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Abstract. When the functioning of a system, that is influenced by random effects, has to be ensured by high reliability then the methods mentioned in the title offer powerful tools to achieve the goal. The paper presents an introduction and a short summary of the relevant most important results. These concern problem formulations, convexity statements, numerical solutions of problems of continuous and discrete variables and applications.
Key words: stochastic programming, probabilistic constraint, logconcavity.

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1 Introduction

The idea to ensure the functioning of a stochastic system, by a large probability, comes from statistics and reliability theory. A statistical test is considered a good one if both the first and second kind errors are small (see Lehman (1959, 1986)). Even more sophisticated methods, e.g., the sequential analysis of Wald (1945) contain the principle that the decision in favor of or against the hypothesis should be made by large probability. In reliability theory the functioning of one single part or a complete system is characterized by its probability and we want to see it to be a number not far from 1.

The use of probabilistic constraint (under the title of "chance constrained programming") was first introduced by Charnes, Cooper and Symonds (1958). In this and subsequent papers by Charnes and his coauthors, individual probabilistic constraint is imposed on each stochastic constraint. Miller and Wagner (1965) used first probabilistic constraint jointly for the stochastic constraints. However, they assumed that the random variables contained in the different stochastic constraints are independent of each other. The general probabilistic constrained stochastic programming model was introduced by Prékopa (1970, 1973b). The introduction of stochastically dependent random elements created challenging mathematical and computational problems.

The basic problems can be formulated in the following manner

$$\begin{aligned} &\text{Minimize} \quad h(\mathbf{x}) \\ &\text{subject to} \\ &\quad h_0(\mathbf{x}) = P\left(g_1(\mathbf{x},\boldsymbol{\xi}) \geq 0, \cdots, g_r(\mathbf{x},\boldsymbol{\xi}) \geq 0\right) \geq p \\ &\quad h_1(\mathbf{x}) \geq p_1, \cdots, h_m(\mathbf{x}) \geq p_m \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &\text{Maximize} \quad h_0(\mathbf{x}) = P(g_1(\mathbf{x},\boldsymbol{\xi}) \geq 0, \cdots, g_r(\mathbf{x},\boldsymbol{\xi}) \geq 0) \\ &\text{subject to} \\ &\quad h_1(\mathbf{x}) \geq p_1, \cdots, h_m(\mathbf{x}) \geq p_m, \end{aligned} \tag{1.2}$$

where $g_1(\mathbf{x},\mathbf{y}), \cdots, g_r(\mathbf{x},\mathbf{y})$ are functions of $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^q$; $h(\mathbf{x}), h_1(\mathbf{x}), \cdots, h_m(\mathbf{x})$ are functions of $\mathbf{x} \in \mathbb{R}^n$; $p, p_1, \cdots, p_m$ are constants, $0 < p < 1$ and $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_q)^T$ is a random vector.

An important special case of problem (1.1) is the following

$$\begin{aligned} &\text{Minimize} \quad \left\{\mathbf{c}^T\mathbf{x} + \sum_{i=1}^{r} q_i^+ E\left[\xi_i - T_i\mathbf{x}\right]_+\right\} \\ &\text{subject to} \\ &\quad P\left(T\mathbf{x} \geq \boldsymbol{\xi}\right) \geq p \\ &\quad A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{1.3}$$

where $T_1, \cdots, T_r$ are the rows of the matrix T.

Introducing the notations

$$\begin{aligned} F(\mathbf{z}) &= P(\boldsymbol{\xi} \leq \mathbf{z}), \qquad \mathbf{z} \in \mathbb{R}^r \\ E &= \{\mathbf{z} \,|\, F(\mathbf{z}) \geq p\}, \end{aligned}$$

problem (1.3) can be written as

$$\begin{aligned} &\text{Minimize} \quad \left\{ \mathbf{c}^T\mathbf{x} + \sum_{i=1}^{r} q_i E\left[\xi_i - T_i\mathbf{x}\right]_+ \right\} \\ &\text{subject to} \\ &T\mathbf{x} \in E \\ &A\mathbf{x} = \mathbf{b}, \qquad \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{1.4}$$

To show how challenging problem (1.3) is, we remark that if the probability distribution of $\boldsymbol{\xi}$ is continuous and $f(\mathbf{z}) = f(z_1, \cdots, z_r)$ is the corresponding probability density function, then the probabilistic constraint in problem (1.3) has the form:

$$\int_{-\infty}^{T_r\mathbf{x}} \cdots \int_{-\infty}^{T_1\mathbf{x}} f(z_1, \cdots, z_r) dz_1 \cdots dz_r \geq p. \tag{1.5}$$

Here we see that the constraining function in (1.3) is an r-dimensional integral that, in the course of the solution of the problem, we will have to compute several times, together with its gradient. An important special case of f is the multivariate normal probability density function, defined as follows:

$$f(\mathbf{z}) = \frac{1}{(2\pi)^{r/2}|C|^{1/2}} e^{-1/2(\mathbf{z}-\boldsymbol{\mu})^T C^{-1}(\mathbf{z}-\boldsymbol{\mu})}, \qquad \mathbf{z} \in \mathbb{R}^q$$

where $\boldsymbol{\mu} = E(\boldsymbol{\xi})$ is the expectation and

$$C = E\left[(\boldsymbol{\xi} - \boldsymbol{\mu})(\boldsymbol{\xi} - \boldsymbol{\mu})^T\right]$$

is the covariance matrix of $\boldsymbol{\xi}$ that is assumed to be positive semi-definite. The symbol $|C|$ means the determinant of C.

Problem (1.3) combines the two most important features that appear in stochastic programming model formulations: the recourse and probabilistic constrained formulations. Without the probabilistic constraint problem (1.3) is called the simple recourse problem. In problem (1.3) we ensure the functioning of the system on a reliability level greater than or equal to p and penalize the violation of the constraint $T\mathbf{x} \geq \boldsymbol{\xi}$ by the sum of the expected componentwise violations multiplied by the weights $q_1^+, \cdots, q_r^+$, respectively.

If we want to avoid the computational difficulties imposed by the probabilistic constraint in problem (1.3), then we may replace it by the constraints

$$E\left[\xi_i - T_i\mathbf{x}\right]_+ = \int_{T_i\mathbf{x}}^{\infty} (z - T_i\mathbf{x}) dF_i(z) = \int_{T_i\mathbf{x}}^{\infty} \left[1 - F_i(z)\right] dz \leq d_i \quad i = 1, \cdots, r \tag{1.6}$$

or by the constraints

$$E\left[\xi_i - T_i\mathbf{x} \,|\, \xi_i - T_i\mathbf{x} > 0\right] \leq d_i, \qquad i = 1, \cdots, r. \tag{1.7}$$

The first one was introduced by Klein Haneveld (1986) and called integrated chance constraint (or probabilistic constraint). The second one was introduced by Prékopa (1973a). The constraints (1.4) can easily be converted into linear constraints. In fact, introducing the functions

$$g_i(t) = \int_t^{\infty} [1 - F_i(z)]\, dz, \qquad i = 1, \cdots, r, \tag{1.8}$$

the constraints (1.6) take the form

$$T_i\mathbf{x} \geq g_i^{-1}(d_i), \qquad i = 1, \cdots, r. \tag{1.9}$$

The constraints (1.7) are also equivalent to linear ones, under some conditions, as it will be shown in the next section.

Finally, we remark that problem (1.3) can be regarded as the stochastic programming problem, formulated on the basis of the "underlying deterministic problem":

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &T\mathbf{x} \geq \boldsymbol{\xi} \\ &A\mathbf{x} = \mathbf{b}, \qquad \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{1.10}$$

This would be the problem if $\boldsymbol{\xi}$ were not random. If in (1.8) we have $T\mathbf{x} = \boldsymbol{\xi}$ rather then $T\mathbf{x} \geq \boldsymbol{\xi}$, then the stochastic programming problem takes the form

$$\begin{aligned} &\text{Minimize} \quad \left\{\mathbf{c}^T\mathbf{x} + \sum_{i=1}^{r} \left(q_i^+ E\left([\xi_i - T_i\mathbf{x}]_+\right) + q_i^- E\left([T_i\mathbf{x} - \xi]_+\right)\right)\right\} \\ &\text{subject to} \\ &A\mathbf{x} = \mathbf{b}, \qquad \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{1.11}$$

Note that in (1.11) there is no constraint of the type $P(T\mathbf{x} = \boldsymbol{\xi}) \geq p$. The reason is that in most cases the probability $P(T\mathbf{x} = \boldsymbol{\xi})$ is either 0 or a very small number.

In this survey paper we consider stochastic programming problems with single objective functions. For stochastic programming with multiple objective functions the reader should consult with the papers by Teghem *et al.*. (1986), Slowinski and Teghem (1988) and the book by Wolf (1983), Stancu-Minasian (1984), Guddat *et al.*. (1985) and the references given there.

For a simple, practice oriented presentation of the various stochastic programming models, see Abel and Thiel (1981).

2 Convexity Theory

A central theoretical question, concerning problems (1.1) and (1.2), is the behavior of the function $h_0(\mathbf{x})$ from the point of view of concavity or generalized concavity, because we want to derive statements for the convexity of the set $\{\mathbf{x} \,|\, h_0(\mathbf{x}) \geq p\}$. We need conditions for the distribution of the random vector $\boldsymbol{\xi}$ and the functions $g_1(\mathbf{x}, \mathbf{y}), \cdots, g_r(\mathbf{x}, \mathbf{y})$ that ensure at least the quasi-concavity of $h_0(\mathbf{x})$. Fortunately, we can derive a very strong type of quasi-concavity: logarithmic concavity of $h_0(\mathbf{x})$, under some conditions which hold in a large variety of practical situations. Before formulating the statements we need some definitions.

A function $f(\mathbf{x}) \geq 0$, $\mathbf{x} \in \mathbb{R}^q$ is said to be logarithmically concave (logconcave) if for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^q$ and $0 < \lambda < 1$, we have

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq [f(\mathbf{x})]^{\lambda}[f(\mathbf{y})]^{1-\lambda}.$$

Following Prékopa (1971) a probability measure P, defined on the Borel-subsets of $\mathbb{R}^q$, is said to be logarithmically concave (logconcave) if for every pair A, B of convex subsets of $\mathbb{R}^q$ and $0 < \lambda < 1$, we have

$$P(\lambda A + (1-\lambda)B) \geq [P(A)]^{\lambda}[P(B)]^{1-\lambda},$$

where $\lambda A + (1-\lambda)B = \{\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \,|\, \mathbf{x} \in A,\, \mathbf{y} \in B\}$.

Logconcave probability density functions, defined in $\mathbb{R}^1$, proved to be very useful in reliability theory (see, e.g., Barlow and Proschan (1981)). Multivariate logconcave functions came up primarily in stochastic programming. The fundamental theorem in this respect, proved by Prékopa (1971, 1973b), is the following

Theorem 2.1 *If the probability measure P, defined on the measurable subsets of $\mathbb{R}^q$, is generated by a logconcave probability density function f, then P is a logconcave measure.*

Theorem 2.1 has a number of important consequences that we formulate below.

Theorem 2.2 *If $g_1(\mathbf{x}, \mathbf{y}), \cdots, g_r(\mathbf{x}, \mathbf{y})$ are quasi-concave functions in $\mathbb{R}^{n+q}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^q$ and the random vector $\boldsymbol{\xi} \in \mathbb{R}^q$ has logconcave probability distribution, then the function*

$$h_0(x) = P(g_1(\mathbf{x}, \boldsymbol{\xi}) \geq 0, \cdots, g_r(\mathbf{x}, \boldsymbol{\xi}) \geq 0) \tag{2.1}$$

is logconcave.

Proof. Let

$$H(\mathbf{x}) = \{\mathbf{y} \,|\, g_1(\mathbf{x}, \mathbf{y}) \geq 0, \cdots, g_r(\mathbf{x}, \mathbf{y}) \geq 0\}$$

and $D = \{\mathbf{x} \,|\, H(\mathbf{x}) \neq \emptyset\}$. The quasi-concavity of the functions $g_i(\mathbf{x}, \mathbf{y})$, $i = 1, \cdots, r$ implies that

$$H(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \supset \lambda H(\mathbf{x}_1) + (1-\lambda)H(\mathbf{x}_2) \tag{2.2}$$

for any $\mathbf{x}_1, \mathbf{x}_2 \in D$ and $0 < \lambda < 1$. Since

$$h_0(\mathbf{x}) = P(\boldsymbol{\xi} \in H(\mathbf{x})), \qquad \mathbf{x} \in \mathbb{R}^n,$$

relation (2.2) implies the assertion. □

In view of Theorem 2.1, the function (2.1) is logconcave if $\boldsymbol{\xi}$ has continuous probability distribution and logconcave density function.

Theorem 2.2 was proved by Prékopa (1973a, 1974) for the case where the $g_i(\mathbf{x}, \mathbf{y})$ functions are concave. Tamm (1976, 1977) remarked that the concavity can be replaced by the weaker quasi-concavity.

Theorem 2.3 *Let P be a logconcave probability measure and $A \subset \mathbb{R}^q$ a convex set. Then the function of the variable $\mathbf{z}$:*

$$P(A + \mathbf{z}), \qquad \mathbf{z} \in \mathbb{R}^q \tag{2.3}$$

is a logconcave function. In particular, the associated probability distribution function

$$F(\mathbf{z}) = P(\{\mathbf{x} \,|\, \mathbf{x} \leq \mathbf{0}\} + \mathbf{z}), \qquad \mathbf{z} \in \mathbb{R}^q \tag{2.4}$$

is logconcave.

Proof. Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^q$ and $0 < \lambda < 1$. Since A is a convex set, we have the equality

$$A + [\lambda \mathbf{z}_1 + (1 - \lambda)\mathbf{z}_2] = \lambda(A + \mathbf{z}_1) + (1 - \lambda)(A + \mathbf{z}_2). \tag{2.5}$$

Equation (2.5) and the logconcavity of P imply the assertion. □

In view of Theorem 2.1, the functions (2.3) and (2.4) are logconcave if the random vector $\boldsymbol{\xi} \in \mathbb{R}^q$ has continuous distribution and logconcave density.

Theorem 2.4 *Assume that the random variable $\xi \in \mathbb{R}^1$ has expectation, continuous probability distribution, and logconcave density f. Then the function*

$$g(z) = E\left(\xi - z \,|\, \xi - z > 0\right), \qquad z \in \mathbb{R}^1 \tag{2.6}$$

is decreasing.

Proof. First we remark that if $P(\xi - z > 0) = 0$, then (2.6) is 0, by definition. Let z_0 be the smallest z such that $F(z) = P(\xi \leq z) = 1$. If there is no such z, we take $z_0 = \infty$. It is easy to derive the equality

$$g(z) = \frac{\int_z^\infty (t - z) f(t) dt}{1 - F(z)} = \frac{\int_z^\infty [1 - F(t)] dt}{1 - F(z)} = -\frac{1}{\dfrac{d}{dz} \ln \displaystyle\int_z^\infty [1 - F(t)] dt} \tag{2.7}$$

that is valid for $z < z_0$.

Theorem 2.2 implies that $1 - F(z) = P(\{x|x > 0\} + z)$ is a logconcave function, hence a suitable application of Theorem 2.3 shows that

$$\int_z^\infty [1 - F(t)]dt$$

is also logconcave. Looking at (2.7) we see that $g(z)$ is decreasing for $z < z_0$. Since $g(z) = 0$ for $z \geq z_0$, the theorem follows. □

Theorem 2.4 tells us that each constraint in (1.7) can be replaced by a linear one provided that $\xi_1, \cdots, \xi_r$ have (individually) continuous distributions with logconcave densities.

The applications of Theorems 2.1—2.3 for the stochastic programming problems introduced in Section 1 is straightforward.

Logconcavity is a quite frequent property enjoyed by many well-known probability distributions: uniform (on a convex set), multivariate normal, Wishart and, with some restrictions on the parameters, the Dirichlet, beta, gamma (among the many multivariate gamma distributions we refer to the one, defined by Prékopa and Szántai (1978)).

Davidovich *et al.*., and Prékopa (1973b) proved that the class of logconcave functions is closed under convolution. Prékopa (1973b) also proved a more general statement that if $f(\mathbf{x}, \mathbf{y}), \mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^q$ is logconcave, then

$$\int_{\mathbb{R}^q} f(\mathbf{x}, \mathbf{y})d\mathbf{y}$$

is also logconcave. Thus, the marginals of a logconcave density function are also logconcave.

Generalizations of the logconcavity theorems are due to Borell (1975), Brascamp and Lieb (1976) and Rinott (1977).

Concerning two positive numbers U and V, let $T_\alpha = [\lambda U^\alpha + (1-\lambda)V^\alpha]^{\frac{1}{\alpha}}$ be defined as

$$T_\alpha = \begin{cases} [\lambda U^\alpha + (1-\lambda)V^\alpha]^{\frac{1}{\alpha}}, & \text{if} \quad -\infty < \alpha < 0, \text{ or } 0 < \alpha < \infty \\ \min(U, V), & \text{if} \quad \alpha = -\infty \\ \max(U, V), & \text{if} \quad \alpha = \infty \\ U^\lambda V^{1-\lambda}, & \text{if} \quad \alpha = 0. \end{cases}$$

Theorem 2.5 *Let f be a probability density function in $\mathbb{R}^q$, satisfying the inequality*

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq [\lambda f^\alpha(\mathbf{x}) + (1-\lambda)f(\mathbf{y})^\alpha]^{\frac{1}{\alpha}}$$

for every $0 < \lambda < 1$, whenever $f(\mathbf{x}) > 0$, $f(\mathbf{y}) > 0$ and P the probability measure generated by f. Then for every pair A, B of convex subsets of $\mathbb{R}^q$ with $P(A) > 0$, $P(B) > 0$ and $0 < \lambda < 1$, we have

$$P(\lambda A + (1-\lambda)B) \geq \left(\lambda[P(A)]^{\frac{\alpha}{1+q\alpha}} + (1-\lambda)[P(B)]^{\frac{\alpha}{1+q\alpha}}\right)^{\frac{1+q\alpha}{\alpha}}.$$

The most important special case, other than the case of a logconcave f, that satisfies the condition of Theorem 2.5 is the one where $1 + q\alpha = 0$, i.e., $\alpha = -1/q$. For this case, as a corollary, we obtain

Theorem 2.6 *If the probability density function f, defined in $\mathbb{R}^q$, has the property that $f^{-1/q}$ is a convex function, and the probability measure P is generated by f, then for every pair A, B of convex subsets of $\mathbb{R}^q$ and $0 < \lambda < 1$, we have*

$$P(\lambda A + (1-\lambda)B) \geq \min[P(A), P(B)]. \tag{2.8}$$

A probability measure satisfying (2.8) is called quasi-concave. An example that satisfies the condition of Theorem 2.6 is the multivariate Student distribution that has, as a special case, the Cauchy distribution, for the case of $q = 1$.

The above generalizations of the notion of logconcavity and the mentioned general theorems are mainly of mathematical interest. While a large variety of known multivariate probability density functions enjoy the logconcavity property, only a few further density functions have been mentioned for which Theorem 2.5 is applicable and we could derive new convexity statements for stochastic programming.

The theorems mentioned so far guarantee the quasi-concavity of $h_0(x)$ in problem (1.1). In fact, every logconcave function is also quasi-concave. Having a probabilistic constraint of the type

$$h_0(x) = P(T\mathbf{x} \geq \boldsymbol{\xi}) \geq p$$

with random $\boldsymbol{\xi}$ and T, our theorems do not apply in a direct manner because the functions $T_1\mathbf{x} - \xi_1, \cdots, T_r\mathbf{x} - \xi_r$ are not quasi-concave functions of the elements and components in $T, \mathbf{x}$ and $\boldsymbol{\xi}$.

For the case of $r = 1$, i.e., a single stochastic constraint $T\mathbf{x} - \boldsymbol{\xi} \geq \mathbf{0}$, van de Panne and Popp (1963) and Kataoka (1963) were the first to obtain the quasi-concavity of the function of the variable $\mathbf{x}$: $P(T\mathbf{x} \geq \boldsymbol{\xi})$. They assumed that the random vector $(T, -\boldsymbol{\xi})$ has a multivariate normal distribution and the function is defined on the set $\{\mathbf{x} \,|\, P(T\mathbf{x} \geq \boldsymbol{\xi}) \geq 0.5\}$.

For the case of $r > 1$, Prékopa (1974) and Burkauskas (1986) proved, among others, that if the elements of the matrix $(T, -\boldsymbol{\xi})$ have a joint normal distribution and the rows or the columns have the property that their covariance matrices are constant multiples of a covariate matrix, then $h_0(x)$ is quasi-concave on the set $\{\mathbf{x} \,|\, h_0(\mathbf{x}) \geq 0.5\}$.

Theorem 2.3 asserts the logconcavity of the joint probability distribution function of random variables having logconcave joint probability density. Bawa (1973) has shown that if the random variables are independent and some further conditions are satisfied, then the joint probability distribution function is concave too, on a suitably chosen convex set.

Continuity and other qualitative properties of the function $h_0(\mathbf{x})$ have been established by Raik (1970, 1971a, b, 1972), Marti (1971), Kall (1976, 1987), Römisch and Schultz (1991, 1993) and others. An approximation theory has been developed by Salinetti (1983). Duality theorems concerning probabilistic constrained stochastic programming problems involving logconcave distributions have been proved by Luc (1983) and Komáromi (1987).

Further references for logconcave and related or more general types of functions can be found in Prékopa (1980a), Dharmadikari and Joag-Dev (1988), Norkin and Roenko (1992) and Norkin (1993).

For further stochastic inequalities that may be useful in stochastic programming see Marshall and Olkin (1979) and Shaked and Tong (1992).

3 Numerical Solution of Probabilistic Constrained Stochastic Programming Problems

We will consider the following special cases of problems (1.1) and (1.2):

$$\begin{aligned} &\text{Minimize} \quad h(\mathbf{x}) \\ &\text{subject to} \\ &G(\mathbf{x}) = P(T\mathbf{x} \geq \boldsymbol{\xi}) \geq p \\ &A\mathbf{x} = \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\text{Maximize} \quad G(\mathbf{x}) = P(T\mathbf{x} \geq \boldsymbol{\xi}) \\ &\text{subject to} \\ &A\mathbf{x} = \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{3.2}$$

The methods that we describe in this section apply to more general problems too. However, we restrict ourselves to problems (3.1) and (3.2) because these are the problems which can be solved by the existing codes.

There are two separate issues to solve these problems: (1) to adapt or create a nonlinear programming technique and (2) to adapt or develop a suitable method to compute the values and the gradients (if necessary) of the function $G(\mathbf{x})$. First we concentrate on issue # 1.

The Method of Feasible Directions:

Historically the first technique, to solve problem (3.1), was developed by Prékopa and Deák (see Prékopa *et al.*(1980)). They used the method of feasible directions developed by Zoutendijk (1960). assuming that an initial feasible solution $\mathbf{x}$ has been found, the method works as follows. Let $\mathbf{x}^1 = \mathbf{x}$.

Step 1. Solve the following direction finding problem:

$$\begin{aligned} &\text{Minimize} \quad z \\ &\text{subject to} \\ &\nabla h(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k) - z \leq 0 \\ &\nabla G(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k) + \theta z \geq 0, \qquad \text{if} \quad G(\mathbf{x}^k) = p \\ &A\mathbf{x} = \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{3.3}$$

where θ is a positive constant, fixed throughout the procedure. Let $(z_{opt}, \mathbf{x}^*)$ be an optimal solution of problem (3.3). If $z_{opt} = 0$ then $\mathbf{x}^*$ is an optimal solution of problem (3.1). If $z_{opt} > 0$ then go to Step 2.

Step 2. Solve the steplength determining problem:

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{subject to} \\ &\quad \lambda \geq 0 \quad \text{and} \\ &\quad \mathbf{x}^k + \lambda(\mathbf{x}^* - \mathbf{x}^k) \qquad \text{is feasible.} \end{aligned} \tag{3.4}$$

Go to Step 1.

The convergence of this procedure was proved by Prékopa (1974b) under the following conditions: h, G are quasi-concave and have continuous gradients; there exists an $\mathbf{x}^0$ such that $G(\mathbf{x}^0) > p$ (Slater's condition); the set $\{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is bounded.

To find an initial feasible solution we may use a simple gradient method to maximize $G(\mathbf{x})$ subject to the constraints $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$. It is needless to carry out the whole procedure, we may stop when an $\mathbf{x}^0$ is encountered that satisfies $G(\mathbf{x}^0) > p$.

The advantage of this method, when solving problem (3.1), is that is provides us with a possibility to handle the determination and use of $G(\mathbf{x})$ as well as $\nabla G(\mathbf{x})$ in a stable manner. The effect of noise when these values are approximated by simulation or bounding methods, can be controlled and, if the result is not satisfactory, the sample size can be increased or the bounds can be improved. This remark applies to all other solution methods too, that we present here concerning problem (3.1).

The Logarithmic Barrier Function Method (SUMT)

If $G(\mathbf{x})$ is a logconcave function in $\mathbb{R}^n$ then so is $G(\mathbf{x}) - p$ on the set $\{\mathbf{x} \mid G(\mathbf{x}) \geq p\}$. This fact suggests the application of the Sequential Unconstrained Minimization Technique to solve problem (3.1).

The method works so that we take a sequence of positive numbers $\{s^k\}$ such that $s^k > s^{k+1}, k = 1, 2, \cdots$ and $\lim_{k\to\infty} s^k = 0$ and solve the problem

$$\begin{aligned} &\text{Minimize} \quad \{h(\mathbf{x}) - s^k \ln(G(\mathbf{x}) - p)\} \\ &\text{subject to} \\ &\quad A\mathbf{x} = \mathbf{b} \\ &\quad \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{3.5}$$

in principle for each k. If $\mathbf{x}^k$ is an optimal solution of problem (3.5) then, under some conditions, we have that

$$\lim_{k\to\infty} h(\mathbf{x}^k) = \min_{\substack{G(\mathbf{x})\geq p \\ A\mathbf{x}=\mathbf{b}, \mathbf{x}\geq\mathbf{0}}} h(\mathbf{x}). \tag{3.6}$$

The conditions are satisfied if h is a continuous convex and G is a continuous logconcave function; $G(\mathbf{x}^0) > p$ for some $\mathbf{x}^0 \in \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and $\{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is a bounded set. Under these conditions the objective function in problem (3.5) is convex for any s. For

a more general convergence theory of the SUMT method we refer to the book by Fiacco and McCormick (1968).

The use of SUMT to solve problems (1.1) and (3.1) was first proposed by Prékopa (1972a) and implemented by Rapcsák (1974), Prékopa *et al.*(1978), Prékopa and Kelle (1978).

The Supporting Hyperplane Method

A variant of Veinott's (1967) supporting hyperplane method is used by Szántai (1988) and Prékopa and Szántai (1978b) to solve problem (3.1).

It is assumed that h, G are quasi-concave and have continuous gradients; there exists a vector $\mathbf{x}^0$ such that $G(\mathbf{x}) > p$, $\mathbf{x}^0 \in \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$; there exists a bounded convex polyhedron K^1 containing the set of feasible solutions of problem (3.1).

In a first phase we find a feasible $\mathbf{x}^0$ satisfying Slater's condition. This can be done by maximizing $G(\mathbf{x})$ subject to $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$, by the use of the method described below. The second phase consists of the following steps.

Step 1. Solve the problem:

$$\begin{aligned} &\text{Minimize} \quad h(\mathbf{x}) \\ &\text{subject to} \\ &\mathbf{x} \in K^s. \end{aligned} \tag{3.7}$$

Let $\mathbf{x}^s$ be an optimal solution. If $\mathbf{x}^s$ is feasible, then Stop, $\mathbf{x}^s$ is an optimal solution of problem (3.1). Otherwise go to Step 2.

Step 2. Let λ^s be the largest $\lambda \geq 0$ such that $\mathbf{x}^0 + \lambda(\mathbf{x}^s - \mathbf{x}^0)$ is feasible and

$$\mathbf{y}^s = \mathbf{x}^0 + \lambda^s(\mathbf{x}^s - \mathbf{x}^0).$$

Choose any constraint that is satisfied with equality sign for $\mathbf{y}^s$. If we have $G(\mathbf{y}^s) = p$, then we define

$$K^{s+1} = \{\mathbf{x} \mid \mathbf{x} \in K^s, \nabla G(\mathbf{y}^s)(\mathbf{x} - \mathbf{y}^s) \geq 0\}.$$

If it is a linear constraint then K^{s+1} is defined as the intersection of K^s and the set determined by this linear constraint.

The Reduced Gradient Method

The application of the method was proposed by Mayer (1979, 1988). Let $h(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ in problem (3.1) that we intend to solve. We assume that $G(\mathbf{x})$ is logconcave, $\nabla G(\mathbf{x})$ is Lipschitz-continuous, Slater's condition holds and the set of feasible solutions is bounded.

At the kth iteration we are given a feasible $\mathbf{x}^k$, a tolerance ϵ^k, the partitioning $\mathbf{x}^k = (\mathbf{y}^k, \mathbf{z}^k)$, $A = (B, R)$, $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_R)$, where B is a nonsingular square matrix and $(\mathbf{y}^k)_j \geq 1$ for all j. We perform the following steps.

Step 1. Solve the direction finding problem

$$
\begin{aligned}
&\text{Minimize} \quad t \\
&\text{subject to} \\
&\quad \mathbf{c}_B^T\mathbf{u} + \mathbf{c}_R^T\mathbf{v} \le t \\
&\quad \nabla G(\mathbf{x}^k)\mathbf{u} + \nabla G(\mathbf{x}^k)\mathbf{v} \ge \theta t, \qquad \text{if} \quad G(\mathbf{x}^k) \le p + \epsilon^k \\
&\quad B\mathbf{u} + R\mathbf{v} = \mathbf{0} \\
&\quad v_j \ge 0, \qquad \text{if} \quad z_j \le \epsilon^k \\
&\quad v_j \le 1, \qquad \text{all} \quad j,
\end{aligned}
\tag{3.8}
$$

where $\mathbf{u}, \mathbf{v}$ and t are the decision variables and θ is a positive constant. An equivalent form of this problem is

$$
\begin{aligned}
&\text{Minimize} \quad t \\
&\text{subject to} \\
&\quad \mathbf{r}^T\mathbf{v} \le t \\
&\quad \mathbf{s}^T\mathbf{v} \ge \theta t, \qquad \text{if} \quad G(\mathbf{x}^k) \le p + \epsilon^k \\
&\quad v_j \ge 0, \qquad \text{if} \quad z_j \le \epsilon^k \\
&\quad v_j \le 1, \qquad \text{all } j,
\end{aligned}
\tag{3.9}
$$

where $\mathbf{r} = \mathbf{c}_R - \mathbf{c}_B^T B^{-1} R$ and $\mathbf{s} = \nabla_z G(\mathbf{x}^k) - \nabla_y G(\mathbf{x}^k) B^{-1} R$ are the reduced gradients of the objective function and the probabilistic constraint, respectively. Let $(\mathbf{v}^*, t^*)$ be an optimal solution of problem (3.9). There are two cases.

Case 1. We have $t^* > \epsilon^k$. Then we compute $\mathbf{u}^* = -B^{-1}R\mathbf{v}^*$ and go to Step 2.

Case 2. We have $t^* \le \epsilon^k$. Then ϵ^k is halved. If the new ϵ^k is smaller than a zero tolerance then we accept $\mathbf{x}^k$ as optimal solution. Otherwise we solve problem (3.8) with the new ϵ^k and to Step 2.

Step 2. Let λ_1 be the largest $\lambda \ge 0$ such that $\mathbf{x}^k + \lambda\mathbf{w}^*$ satisfies the linear constraints, where $\mathbf{w}^* = (\mathbf{u}^*, \mathbf{v}^*)$. Let λ_2 be any λ satisfying

$$p \le G(\mathbf{x}^k + \lambda\mathbf{w}^*) \le p + \epsilon^k$$

and $\lambda^* = \min(\lambda_1, \lambda_2)$. Define $\mathbf{x}^{k+1} = \mathbf{x}^k + \lambda^*\mathbf{w}^*$ and go to Step 3.

Step 3. If $(\mathbf{y}^{k+1})_j \le \epsilon^k$ for some j then the nondegeneracy assumption is violated. Find a new partition for which the nondegeneracy assumption holds. If necessary, reduce ϵ^k to meet this assumption. Go to Step 1.

A Primal-Dual Method

A primal-dual method has been developed by Komáromi (1986) to solve the following problem

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &F(\mathbf{y}) \geq p \\ &T\mathbf{x} \geq \mathbf{y} \\ &D\mathbf{x} \geq \mathbf{d}, \end{aligned} \tag{3.10}$$

where F is the probability distribution function of the random vector $\boldsymbol{\xi}$: $F(\mathbf{y}) = P(\boldsymbol{\xi} \leq \mathbf{y})$. Problem (3.10) comes from the problem

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &P(T\mathbf{x} \geq \boldsymbol{\xi}) \geq p \\ &D\mathbf{x} \geq \mathbf{d}, \end{aligned} \tag{3.11}$$

to which it is equivalent. Let $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^r$ and suppose F is a strictly logconcave probability distribution function, i.e., for every pair $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^r$, $\mathbf{y}_1 \neq \mathbf{y}_2$, and $0 < \lambda < 1$, we have

$$F(\lambda\mathbf{y}_1 + (1-\lambda)\mathbf{y}_2) > [F(\mathbf{y}_1)]^{\lambda}[F(\mathbf{y}_2)]^{1-\lambda}.$$

The advantage of problem (3.10) over problem (3.11) is that the probabilistic constraint involves only the probability distribution function of $\boldsymbol{\xi}$ and not the composite function $F(T\mathbf{x})$.

Starting to solve problem (3.10), we associate with it a "dual" problem (not a dual problem in the classical sense):

$$\begin{aligned} &\text{Maximize} \quad \left[\min_{F(\mathbf{y})\geq p} \mathbf{u}^T\mathbf{y} + \mathbf{v}^T\mathbf{d}\right] \\ &\text{subject to} \\ &T^T\mathbf{u} + D^T\mathbf{v} = \mathbf{c} \\ &\mathbf{u} \geq \mathbf{0}, \quad \mathbf{v} \geq \mathbf{0}. \end{aligned} \tag{3.12}$$

The procedure works in the following manner. First we assume that a pair of vectors $(\mathbf{u}^{(1)}, \mathbf{v}^{(1)})$ is available, for which

$$\left(\mathbf{u}^{(1)}, \mathbf{v}^{(1)}\right) \in V = \{(\mathbf{u}, \mathbf{v}) \,|\, T^T\mathbf{u} + D^T\mathbf{v} = \mathbf{c}, \quad \mathbf{v} \geq \mathbf{0}\}.$$

Suppose that $(\mathbf{u}^{(k)}, \mathbf{v}^{(k)})$ has already been chosen, where $\mathbf{u}^{(k)} \geq \mathbf{0}$. Then we perform the following steps.

Step 1. Solve the problem

$$\begin{aligned} &\text{Minimize} \quad (\mathbf{u}^{(k)})^T\mathbf{y} \\ &\text{subject to} \\ &\quad F(\mathbf{y}) \geq p. \end{aligned} \tag{3.13}$$

Let $\mathbf{y}(\mathbf{u}^{(k)})$ designate the optimal solution. Then we solve the direction finding problem

$$\begin{aligned} &\text{Maximize} \quad \{\mathbf{u}^T\mathbf{y}(\mathbf{u}^{(k)}) + \mathbf{d}^T\mathbf{v}\} \\ &\text{subject to} \\ &\quad (\mathbf{u}, \mathbf{v}) \in V. \end{aligned} \tag{3.14}$$

Let $(\mathbf{u}_k^*, \mathbf{v}_k^*)$ be an optimal solution to this problem. If $\mathbf{u}_k^* = \rho\mathbf{u}^{(k)}$ then $(\mathbf{u}_k^*, \mathbf{v}_k^*)$ is an optimal solution of the dual problem (3.12) and the pair $\hat{\mathbf{x}}, \mathbf{y}(\mathbf{u}^{(k)})$ is an optimal solution to the primal problem (3.10), where $\hat{\mathbf{x}}$ is an optimal solution of the LP:

$$\begin{aligned} &\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &\quad T\mathbf{x} \geq \mathbf{y}(\mathbf{u}^{(k)}) \\ &\quad D\mathbf{x} \geq \mathbf{d}. \end{aligned}$$

Otherwise go to Step 2.

Step 2. Find $\lambda^{(k)}$ $(0 < \lambda < 1)$ satisfying

$$(\mathbf{u}_k^*)^T\mathbf{y}\left(\frac{\lambda^{(k)}}{1-\lambda^{(k)}}\mathbf{u}^{(k)} + \mathbf{u}_k^*\right) > \left(\mathbf{u}^{(k)}\right)^T y\left(\mathbf{u}^{(k)}\right) + \left(\mathbf{v}^{(k)}\right)^T \mathbf{d}.$$

Then we define

$$\mathbf{u}^{(k+1)} = \lambda^{(k)}\mathbf{u}^{(k)} + (1-\lambda^{(k)})\mathbf{u}_k^*$$

$$\mathbf{v}^{(k+1)} = \lambda^{(k)}\mathbf{v}^{(k)} + (1-\lambda^{(k)})\mathbf{v}_k^*$$

If the procedure is infinite, then the sequence $(\mathbf{u}^{(k)}, \mathbf{v}^{(k)})$ converges and the limiting pair has the same property as $(\mathbf{u}_k^*, \mathbf{v}_k^*)$ in Step 1.

The second computational burden when solving a probabilistic constrained stochastic programming problem arises when finding values and gradients of the constraining function $F(T\mathbf{x})$.

When the probability distribution of the random vector $\boldsymbol{\xi}$ is continuous then it means finding the values and gradients of the multivariate probability distribution function $F(\mathbf{z}) = P(\boldsymbol{\xi} \leq \mathbf{x})$ in problem (3.1).

There are three distributions for which efficient numerical techniques are available to accomplish the above mentioned job: the multivariate normal, the multivariate Dirichlet and a multivariate gamma distribution, defined by Prékopa and Szántai (1978a).

Deák (1980, 1986, 1988, 1990) developed efficient methods to generate multivariate normal random samples. These can be used directly to find the probability distribution functions and their gradients. However, Szántai (1985, 1986) and Gassman (1988) developed other techniques that use the fast random number generators of Deák but considerably reduce the cost when estimating $F(\mathbf{z})$ and $\nabla F(\mathbf{z})$.

Szántai takes the complementary of the event $\{\xi_1 \leq z_1\} \cap \cdots \cap \{\xi_n \leq z_n\}$, applies the inclusion-exclusion formula, computes exactly the univariate and bivariate distribution function values there and applies simulation for the rest. This can be done for any distribution, not only for the normal or the other two ones mentioned above. For the case of the normal distribution Gassman combined Deák's and Szántai's techniques in an efficient way. On a 486 and 33 MHz PC we can compute values of 10-dimensional probability distribution functions in a few seconds, by the use of this method.

The gradients of the probability distribution functions of the above mentioned three continuous probability distributions can be obtained by the same method that produces the function values. See, e.g., Prékopa (1970), where it is shown that the gradient of an m-dimensional normal probability distribution function (at a given point) can be obtained by computing m function values of an $m-1$-dimensional normal probability distribution function. Similar assertion holds for the gamma distribution of Prékopa and Szántai (1978a) and also for the Dirichlet distribution.

Further ideas pertaining to the numerical solution of probabilistic constrained stochastic programming problems can be found in Prékopa (1988).

4 Applications

A large number of applications have been carried out on various fields. Below we mention a sample of them. We concentrate on problems where the probabilistic constraint is taken jointly for the stochastic constraints, for two reasons. First, this is the correct way of model construction if the random variables involved are stochastically dependent. Second, if the random variables are on the right hand side and individual probabilistic constraints are used, then the problem can be converted, in a trivial way, into a deterministic one that is most frequently an LP. This means that the stochastic programming problem in question does not contain new difficulties, neither from the mathematical, nor from the computational point of view, than does the underlying deterministic problem.

The first large scale application using joint probabilistic constraint was done by Prékopa *et al.*(1980) concerning a five-year plan of the Hungarian power industry. The problem is of type (3.1), where $\mathbf{x}$ has 46, $\boldsymbol{\xi}$ has 4 components and A has 106 rows. The components of the random vector $\boldsymbol{\xi}$ essentially mean demands for electricity and are supposed to have joint normal distribution with nonzero correlations.

An interesting feature of the problem is that the cost arising from the optimal solution of the stochastic programming problem is almost equal to the cost that one obtains when solving the LP, replacing $E(\xi_i)$ for $\xi_i, i = 1, 2, 3, 4$, on the right hand side of the underlying problem. However, the two optimal solutions, $\mathbf{x}_{lin}$ and $\mathbf{x}_{stoch}$ are significantly different. Different fuel

composition and investment figures are optimal in the two problems. On the other hand, the reliability level of the deterministic solution, i.e., the probability $P(T\mathbf{x}_{lin} \geq \boldsymbol{\xi})$ is only 0.09, whereas for about the same cost the reliability level 0.99 can be ensured by the solution of the probabilistic constrained problem with $p = 0.99$.

Power system expansion problems have been formulated in probabilistic constrained or similar forms by Prékopa (1980b, 1985), Bloom (1983), Bloom *et al.*(1984). In the papers by Prékopa the transmission system is also included, thus, the problem is a stochastic network design problem too.

Recently a probabilistic constrained stochastic programming problem has been formulated and solved by Guddat *et al.*(1992) for an optimal power dispatching problem.

Applications in electronic design include the papers by Styblinski and Ruszczynski (1983) and Singh, Abraham and Akella (1990). Both solve probability maximization problems. About the latter one some more will be presented in the next section.

The solution of an inventory control problem by Prékopa (1965, 1972b) and Ziermann (1964) and its more general variant, presented by Prékopa and Kelle (1978), resulted in big savings in the Hungarian economy. Safety stocks, corresponding to given service levels, have been determined for thousands of raw materials and half ready products. These ensure continuous production if the delivery process has uncertainty in it or ensure that all demands will be met if there is uncertainty in the production or demand. These ideas have also been used by Kelle (1985) to optimize in multi-stage production inventory systems.

Among the applications to water resources problems a very successful one is the optimal water level regulation method developed by Prékopa and Szántai (1979). Mathematically it is a sequential probability maximization problem. Given the random water inputs $\xi_1, \xi_2, \cdots$, we have to find optimal water amounts $z_1, z_2, \cdots$ to be released through the lock so that the probability

$$P\left(a_k \leq \sum_{i=1}^{k} \xi_i - \sum_{i=1}^{k} z_i \leq b_k,\ k = 1, \cdots, N\right)$$

be as large as possible, given that $0 \leq z_1, \cdots, z_N \leq K$, where K is the capacity of the lock. Such problems are solved sequentially, conditioning on those random variables that have already been realized. While the former lock operation policy ensured that the water level stayed within the prescribed limits in 80% of the periods, the stochastic programming methodology increased this level to 97.5%.

Probabilistic constrained stochastic programming models have successfully been applied to reservoir system design problems by Prékopa, and Szántai (1978b), Prékopa *et al.*(1978), Yakowitz (1982), Kelman *et al.*(1989) and Morgan *et al.*(1993), among others. For water quality management models see, e.g., Somlyódy and Wets (1988) and Pintér (1991).

A probabilistic constrained diet model has been developed by Balintfy and Armstrong (1980). In this paper the nutrient requirement vector has the random variables and the model ensures that a given percent of the population receives all nutrients on the required level. For the development of the diet models and a comprehensive bibliography, see Lancaster (1992).

Animal feed problems can also be formulated as probabilistic constrained stochastic programming problems. A good example is the one presented by van de Panne and Popp (1963).

Designing safe engineering structures is also a probabilistic constrained stochastic programming problem. Randomness may occur in the load or material inhomogeneity. For the problem formulations see Thoft-Christensen and Murotsu (1986).

Kibzun and Kurbakovskiy (1991) have applied probabilistic constrained stochastic programming to find the optimal size of an airport runway.

Finance is one important field of application of probabilistic constrained programming and probability maximization. Already the first classical finance model of Roy (1952), Markowitz (1952) and others required safety in the optimal financial plan. Among the more recent model constructions we mention those by Pickens *et al.*(1991) and Mclean and Ziemba (1991).

5 Programming Under Probabilistic Constraint with Discrete Variables

If both the random and decision variables are discrete, then we are led to solve nonsmooth nonlinear programming problems with discrete variables. There are only a few papers that handle such problems.

Looking at the probabilistic constraint

$$P(T\mathbf{x} \geq \boldsymbol{\xi}) \geq p, \tag{5.1}$$

we observe that it has an equivalent formulation by the aid of the "p-level efficient points", or in abbreviated form: PLEP's. This concept has been introduced by Prékopa (1990).

A point z is called a PLEP if

$$F(z) = P(\xi \leq z) \geq p$$

and for no $y \leq z, y \neq z$ do we have $F(y) \geq p$. If ξ has a finite number of possible values, then for any p the number of PLEP's is finite.

Assume that for a given p, the set of PLEP's is $\{z_1, \cdots, z_N\}$. Then the constraint (5.1) can be written in the equivalent form:

$$T\mathbf{x} \geq z_i, \qquad \text{for at least one } \; i \in \{1, \cdots, N\}. \tag{5.2}$$

Replacing the constraint (5.2) for (5.1), problem (3.1) takes the form:

$$\text{Minimize} \quad \mathbf{c}^T\mathbf{x}$$

subject to

$$T\mathbf{x} \geq z_i, \qquad \text{for at least one } \; i \in \{1, \cdots, N\} \tag{5.3}$$

$$A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}.$$

Enumerating all PLEP's, the straightforward solution to the problem is to use one at a time out of the inequalities making the first constraint in (5.3) and thereby solve N linear programming problems.

This is more complicated if the objective function contains penalty terms, i.e., the stochastic programming problem is

$$\begin{aligned} &\text{Minimize} \quad \left\{ \mathbf{c}^T\mathbf{x} + \sum_{i=1}^{r} q_i E\left([\xi_i - T_i\mathbf{x}]_+\right) \right\} \\ &\text{subject to} \\ &P(T\mathbf{x} \geq \boldsymbol{\xi}) \geq p \\ &A\mathbf{x} = \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{5.4}$$

Prékopa (1990) suggested a solution technique to problem (5.4), using the constraints (5.2) and reformulating the penalty term in the objective function for each PLEP.

Sen (1992) observed that (5.3) belongs to the class of disjunctive programming problems, developed by Balas (1979), and obtained a bounding technique for the value of the objective function.

An interesting practical problem, involving probability maximization under constraints, has been solved by Singh *et al.*(1990). It concerns wafer design in semiconductor manufacturing.

Suppose we have S chip sites available that we want to allocate for the production of m different types of chips. We want to produce a set of chips that consists of a_j chips of type $j, j = 1, \cdots, m$. The probability that a chip site assigned to type j chip will yield a good chip is p_j. If x_j chip sites are allocated to type j chips then the probability that there will be at least a_j good chips is:

$$\sum_{i=a_j}^{x_j} \binom{x_j}{i} p_j^i (1-p_j)^{x_j - i}. \tag{5.5}$$

The product of the probabilities in (5.5) gives us the probability that a complete set will be obtained. Thus, we have arrived at the following optimization problem:

$$\begin{aligned} &\text{Maximize} \quad \prod_{j=1}^{m} \sum_{i=a_j}^{x_j} \binom{x_j}{i} p_j^i (1-p_j)^{x_j - i} \\ &\text{subject to} \\ &\sum_{j=1}^{m} x_j \leq S \\ &x_j \text{ integer}, \;\; x_j \geq 0, \;\; j = 1, \cdots, m. \end{aligned} \tag{5.6}$$

In the proposed solution method the first step is to fit a smooth logconcave function to each function in (5.5). Then we carry out the optimization with these smooth functions. Finally, a sensitivity analysis procedure terminates the solution.

Numerical examples show that the naive solutions do not give right answer. The allocation of the sites according to the proportions of $a_1, \cdots, a_m$ does not give the largest probability, in general. Moreover, if one optimal allocation has been found and the number of available chip sites is doubled, then the optimal allocation is not necessarily the double of the former one.

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