-20800x_{99}-20800x_{100}\\
&-20800x_{101}-20800x_{102}-20800x_{103}-20800x_{104}=0\\
&constraint\ 4:\\
&x_3+x_4+x_5+x_{13}+x_{14}+x_{15}+y_{2,3}+y_{5,6}+y_{5,8}+y_{9,15}\\
&-28540x_{16}-28540x_{17}-28540x_{18}-26400x_{19}-28540x_{20}\\
&-28540x_{21}-28540x_{22}-28540x_{23}-28540x_{24}-28540x_{25}\\
&-28540x_{26}-22400x_{27}-22400x_{28}-22400x_{29}-22400x_{30}\\
&-22400x_{31}-22400x_{32}-22400x_{33}-22400x_{34}-22400x_{35}\\
&-22400x_{36}-24400x_{37}-24400x_{38}-24400x_{39}-24400x_{40}\\
&-24400x_{41}-24400x_{42}-24400x_{43}-24400x_{44}-24400x_{45}\\
&-24400x_{46}-24400x_{47}-24400x_{48}-24400x_{49}-24400x_{50}\\
&-24400x_{51}-24400x_{52}-24400x_{53}-24400x_{54}-24400x_{55}\\
&-24400x_{56}-26000x_{57}-26000x_{58}-26400x_{59}-26400x_{60}\\
&-26400x_{61}-26400x_{62}-26400x_{63}-26400x_{64}-26400x_{65}\\
&-26400x_{66}-26400x_{67}-26400x_{68}-26400x_{69}-26400x_{70}\\
&-26400x_{71}-26400x_{72}-26400x_{73}-26400x_{74}-26400x_{75}\\
&-26400x_{76}-26400x_{77}-26400x_{78}-26400x_{79}-26400x_{80}\\
&-26400x_{81}-26400x_{82}-26400x_{83}-26400x_{84}-22400x_{85}\\
&-22400x_{86}-22400x_{87}-22400x_{88}-22400x_{89}-22400x_{90}\\
&-22400x_{91}-22400x_{92}-22400x_{93}-22400x_{94}-24400x_{95}\\
&-24400x_{96}-24400x_{97}-24400x_{98}-24400x_{99}-24400x_{100}\\
&-24400x_{101}-24400x_{102}-24400x_{103}-24400x_{104}=0
\end{aligned}
$$

constraint 5 :

$x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}+x_{12}+x_{13}+x_{14}+x_{15}+y_{2,3}$
$-41200x_{16}-41200x_{17}-41200x_{18}-41200x_{19}-33200x_{20}$
$-41200x_{21}-41200x_{22}-41200x_{23}-41200x_{24}-41200x_{25}$
$-41200x_{26}-37200x_{27}-37200x_{28}-37200x_{29}-37200x_{30}$
$-37200x_{31}-39200x_{32}-39200x_{33}-39200x_{34}-39200x_{35}$
$-39200x_{36}-35200x_{37}-35200x_{38}-35200x_{39}-35200x_{40}$
$-35200x_{41}-35200x_{42}-35200x_{43}-35200x_{44}-37200x_{45}$
$-37200x_{46}-37200x_{47}-37200x_{48}-37200x_{49}-37200x_{50}$
$-39200x_{51}-39200x_{52}-39200x_{53}-41200x_{54}-41200x_{55}$
$-41200x_{56}-35200x_{57}-35200x_{58}-35200x_{59}-35200x_{60}$
$-35200x_{61}-35200x_{62}-35200x_{63}-35600x_{64}-37200x_{65}$
$-37200x_{66}-37200x_{67}-37200x_{68}-37600x_{69}-39200x_{70}$
$-39200x_{71}-39200x_{72}-39200x_{73}-39200x_{74}-39200x_{75}$
$-39200x_{76}-39200x_{77}-39200x_{78}-39200x_{79}-39200x_{80}$
$-39200x_{81}-39200x_{82}-39200x_{83}-39200x_{84}-33200x_{85}$
$-33200x_{86}-33200x_{87}-33200x_{88}-35200x_{89}-35200x_{90}$
$-35200x_{91}-35200x_{92}-35200x_{93}-37200x_{94}-33200x_{95}$
$-33200x_{96}-33200x_{97}-33200x_{98}-33200x_{99}-33600x_{100}$
$-35200x_{101}-35200x_{102}-35200x_{103}-35200x_{104}=0$

constraint 6 :

$x_1+x_2+x_3+x_{13}+x_{14}+x_{15}+y_{3,4}+y_{3,5}+y_{5,13}+y_{5,14}+y_{9,15}$
$-30500x_{16}-30500x_{17}-30500x_{18}-30500x_{19}-30500x_{20}$
$-22400x_{21}-30500x_{22}-30500x_{23}-30500x_{24}-30500x_{25}$
$-30500x_{26}-24400x_{27}-24400x_{28}-26400x_{29}-26400x_{30}$
$-28400x_{31}-24400x_{32}-24400x_{33}-24400x_{34}-26400x_{35}$
$-26400x_{36}-22400x_{37}-22400x_{38}-22400x_{39}-22400x_{40}$
$-24400x_{41}-24400x_{42}-24400x_{43}-24400x_{44}-22400x_{45}$
$-22400x_{46}-22400x_{47}-24400x_{48}-24400x_{49}-24400x_{50}$
$-22400x_{51}-22400x_{52}-22400x_{53}-22400x_{54}-22800x_{55}$
$-22800x_{56}-26400x_{57}-26400x_{58}-22400x_{59}-22400x_{60}$
$-22400x_{61}-24400x_{62}-24400x_{63}-24400x_{64}-22400x_{65}$
$-22400x_{66}-22400x_{67}-24400x_{68}-24400x_{69}-22400x_{70}$
$-22400x_{71}-22400x_{72}-24400x_{73}-24400x_{74}-24400x_{75}$
$-24400x_{76}-24400x_{77}-24400x_{78}-24400x_{79}-24400x_{80}$
$-24400x_{81}-26400x_{82}-26400x_{83}-26400x_{84}-22400x_{85}$
$-24400x_{86}-24400x_{87}-24400x_{88}-22400x_{89}-22400x_{90}$
$-24400x_{91}-24400x_{92}-24400x_{93}-22400x_{94}-22400x_{95}$
$-22400x_{96}-22400x_{97}-24400x_{98}-24400x_{99}-22400x_{100}$
$-22400x_{101}-22400x_{102}-22400x_{103}-22400x_{104}=0$

constraint 7 :

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_{13} + y_{5,6} + y_{5,8} + y_{5,14} + y_{13,14}$$
$$-28170x_{16} - 28170x_{17} - 28170x_{18} - 28170x_{19} - 28170x_{20}$$
$$-28170x_{21} - 20400x_{22} - 28170x_{23} - 28170x_{24} - 28170x_{25}$$
$$-28170x_{26} - 24400x_{27} - 24400x_{28} - 22400x_{29} - 24400x_{30}$$
$$-26400x_{31} - 22400x_{32} - 22400x_{33} - 24400x_{34} - 22400x_{35}$$
$$-24400x_{36} - 24400x_{37} - 24400x_{38} - 24400x_{39} - 26400x_{40}$$
$$-22400x_{41} - 22400x_{42} - 24400x_{43} - 24400x_{44} - 22400x_{45}$$
$$-24400x_{46} - 24400x_{47} - 22400x_{48} - 22400x_{49} - 22400x_{50}$$
$$-22400x_{51} - 22800x_{52} - 22800x_{53} - 22400x_{54} - 22400x_{55}$$
$$-22400x_{56} - 24400x_{57} - 24400x_{58} - 24400x_{59} - 24400x_{60}$$
$$-24400x_{61} - 22400x_{62} - 24400x_{63} - 24400x_{64} - 22400x_{65}$$
$$-22400x_{66} - 22400x_{67} - 22400x_{68} - 22400x_{69} - 22400x_{70}$$
$$-26400x_{71} - 26800x_{72} - 24400x_{73} - 24400x_{74} - 24400x_{75}$$
$$-24400x_{76} - 24400x_{77} - 24400x_{78} - 24800x_{79} - 24800x_{80}$$
$$-26400x_{81} - 22400x_{82} - 22400x_{83} - 22400x_{84} - 24400x_{85}$$
$$-22400x_{86} - 24400x_{87} - 24400x_{88} - 22400x_{89} - 22400x_{90}$$
$$-20400x_{91} - 22400x_{92} - 22400x_{93} - 20400x_{94} - 22400x_{95}$$
$$-22400x_{96} - 22400x_{97} - 22400x_{98} - 22400x_{99} - 24400x_{100}$$
$$-20400x_{101} - 22400x_{102} - 22400x_{103} - 24000x_{104} = 0$$

constraint 8 :

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_{13} + x_{14} + y_{5,6} + y_{5,8} + y_{14,15}$$
$$-30695x_{16} - 30695x_{17} - 30695x_{18} - 30695x_{19} - 30695x_{20}$$
$$-30695x_{21} - 30695x_{22} - 22800x_{23} - 30695x_{24} - 30695x_{25}$$
$$-30695x_{26} - 26800x_{27} - 26800x_{28} - 26800x_{29} - 26800x_{30}$$
$$-24800x_{31} - 26800x_{32} - 26800x_{33} - 26800x_{34} - 26800x_{35}$$
$$-28800x_{36} - 26800x_{37} - 26800x_{38} - 28800x_{39} - 26800x_{40}$$
$$-24800x_{41} - 26800x_{42} - 24800x_{43} - 28800x_{44} - 26800x_{45}$$
$$-24800x_{46} - 26800x_{47} - 24800x_{48} - 26800x_{49} - 26800x_{50}$$
$$-28800x_{51} - 24800x_{52} - 24800x_{53} - 28800x_{54} - 24800x_{55}$$
$$-24800x_{56} - 26800x_{57} - 26800x_{58} - 26800x_{59} - 26800x_{60}$$
$$-26800x_{61} - 24800x_{62} - 26800x_{63} - 26800x_{64} - 24800x_{65}$$
$$-26800x_{66} - 26800x_{67} - 24800x_{68} - 24800x_{69} - 26800x_{70}$$
$$-28800x_{71} - 24800x_{72} - 26800x_{73} - 26800x_{74} - 28800x_{75}$$
$$-28800x_{76} - 28800x_{77} - 29200x_{78} - 24800x_{79} - 24800x_{80}$$
$$-26800x_{81} - 24800x_{82} - 24800x_{83} - 28800x_{84} - 26800x_{85}$$
$$-24800x_{86} - 26800x_{87} - 26800x_{88} - 24800x_{89} - 24800x_{90}$$
$$-24800x_{91} - 22800x_{92} - 24800x_{93} - 24800x_{94} - 24800x_{95}$$
$$-26800x_{96} - 26800x_{97} - 24800x_{98} - 26800x_{99} - 26800x_{100}$$
$$-24800x_{101} - 22800x_{102} - 24800x_{103} - 22800x_{104} = 0$$

$constraint\ 9:$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_{13} + x_{14} + x_{15} + y_{5,6} + y_{5,8} + y_{9,15}$
$-33200x_{16} - 33200x_{17} - 33200x_{18} - 33200x_{19} - 33200x_{20}$
$-33200x_{21} - 33200x_{22} - 33200x_{23} - 27200x_{24} - 33200x_{25}$
$-33200x_{26} - 29200x_{27} - 29200x_{28} - 31200x_{29} - 29200x_{30}$
$-33200x_{31} - 31200x_{32} - 31200x_{33} - 29200x_{34} - 29200x_{35}$
$-31200x_{36} - 31200x_{37} - 31200x_{38} - 31200x_{39} - 31200x_{40}$
$-29200x_{41} - 29200x_{42} - 29200x_{43} - 27200x_{44} - 29200x_{45}$
$-29200x_{46} - 29200x_{47} - 29200x_{48} - 27200x_{49} - 29200x_{50}$
$-33200x_{51} - 29200x_{52} - 29200x_{53} - 31200x_{54} - 29200x_{55}$
$-29200x_{56} - 27200x_{57} - 31200x_{58} - 29200x_{59} - 29200x_{60}$
$-31200x_{61} - 29200x_{62} - 27200x_{63} - 29200x_{64} - 29200x_{65}$
$-31200x_{66} - 31600x_{67} - 27200x_{68} - 31200x_{69} - 29200x_{70}$
$-27200x_{71} - 31200x_{72} - 31200x_{73} - 31600x_{74} - 29200x_{75}$
$-29600x_{76} - 31200x_{77} - 29200x_{78} - 27200x_{79} - 29200x_{80}$
$-29200x_{81} - 27200x_{82} - 29200x_{83} - 31200x_{84} - 31200x_{85}$
$-29200x_{86} - 27200x_{87} - 31200x_{88} - 27200x_{89} - 27200x_{90}$
$-29200x_{91} - 29200x_{92} - 27200x_{93} - 29200x_{94} - 29200x_{95}$
$-27200x_{96} - 31200x_{97} - 27200x_{98} - 29200x_{99} - 29200x_{100}$
$-27200x_{101} - 27200x_{102} - 27200x_{103} - 29200x_{104} = 0$

$constraint\ 10:$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + y_{5,14} + y_{9,15} + y_{13,14}$
$-44690x_{16} - 44690x_{17} - 44690x_{18} - 44690x_{19} - 44690x_{20}$
$-44690x_21 - 44690x_{22} - 44690x_{23} - 44690x_{24} - 33200x_{25}$
$-44690x_26 - 35200x_{27} - 37200x_{28} - 37200x_{29} - 35200x_{30}$
$-39200x_31 - 37200x_{32} - 41200x_{33} - 35200x_{34} - 37200x_{35}$
$-37200x_36 - 37200x_{37} - 39200x_{38} - 37200x_{39} - 37200x_{40}$
$-35200x_41 - 35200x_{42} - 35200x_{43} - 37200x_{44} - 35200x_{45}$
$-35200x_46 - 35200x_{47} - 35200x_{48} - 35200x_{49} - 33200x_{50}$
$-39200x_51 - 35200x_{52} - 37200x_{53} - 37200x_{54} - 35200x_{55}$
$-37200x_56 - 37200x_{57} - 33200x_{58} - 35200x_{59} - 37200x_{60}$
$-35200x_61 - 35200x_{62} - 35200x_{63} - 33200x_{64} - 35200x_{65}$
$-37200x_66 - 35200x_{67} - 35200x_{68} - 33200x_{69} - 35200x_{70}$
$-37200x_71 - 37200x_{72} - 37200x_{73} - 33200x_{74} - 37200x_{75}$
$-33200x_76 - 35200x_{77} - 33200x_{78} - 35200x_{79} - 33200x_{80}$
$-37200x_81 - 35200x_{82} - 33200x_{83} - 37200x_{84} - 37200x_{85}$
$-35200x_86 - 35200x_{87} - 33200x_{88} - 33200x_{89} - 35200x_{90}$
$-33200x_91 - 35200x_{92} - 33200x_{93} - 35200x_{94} - 35200x_{95}$
$-35200x_96 - 33200x_{97} - 33200x_{98} - 35200x_{99} - 33200x_{100}$
$-33200x_{101} - 35200x_{102} - 33200x_{103} - 33200x_{104} = 0$

constraint 11 :

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15}$
$-46000x_{16} - 46000x_{17} - 46000x_{18} - 46000x_{19} - 46000x_{20}$
$-46000x_{21} - 46000x_{22} - 46000x_{23} - 46000x_{24} - 46000x_{25}$
$-38000x_{26} - 44000x_{27} - 42000x_{28} - 42000x_{29} - 42000x_{30}$
$-46000x_{31} - 44000x_{32} - 42000x_{33} - 42000x_{34} - 42000x_{35}$
$-40000x_{36} - 44000x_{37} - 42000x_{38} - 42000x_{39} - 42000x_{40}$
$-42000x_{41} - 40000x_{42} - 40000x_{43} - 42000x_{44} - 42000x_{45}$
$-42000x_{46} - 40000x_{47} - 40000x_{48} - 40000x_{49} - 42000x_{50}$
$-40000x_{51} - 44000x_{52} - 42000x_{53} - 40000x_{54} - 44000x_{55}$
$-42000x_{56} - 44000x_{57} - 42000x_{58} - 44000x_{59} - 42000x_{60}$
$-42000x_{61} - 40000x_{62} - 42000x_{63} - 42000x_{64} - 42000x_{65}$
$-40000x_{66} - 40000x_{67} - 42000x_{68} - 42000x_{69} - 40000x_{70}$
$-42000x_{71} - 40000x_{72} - 38000x_{73} - 40000x_{74} - 38000x_{75}$
$-40000x_{76} - 38000x_{77} - 40000x_{78} - 40000x_{79} - 40000x_{80}$
$-38000x_{81} - 40000x_{82} - 40000x_{83} - 38000x_{84} - 42000x_{85}$
$-40000x_{86} - 40000x_{87} - 42000x_{88} - 40000x_{89} - 38000x_{90}$
$-40000x_{91} - 40000x_{92} - 38000x_{93} - 40000x_{94} - 40000x_{95}$
$-40000x_{96} - 42000x_{97} - 40000x_{98} - 38000x_{99} - 40000x_{100}$
$-40000x_{101} - 40000x_{102} - 38000x_{103} - 42000x_{104} = 0$

constraint 12 :

$x_{10} + x_{11} + x_{12} + y_{7,10} + y_{8,11} + y_{9,10} + y_{9,11} <= 14040$

constraint 13 :

$x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{15} + y_{5,8} + y_{6,7} + y_{14,15} <= 28520$

constraint 14 :

$x_3 + x_4 + x_5 + x_{13} + x_{14} + x_{15} + y_{2,3} + y_{5,6} + y_{5,8} + y_{9,15} <= 28540$

constraint 15 :

$x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + y_{2,3} <= 41200$

constraint 16 :

$x_1 + x_2 + x_3 + x_{13} + x_{14} + x_{15} + y_{3,4} + y_{3,5} + y_{5,13} + y_{5,14} + y_{9,15} <= 30500$

constraint 17 :

$x_1 + x_2 + x_3 + x_4 + x_5 + x_{13} + y_{5,6} + y_{5,8} + y_{5,14} + y_{13,14} <= 28170$

constraint 18 :

$x_1 + x_2 + x_3 + x_4 + x_5 + x_{13} + x_{14} + y_{5,6} + y_{5,8} + y_{14,15} <= 30695$

constraint 19 :

$x_1 + x_2 + x_3 + x_4 + x_5 + x_{13} + x_{14} + x_{15} + y_{5,6} + y_{5,8} + y_{9,15} <= 33200$

constraint 20 :

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + y_{5,14} + y_{9,15} + y_{13,14}$
$<= 44690$

constraint 21 :

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} <= 46000$

Bounds

$0 <= x_1 <= 3125$
$0 <= x_2 <= 3030$
$0 <= x_3 <= 3020$
$0 <= x_4 <= 3200$
$0 <= x_5 <= 3770$
$0 <= x_6 <= 2500$
$0 <= x_7 <= 3800$
$0 <= x_8 <= 3900$
$0 <= x_9 <= 3415$
$0 <= x_{10} <= 1500$
$0 <= x_{11} <= 2880$
$0 <= x_{12} <= 1525$
$0 <= x_{13} <= 2800$
$0 <= x_{14} <= 3500$
$0 <= x_{15} <= 2500$
$2000 <= y_{1,2} <= 3000$
$1500 <= y_{2,3} <= 2250$
$1800 <= y_{3,4} <= 2700$
$1800 <= y_{3,5} <= 2700$
$1800 <= y_{3,13} <= 2700$
$1800 <= y_{4,5} <= 2700$
$1500 <= y_{5,6} <= 2250$
$2000 <= y_{5,8} <= 3000$
$1800 <= y_{5,13} <= 2700$
$1450 <= y_{5,14} <= 2175$
$2000 <= y_{6,7} <= 3000$
$2000 <= y_{7,8} <= 3000$
$1200 <= y_{7,10} <= 1800$
$3000 <= y_{8,9} <= 4500$
$1500 <= y_{8,11} <= 2250$
$1225 <= y_{9,10} <= 1835$
$1500 <= y_{9,11} <= 2250$
$1500 <= y_{9,15} <= 2250$
$2450 <= y_{10,11} <= 3675$
$1675 <= y_{11,12} <= 2500$
$1200 <= y_{13,14} <= 1800$
$2000 <= y_{14,15} <= 3000$
$x_j >= 0, \quad j = 16, ..., 104$

The optimal solution of the problem.
Solution value = 78273.333333
The dimension of **x** is 126.
x =

1683.33	3030.00	3020.00	3200.00	3770.00	2500.00	3800.00	450.00
3415.00	1500.00	2880.00	1525.00	2800.00	3500.00	2500.00	2000.00
1500.00	1800.00	1800.00	1800.00	1800.00	1500.00	2000.00	1800.00
1450.00	2000.00	2000.00	1200.00	3000.00	1500.00	1225.00	1500.00
1500.00	2450.00	1675.00	1200.00	2000.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.14
0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.30	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.06	0.00	0.00	0.00	0.00	0.00		

Tx is NOT DOMINATED by any pLEP.
The dimension of **Tx** is 10.
Tx =

11330.00	22070.00	25290.00	36360.00	24883.33	23653.33	26503.33	28503.33
37723.33	39573.33						

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