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On the relationship between probabilistic constrained, disjunctive and multiobjective programming

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Abstract. A probabilistic constrained stochastic programming model is formulated, where one term in the objective function, to be minimized, is the maximum of a finite or infinite number of linear functions. The model is reformulated as a finite or semi-infinite disjunctive programming problem. Duality relationships are established for both the original and the convexified problems. Numerical solution techniques are presented for both the finite and semi-infinite problems that provide us with lower and upper bounds for the optimum value.

1 Introduction

Programming under probabilistic constraint, or chance constrained programming, has been introduced by Charnes, Cooper, Symonds (1958), Miller, Wagner (1965) and Prékopa (1970, 1973b).

When we formulate a stochastic programming problem, then in the first step we formulate a deterministic problem that would be our optimization problem if we did not have randomness in it. That problem is called underlying deterministic problem or base problem. After we have identified the random variables in the problem, we observe that it loses its original meaning. Then we reformulate it, by the use of some decision principle. The resulting problem is a stochastic programming problem. Consider the following base problem:

$$\begin{aligned} &\min c^T x \\ &\text{subject to} \\ &Ax \geq b \\ &Tx \geq \xi \\ &\quad x \geq 0, \end{aligned} \tag{1}$$

where A is an $m \times n$, T is an $r \times n$ matrix, x, c, b, ξ are vectors of suitable sizes, ξ is a random variable. The decision variable is x. Based on this we formulate the stochastic programming problem:

$$\begin{aligned} &\min c^T x \\ &\text{subject to} \\ &\quad Ax \geq b \\ &P(Tx \geq \xi) \geq p \\ &\quad x \geq 0, \end{aligned} \tag{2}$$

where p $(0 < p < 1)$ is a prescribed probability. We call (2) probabilistic constrained problem.

There are more general probabilistic constrained problem formulations too (see Prékopa, 1995) but in this paper we take problem (2) as the starting point of our investigation.

The two important special cases of problem (2) are those, where ξ has continuous distribution and where ξ is a discrete random vector. First, in order to introduce the type of problem we are dealing with in this paper, we look at the discrete case.

In the history of the programming under probabilistic constraint, or more generally, probabilistic programming more attention was paid to the case of a continuously distributed ξ.

Even though many papers have been written about it, the fact that a problem solution involves not only the execution of a nonlinear programming method but also the calculation of multivariate integrals, to obtain function and gradient values in each iteration, made the application difficult. The situation changed by the introduction of the concept

of p-efficient point, by Prékopa (1990). Since then probabilistic constrained problems with discrete random variables has generated lot of interest, development of new algorithms as well as applications. Below we recall the definition of that concept. Let S designate the support of this discrete random variable ξ. In what follow we designate the c.d.f. of ξ by F.

Definition 1. The point $s \in S$ is a p-level efficient (or briefly a p-efficient) point of S if $F(s) \geq p$ and there is no $y \in S$ such that $y \leq s$, $y \neq s$, $F(y) \geq p$.

Let $\{s_1, \ldots, s_N\}$ be the set of p-efficient points. Then problem (2) is equivalent to the following:

$$\begin{aligned} &\min c^T x \\ &\text{subject to} \\ &Ax \geq b \\ &Tx \in \bigcup_{i=1}^{N} (s_i + R^r_+) \\ &\quad x \geq 0. \end{aligned} \tag{3}$$

Problem (3) is a disjunctive programming problem. A relaxation of it (see Balas 1979, 1998; Prékopa, Vizvári, Badics, 1998) is:

$$\begin{aligned} &\min c^T x \\ &\text{subject to} \\ &Ax \qquad\qquad \geq b \\ &Tx - \sum_{i=1}^{N} \lambda_i s_i \geq 0 \\ &\qquad \sum_{i=1}^{N} \lambda_i \quad = 1 \\ &x \geq 0,\ \lambda \geq 0. \end{aligned} \tag{4}$$

The dual of this problem is:

$$\begin{aligned} &\max\{v + b^T u\} \\ &\text{subject to} \\ &\quad v - s_i^T z \leq 0, \quad i = 1, \ldots, N \\ &A^T u + T^T z \leq c \\ &\quad u \geq 0,\ z \geq 0. \end{aligned} \tag{5}$$

The decision variables are u, z and v. An equivalent form of problem (5) is the following:

$$\begin{aligned} &\max \left\{ \min_{1 \leq i \leq N} s_i^T z + b^T u \right\} \\ &\text{subject to} \\ &A^T u + T^T z \leq c \\ &\quad u \geq 0,\ z \geq 0, \end{aligned} \tag{6}$$

where the variable v does not appear.

The $s_1, \ldots, s_N$ p-efficient points that appear in the probabilistic constraint in problem (3), produce a multi-objective optimization problem (6), as its dual. Now we mention the following

Theorem 1. (Prékopa, Vizvári, Badics, 1998) *If $\{s_1, \ldots, s_N\}$ is an antichain, i.e., none of them dominates any other one in the set, then for every $0 < p < 1$ there exists a c.d.f. such that $\{s_1, \ldots, s_N\}$ is the set of its p-efficient points.*

Now, let us start from problem (6), where we assume that the set $\{s_1, \ldots, s_N\}$ is an antichain. If we reformulate it in the form of problem (5) and take the dual of the latter, then we come to problem (4) that is a relaxation of the probabilistic constrained problem (3), written up (based on the above theorem) with $\{c_1, \ldots, c_N\}$ as the set of p-efficient points of the probability distribution of a random vector ξ.

Thus, not the original probabilistic constrained problem but its relaxation is in a primal-dual relationship with a multi-objective optimization problem.

The present paper is related, in addition to earlier works of the author, to the works of Balas (1979, 1998) on disjunctive programming and to a paper by Komáromi (1986), where a duality theorem is proved for probabilistic constrained problems with continuously distributed random vectors. Our problems are more general than Komáromi's, allow for the use of discrete and continuous random variables, and, most importantly, we develop efficient algorithms to solve our problems.

The structure of the paper is as follows. In Section 2, we present a pair of primal-dual, multiobjective, probabilistic constrained stochastic programming problems for the case of discrete random variables. The counterparts, for continuously distributed random vectors, are formulated in Section 3. Limiting procedure is used to obtain the duality theorem for the continuous case. In Section 4 we recall formerly developed solution algorithms for problem (2), one for the discrete, one for the continuous case. A hybrid of the two solves problem (2) with continuously distributed ξ in such a way that in each iteration we obtain a lower and an upper bound for the optimum value. In Section 5 we present algorithms to solve the other types of problems, where discrete and continuous random vectors may appear simultaneously and probabilistic constraints appear along with multiobjective terms in the objective functions. Finally, in Section 6 some results are mentioned for the case, where ξ has independent components and it is shown that the problem with dependent random variables can be relaxed to the former case, allowing for easier calculation of an approximate solution.

2 A general, multi-objective, probabilistic constrained model and a duality relationship for the case of a discrete ξ

Consider the following optimization problem:

$$\begin{aligned}
&\min \left\{ \max_{1\le i\le M} c_i^T x + q^T y \right\} \\
&\text{subject to} \\
&\quad Ax + B\ y \ge b \\
&P(Tx + Wy \ge \xi) \ge p_0 \\
&\qquad x \ge 0,\ y \ge 0,
\end{aligned} \tag{7}$$

where the decision variables are x and y. In the above problem we have simultaneously multiobjective function and probabilistic constraint.

Suppose that ξ is discrete and the set of its p_0-efficient points is $\{s_1, \ldots, s_N\}$. The following problem is equivalent to (7):

$$\begin{aligned}
&\min \left\{ \max_{1\le i\le M} c_i^T x + q^T y \right\} \\
&\text{subject to} \\
&Ax + \ By \ge b \\
&Tx + Wy \in \bigcup_{i=1}^{N} (s_i + R_+^r) \\
&\qquad x \ge 0,\ y \ge 0.
\end{aligned} \tag{8}$$

A relaxation of this problem is:

$$\begin{aligned}
&\min \left\{ \max_{1\le i\le M} c_i^T x + q^T y \right\} \\
&\text{subject to} \\
&Ax + \ By \qquad\qquad \ge b \\
&Tx + Wy - \sum_{i=1}^{N} \lambda_i s_i \ge 0 \\
&\qquad\qquad \sum_{i=1}^{N} \lambda_i \ = 1 \\
&\qquad x \ge 0,\ y \ge 0,\ \lambda \ge 0.
\end{aligned} \tag{9}$$

Introducing the variable t, (9) can be written in the equivalent form:

$$
\begin{aligned}
&\min\{t + q^T y\} \\
&\text{subject to} \\
&t - c_i^T x \qquad\qquad\qquad \geq 0, \quad i = 1, \ldots, M \\
&\quad Ax + \ By \qquad\qquad\quad \geq b \\
&\quad Tx + Wy - \sum_{i=1}^{N} \lambda_i s_i \geq 0 \\
&\qquad\qquad x \geq 0, \ y \geq 0, \ \lambda \geq 0.
\end{aligned}
\tag{10}
$$

The dual of the last problem is:

$$
\begin{aligned}
&\max\{v + b^T u\} \\
&\text{subject to} \\
&v - \qquad s_i^T z \qquad\qquad \leq 0, \quad i = 1, \ldots, N \\
&\quad B^T u + W^T z \qquad\quad \leq q \\
&\quad A^T u + \ T^T z - \sum_{i=1}^{M} \mu_i c_i \leq 0 \\
&\qquad\qquad \sum_{i=1}^{M} \mu_i \quad = 1 \\
&\qquad\qquad u \geq 0, \ z \geq 0, \ \mu \geq 0.
\end{aligned}
\tag{11}
$$

We assume that the set $\{c_1, \ldots, c_M\}$ is an antichain. Then the set $\{-c_1, \ldots, -c_M\}$ is also an antichain. It follows from the proof of Theorem 1. (see Prékopa, Vizvári, Badics 1998) that if we supplement to it a suitable vector d, the obtained $M + 1$ points $-c_1, \ldots, -c_M$, d may be regarded as the support of a random vector η whose p_1-efficient points are $-c_1, \ldots, -c_M$, where p_1 is an arbitrarily chosen probability, $0 < p_1 < 1$. If we use this, then we can write (11) in the following equivalent form:

$$
\begin{aligned}
&\max \left\{ \min_{1 \leq i \leq N} s_i^T z + b^T u \right\} \\
&\text{subject to} \\
&\quad W^T z + B^T u \leq q \\
&P(-T^T z - A^T u \geq \eta) \geq p_1 \\
&\quad z \geq 0, \ u \geq 0.
\end{aligned}
\tag{12}
$$

Thus, the dual of the convexified problem (11) is of the same type as the original problem (7). In this sense we consider problem (7) and (12) a pair of primal-dual probabilistic constrained multi-objective stochastic programming problems. Below we present side by

side the pair of primal-dual problems which are the following

$$\begin{aligned} &\min \left\{ \max_{1\le i\le M} c_i^T x + q^T y \right\} \\ &\text{subject to} \\ &\quad Ax + B\ y \ge b \\ &P(Tx + Wy \ge \xi) \ge p_0 \\ &\quad x \ge 0,\ y \ge 0. \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\max \left\{ \min_{1\le i\le N} s_i^T z + b^T u \right\} \\ &\text{subject to} \\ &\quad W^T z + B^T u \le q \\ &P(-T^T z - A^T u \ge \eta) \ge p_1 \\ &\quad z \ge 0,\ u \ge 0. \end{aligned} \tag{14}$$

If A, B, W, b, q are 0 matrices and vectors, respectively, then the pair of primal-dual problems is:

$$\begin{aligned} &\min \left(\max_{1\le i\le M} c_i^T x \right) \\ &\text{subject to} \\ &P(Tx \ge \xi) \ge p_0 \\ &\quad x \ge 0 \end{aligned} \tag{15}$$

$$\begin{aligned} &\max \left(\min_{1\le i\le N} s_i^T z \right) \\ &\text{subject to} \\ &P_1(-T^T z \ge \eta) \ge p_1 \\ &\quad z \ge 0. \end{aligned} \tag{16}$$

These are discrete variants of Komáromi's primal-dual problems (see Komáromi, 1986).

3 The case of continuously distributed random vectors

Consider again problem (2) but now assume that ξ is a continuously distributed random vector. While in the discrete case the set of feasible solutions to the problem is nonconvex, in general, in the continuous case a class of probability distributions has been identified such that if the distribution of ξ belongs to that class, then problem (2) is convex. Before stating the relevant theorem we recall two definitions.

Definition 2. A (point) function $f(z) \geq 0$, $z \in R^r$ is called logarithmically concave (logconcave) if for any $x, y \in R^r$ and $0 < \lambda < 1$ we have the relation:

$$f(\lambda x + (1-\lambda)y) \geq [f(x)]^{\lambda}[f(g)]^{1-\lambda}. \tag{17}$$

Definition 3. (Prékopa, 1971) A probability measure P, defined on the Borel sets of R^r is called logarithmically concave (logconcave) if for any convex subsets A, B of R^r and $0 < \lambda < 1$ we have the relation:

$$P(\lambda A + (1-\lambda)B) \geq [P(A)]^{\lambda}[P(B)]^{1-\lambda}. \tag{18}$$

The basic theorem of logconcave measures (Prékopa, 1971, 1973a) is the following:

Theorem 2. *If P is a probability measure generated by a logconcave probability density function, then the measure P is logconcave.*

Theorem 2. implies that if the probability measure P is continuous and is generated by a logconcave density function, further A is a convex set, then $P(A+z)$, $z \in R^r$ is a logconcave (point) function. If we choose $A = \{t \mid t \leq z\}$, then

$$P(A+z) = P(\xi \leq z) = F(z)$$

and we have that the c.d.f. of ξ is a logconcave (point) function. This fact readily implies

Theorem 3. *The set of x vectors satisfying the probabilistic constraint in problem* (2) *is convex, for any fixed probability p.*

A sharper form of Theorem 3. is also proved in Prékopa (1973a). It is the following:

Theorem 4. *Let f be a logconcave p.d.f. in R^r and F the corresponding c.d.f. If f is positive and strictly logconcave in an open convex set D, then F is also strictly logconcave in the set D.*

Strict logconcavity of a point function f means that the inequality (17) holds strictly whenever $x \neq y$.

Under the condition of Theorem 4. the relation $F(z) = p$ determines a strictly convex surface in D that can be represented by strictly convex functions in such a way that we take $r-1$ of the variables $z_1, \ldots, z_r$ and then determine the remaining one as the unique solution of the above equation. An important special case is the nondegenerate multivariate normal distribution the p.d.f. of which is strictly logconcave in the entire space, thus we can choose $D = R^r$.

Assume, for the sake of simplicity, that both random vectors ξ, η have strictly logconcave p.d.f.'s in the entire space R^r and let F, G designate the corresponding c.d.f's, respectively. It follows that for any $0 < p_0$, $p_1 < 1$ the sets

$$S = \{w \mid F(w) \geq p_0\}$$
$$C = \{w \mid G(w) \geq p_1\}$$

are compact, convex sets and their boundary sets are

$$\tilde{S} = \{w \mid F(w) = p_0\}$$
$$\tilde{C} = \{w \mid G(w) = p_1\},$$

respectively. The sets $\tilde{S}$, $\tilde{C}$ are strictly convex surfaces that can also be represented in functional forms, according to the above made remark.

Consider the following pair of multiobjective, probabilistic constrained semi-infinite programming problems:

$$\begin{aligned} &\min \left\{ \max_{c \in \tilde{C}} c^T x + q^T y \right\} \\ &\text{subject to} \\ &\quad Ax + By \geq b \\ &P(Tx + Wy \geq \xi) \geq p_0 \\ &\quad x \geq 0, \ y \geq 0 \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\max \left\{ \min_{s \in \tilde{S}} s^T z + b^T u \right\} \\ &\text{subject to} \\ &\quad W^T z + B^T u \leq q \\ &P(-T^T z - A^T u \geq \eta) \geq p_1 \\ &\quad z \geq 0, \ u \geq 0. \end{aligned} \tag{20}$$

These problems can be written in the following alternative forms:

$$\begin{aligned} &\min \left\{ \max_{c \in \tilde{C}} c^T x + q^T y \right\} \\ &\text{subject to} \\ &Ax + By \geq b \\ &Tx + Wy \in \bigcup_{s \in \tilde{S}} (s + R^r_+) \\ &\quad x \geq 0, \ y \geq 0 \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\max \left\{ \min_{s \in \tilde{S}} s^T z + b^T u \right\} \\ &\text{subject to} \\ &W^T z + B^T u \leq q \\ &-T^T z - A^T u \in \bigcup_{c \in \tilde{C}} (-c + R^n_+) \\ &\quad z \geq 0, \ u \geq 0. \end{aligned} \tag{22}$$

Problems (21) and (22) are disjunctive, semi-infinite programming problems, where the decision vectors are finite dimensional but there are infinitely many constraints in them with continuum cardinality. Both pairs of problems are in primal-dual relationship as we clarify below.

Let us choose a dense sequence $\{s_i\}$ in the set $\tilde{S}$ and another dense sequence $\{-c_i\}$ in the set $\tilde{C}$ and introduce the notations:

$$\begin{aligned} S_N &= \bigcup_{i=1}^{N} \{w \in R^r \mid w \geq s_i\} \\ C_M &= \bigcup_{i=1}^{M} \{w \in R^m \mid w \geq -c_i\} \\ \tilde{S}_N &= \{s_1, \ldots, s_N\} \\ \tilde{C}_M &= \{-c_1, \ldots, -c_M\}. \end{aligned}$$

If in problems (21) and (22) we replace $\tilde{S}_N$ for $\tilde{S}$ and $\tilde{C}_M$ for $\tilde{C}$, then we obtain a pair of problems of the type (13), (14). It is easy to see that

$$\begin{aligned} \text{cl}\left(\lim_{N\to\infty} S_N\right) &= S \\ \text{cl}\left(\lim_{M\to\infty} C_M\right) &= C, \end{aligned}$$

where cl means closure. It follows that similar limiting relations hold for the sets of feasible solutions of problems (19), (20) or problems (21), (22): the closures of the limits ($M \to \infty$, $N \to \infty$) of the feasible sets, taken with $\tilde{S}_N$, $\tilde{C}_M$ are equal to the sets of feasible solutions of problems (19), (20) or (21), (22).

Incidentally we mention that the strict logconcavity of the functions F and G implies that $\tilde{S}_N$ and $\tilde{C}_M$ are discrete convex sets. In our context discrete convexity of a finite set D means that if $h \in D$, then $h \in \text{riconv}\,(D \backslash \{h\})$.

The discrete convexity of the sets $\tilde{S}_N$ and $\tilde{C}_M$ implies that if in problems (19), (20) or (21), (22) we replace $\tilde{S}_N$, $\tilde{C}_M$ for $\tilde{S}$, $\tilde{C}$, respectively, then we can write up the first terms of the objective functions in the forms: $\max\limits_{1\leq i\leq M} c_i^T x$, $\min\limits_{1\leq i\leq N} s_i^T z$, respectively. In this case the pair of problems (21), (22) is the following:

$$\begin{aligned} &\min \left\{ \max_{1\leq i\leq M} c_i^T x + q^T y \right\} \\ &\text{subject to} \\ &Ax + By \geq b \\ &Tx + Wy \in \bigcup_{i=1}^{N} (s_i + R_+^r) \\ &\qquad x \geq 0,\ y \geq 0 \end{aligned} \tag{23}$$

and

$$
\begin{aligned}
&\max \left\{ \min_{1 \le i \le N} c_i^T z + b^T v \right\} \\
&\text{subject to} \\
&\quad W^T z + B^T v \le q \\
&\quad -T^T z - A^T v \in \bigcup_{i=1}^{M} (-c_i + R_+^n) \\
&\qquad z \ge 0, \ v \ge 0.
\end{aligned} \tag{24}
$$

It is easy to see that if $M \to \infty$, $N \to \infty$, then the optimum values of problems (23), (24) converge to those of problems (19), (20) or (21), (22), respectively.

The above reasoning remains valid even if the supports of the random vectors are not the entire spaces and we only require that they are compact, convex sets. Then S, C, $\tilde{S}$, $\tilde{C}$, S_N, C_M, $\tilde{S}_N$, $\tilde{C}_M$ have to be defined as restricted to the support sets. The closures of the above set sequences are obviously independent of the choices of the sets $\{s_i\}$, $\{-c_i\}$. The following theorem holds true:

Theorem 5. *Assume that the random vectors in problems* (19), (20) *or* (21), (22) *have compact, convex supports and their c.d.f.'s on these sets are strictly quasi-concave (this condition is satisfied if they are strictly logconcave). Then if one of the two problems has feasible solution and finite optimum, then so does the other one and the optimum values are equal.*

Proof. Assume that problem (19) or (21) has feasible solution: (x_0, y_0) and consider first the case, where the probabilistic constraint holds with equality sign for that vector. Let $s_1 = Tx_0 + Wy_0$. Then problem (23) has feasible solution for any N. Since the set of feasible solutions of problem (23) is increasing with N and converges to that of problem (19) or (21) and the latter problem has finite optimum, it follows that problem (24) has feasible solution and finite optimum, for any M. This implies that problem (20) or (22) also has feasible solution, e.g., any feasible solution of problem (24), for any M, has that property. The convergence of the optimum values of the finite problems to those of the continuous problems has already been established. Since for any N, M the optimum values of the finite problems are equal, so are the optimum values of the continuous problems.

If the probabilistic constraint holds with strict inequality in case of x_0, y_0, then for sufficiently large N all problems (24) have feasible solution and the rest of the proof is the same as before. The same proof applies if we start from problem (20) or (22). □

The same theory can be developed for the hybrid case, where in the objective function we have a finite number of linear functions in the multiobjective term, while a continuously distributed random vector is in the probabilistic constraint or vice versa.

4 An algorithm to solve problem (2) in case of a continuously distributed random vector

The algorithm is based on two existing algorithms that we will use alternatingly in such a way that we produce a decreasing outer approximation sequence and an increasing inner approximation sequence to the feasible set. The two sequences of optimum values then converge to the optimal value to the original problem. In the first two subsections we review a cutting plane method of Prékopa, Vizvári, Badics (1998) that solves problem (2) with discrete ξ. Then we briefly describe the supporting hyperplane method of Veinott (1967) and Szántai (1988) for the solution of problem (2) with continuously distributed ξ. Finally, in the first subsection we combine the two methods and present one for the solution of the problem with continuously distributed ξ.

4.1 Cutting plane method to solve problem (2) with discrete random variable

Let $s_1, \ldots, s_N$ be the p-efficient points of the c.d.f. of ξ and suppose that they have already been enumerated. Consider the relaxed problem:

$$\begin{aligned}
&\min c^T x \\
&\text{subject to} \\
&\quad Ax \geq b \\
&\quad Tx \geq \sum_{i=1}^{N} s_i \lambda_i \\
&\sum_{i=1}^{N} \lambda_i = 1 \\
&x \geq 0, \quad \lambda \geq 0.
\end{aligned} \tag{25}$$

If we introduce slack variables, then the problem becomes

$$\begin{aligned}
&\min c^T x \\
&\text{subject to} \\
&\qquad\qquad Ax \geq b \\
&Tx - u - \sum_{i=1}^{N} s_i \lambda_i = 0 \\
&\qquad\qquad \sum_{i=1}^{N} \lambda_i = 1 \\
&x \geq 0, \quad u \geq 0, \quad \lambda \geq 0.
\end{aligned} \tag{26}$$

Sometimes the p-efficient points are lying in a manifold with dimension smaller than r. A preprocessing step checks on it and identifies the smallest manifold that contains all p-efficient

points. This is done in the following way. We pick a vector $\overline{s}$ that is in the relative interior of conv $\{s_i, \ldots, s_n\}$. Let us choose the vector

$$\overline{s} = \frac{1}{N} \sum_{i=1}^{N} s_i$$

that has the required property. Let us rewrite the second constraint in problem (26) in the following way:

$$Tx - u - \overline{s} - \sum_{i=1}^{N} (s_i - \overline{s})\lambda_i = 0.$$

The (x, u) vectors satisfying this equation with some λ vector are those that satisfy

$$w_i^T (Tx - u - \overline{s}) = 0, \quad i = 1, \ldots, h,$$

where $w_1, \ldots, w_h$ are linearly independent vectors that span the subspace L given by

$$L = \{w \mid w^T (s_i - \overline{s}) = 0\}.$$

Problem (26) is equivalent to the following:

$$\begin{aligned}
&\min c^T x \\
&\text{subject to} \\
&\qquad\qquad Ax \geq b \\
&\qquad w_i^T (Tx - u - \overline{s}) = 0, \quad i = 1, \ldots, h \\
&Tx - u - \overline{s} - \sum_{i=1}^{N} (s_i - \overline{s})\lambda_i = 0 \\
&\qquad\qquad \sum_{i=1}^{N} \lambda_i = 1 \\
&\qquad x \geq 0, \quad u \geq 0, \quad \lambda \geq 0.
\end{aligned} \tag{27}$$

Here the second set of constraints is redundant but it becomes essential in the next step.

If $\{s_1, \ldots, s_N\}$ is a discrete convex set, then the convex polyhedron

$$K = \left\{ z = \sum_{i=1}^{N} (s_i - \overline{s})\lambda_i \;\middle|\; \sum_{i=1}^{N} \lambda_i = 1, \quad \lambda \geq 0 \right\}$$

can be determined as the intersection of the sets

$$v_i^T z \geq 0, \qquad i = 1, \ldots, M,$$

where $v_1, \ldots, v_M$ are the inward pointing normals of the facets of K inside the subspace L. The v_i vectors can be obtained in the following way. Solve the LP:

$$\begin{aligned} &\max\{\lambda_1 + \cdots + \lambda_N\} = \alpha \\ &\text{subject to} \\ &(s_1 - \overline{s})\lambda_1 + \cdots + (s_N - \overline{s})\lambda_N = \beta, \\ &\qquad \lambda_1 \geq 0, \ldots, \lambda_N \geq 0, \end{aligned} \tag{28}$$

where β is fixed and specified at an earlier step. If $s_{j_1} - \overline{s}, \ldots, s_{r-h} - \overline{s}$ is an optimal basis to problem (28), then find a $v \neq 0$ satisfying

$$\begin{aligned} v^T w_i &= 0, \qquad i = 1, \ldots, h \\ v^T(s_{j_i} - s_{j_1}) &= 0, \qquad i = 2, \ldots, r-h. \end{aligned} \tag{29}$$

Equations (28) determine v up to a constant factor. Choose v such that it is pointing inwards to the convex polyhedron K. As β varies in the subspace L, we obtain all inequalities (**??**), the number of which is finite. Thus, we can write problem (26) in the new form:

$$\begin{aligned} &\min c^T x \\ &\text{subject to} \\ &\qquad Ax \geq b \\ &w_i^T(Tx - u - \overline{s}) = 0, \quad i = 1, \ldots, h \\ &v_i^T(Tx - u - \overline{s}) \geq 0, \quad i = 1, \ldots, M \\ &\qquad x \geq 0, \quad u \geq 0. \end{aligned}$$

While the second set of constraints is determined at the beginning of the algorithm, those, belonging to the third set are generated, one at a time, in the subsequent steps of the cutting plane algorithm. Below we present its summary. See Figure 1 for illustration.

PVB (Prékopa, Vizvári, Badics) algorithm

Step 0. Enumerate all p-efficient points $s_1, \ldots, s_N$. Initialize $k \leftarrow 0$ and go to Step 1.

Step 1. Determine the vectors $w_1, \ldots, w_h$.

Step 2. Solve the LP:

$$\begin{aligned} &\min c^T x \\ &\text{subject to} \\ &\qquad Ax \geq b \\ &w_l^T(Tx - u - \overline{s}) = 0, \quad l = 1, \ldots, h \\ &v_i^T(Tx - u - \overline{s}) \geq 0, \quad i = 1, \ldots, k \\ &\qquad x \geq 0, \quad u \geq 0. \end{aligned} \tag{30}$$

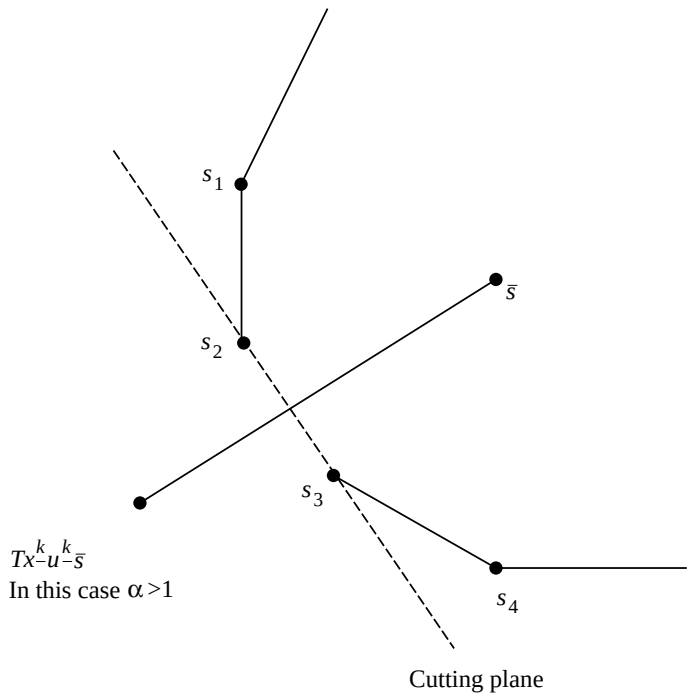

Figure 1: Illustration of the cutting plane method

If $k = 0$, then ignore the constraint for $i = 1, \ldots, k$. Let (x^k, u^k) be an optimal solution. Go to Step 3.

Step 3. For $t = Tx^k - u^k - \overline{s}$ solve problem (28) and let α be the optimum value. If $\alpha \leq 1$, Stop, (x^k, u^k) is an optimal solution to problem (26). Otherwise go to Step 4.

Step 4. Create the cut

$$v_{k+1}^T(Tx - u - \overline{s}) \geq 0,$$

set $k \leftarrow k + 1$ and go to Step 2.

Under the condition that none of the p-efficient points is in the interior of a section determined by any other two of them, the algorithm terminates in a finite number of iterations.

4.2 The supporting hyperplane algorithm

The supporting hyperplane method, developed originally by Veinott (1967) was adapted by Szántai (1988) to solve problem (2). We assume that the random vector ξ has continuous distribution with logconcave p.d.f. Let us introduce the notation: $h(x) = P(Tx \geq \xi) - p$. Suppose also that the convex polyhedron $K^0 = \{x \mid Ax \geq b,\ x \geq 0\}$ is bounded and Slater's condition is satisfied: there exists an $x^0 \in K_0$ such that $h(x^0) > 0$. Throughout the

algorithm we use a fixed x^0 that has the above property. Such an x^0 can be found, e.g., by the use of a few steps in the solution of the problem:

$$\begin{aligned} &\max P(Tx \geq \xi) \\ &\text{subject to} \\ &Ax \geq b \\ &\quad x \geq 0. \end{aligned} \tag{31}$$

If we encounter an x for which $h(x) > 0$, then we may stop the algorithm and choose $x^0 = x$. The supporting hyperplane algorithm can be summarized as follows.

Step 0. Find x^0 satisfying $Ax^0 \geq b$, $x^0 \geq 0$, $h(x^0) > 0$. Go to Step 1.

Step 1. Solve the LP:

$$\begin{aligned} &\min c^T x \\ &\text{subject to} \\ &\qquad\qquad Ax \geq b \\ &\nabla h(x^i)(x - x^0) \geq 0, \quad i = 1, \ldots, k \\ &\qquad\qquad x \geq 0. \end{aligned}$$

Let x^{*k} be an optimal solution. Go to Step 2.

Step 2. Check for the sign of $h(x^{*k})$. If $h(x^{*k}) \geq 0$, Stop, optimal solution to problem (2) has been found. Otherwise go to Step 3.

Step 3. Find λ^k such that $0 < \lambda^k < 1$ and

$$h(x^0 + \lambda^k(x^{*k} - x^0)) = 0.$$

Define $x^{k+1} = x^0 + \lambda^k(x^{*k} - x^0)$ and go to step 4.

Step 4. Introduce the cut:

$$\nabla h(x^{k+1})(x - x^0) \geq 0,$$

set $k \leftarrow k + 1$ and go to Step 1.

If, in addition to the already mentioned assumption, the function h has continuous gradient, then the sequence of the optimum values converges to the optimum value of problem 2.

4.3 The hybrid algorithm

The combined application of the PVB and VS algorithms produces a solution method for problem (2), where ξ has continuous distribution. At each iteration both lower and upper bounds for the optimum value of problem (2) are available, and, if they are close, then we may stop. We assume that all conditions that we required in Section 4.2 are satisfied. The algorithm can be summarized as follows. See Figure 2 for illustration.

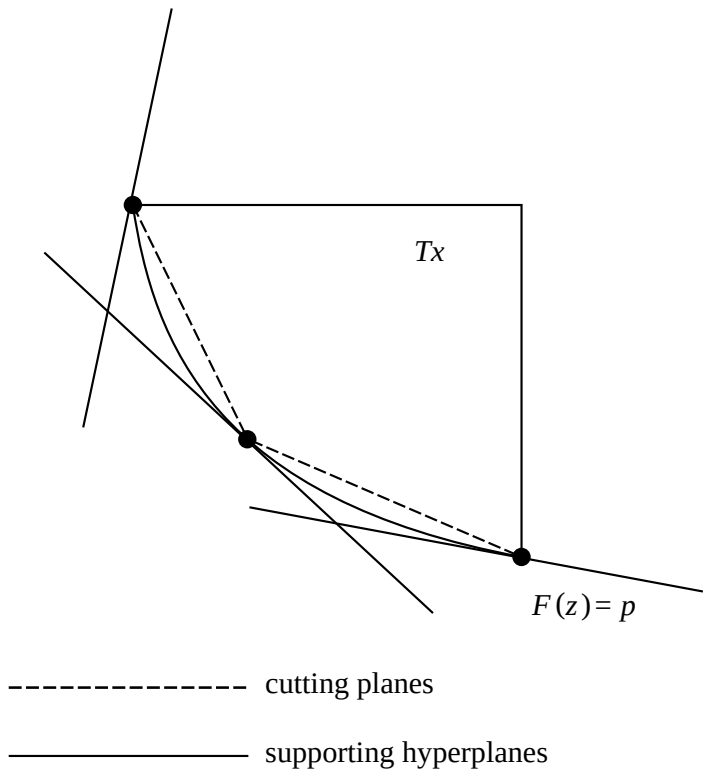

Figure 2: Illustration of the hybrid method

Step 0. Find x^0 satisfying Slater's condition. Initialize $k \leftarrow 0$.

Step 1. Choose $x^i, \ldots, x^r$ linearly independent feasible vectors such that $F(Tx^i) = p$, $i = 1, \ldots, r$. Set $k \leftarrow r$.

Step 2. Solve the LP:

$$\begin{aligned} &\min c^T x \\ &\text{subject to} \\ &\qquad\qquad Ax \geq b \\ &\nabla h(x^i)(x - x^0) \geq 0, \quad i = 1, \ldots, k \\ &\qquad\qquad x \geq 0. \end{aligned}$$

Let x^{*k} be an optimal solution. If $F(Tx^{*k}) \geq p$, then Stop, x^{*k} is an optimal solution to problem (2). Otherwise let $x^{k+1} = x^0 + \lambda^k(x^{*k} - x^0)$, where λ^k is chosen in such a way that $F(T(x^0 + \lambda^k(x^{*k} - x^0))) = p$. Set $k \leftarrow k + 1$ and let $U_k = c^T x^{k+1}$. Clearly, U_k is an upper bound on the optimum value of problem (2).

Remark. 1 When Step 1 is executed for the first time then $k = r$.

Step 3. Solve optimally, by the PVB algorithm, the LP:

$$\begin{aligned} &\min c^T x \\ &\text{subject to} \\ &\qquad Ax \geq b \\ &Tx - u - \overline{s} = \sum_{i=1}^{k} (s_i - \overline{s})\lambda_i \\ &\qquad \sum_{i=1}^{k} \lambda_i = 1 \\ &x \geq 0, \quad u \geq 0, \quad \lambda \geq 0, \end{aligned} \tag{32}$$

where $s_i = Tx^i$, $i = 1, \ldots, k$; $s = (1/k)\sum_{j=1}^{k} s_j$. Note that the vectors $s_1, \ldots, s_k$ span the whole space R^r, thus the subspace L reduces to the zero vector. No additional constraints are needed in problem (32). Let L_k be the optimum value of problem (32). Clearly, L_k is a lower bound on the optimum value of problem (2).

Step 4. If $U_k - L_k \leq \varepsilon$, where ε is a previously chosen tolerance level, then Stop, optimal solution with required precision has been found. Otherwise go to Step 5.

Step 5. Having solved problem (32), we append as many further vectors to the current set $\{s_1, \ldots, s_k\}$ as the number of cutting planes encountered in the course of the solution. Let l be their number. These new vectors satisfy the probabilistic constraint with equality sign.

This is best done in such a way that first we identify those bases that turned out to be optimal in the auxiliary problem (28), with optimum values greater than 1. Then take the intersections of the normals of the corresponding cutting planes, originating from the centers of the simplices (spanned by the basic vectors), with the p-efficient surface of the distribution function F of the random vector ξ. Set $k \leftarrow k + l$ and go to Step 2.

The convergence of the procedure is ensured by the convergence of the supporting hyperplane method. In fact, in Step 2 a new supporting hyperplane is appended to the already existing ones and those that are returned by the PVB algorithm.

The PVB algorithm and the supporting hyperplane algorithm have been implemented (see Prékopa, Vizvári, Badics 1998 and Szántai 1988). We have no implementation so far for the hybrid algorithm.

5 Solutions of the multiobjective probabilistic constrained stochastic programming problems

There are two types of the multiobjective terms finitely and infinitely many of them and two types of the probabilistic constraints: with discrete and continuous distributions. All four combinations many come up in the applications.

5.1 Finitely many functions in the multiobjective term and discrete distribution in the probabilistic constraint

In this case we have problem (9) or the equivalent problem (10) to solve. The solution algorithm is a simple modification of the PVB algorithm. Below we briefly describe it.

Step 0. Enumerate all p-efficient points $s_1, \ldots, s_N$. Initialize $k \leftarrow 0$.

Step 1. Determine the vectors $w_1, \ldots, w_h$.

Step 2. Solve the LP:

$$\begin{aligned}
&\min \{t + c^T x\} \\
&\text{subject to} \\
&\qquad\qquad Ax \geq b \\
&w_l^T(Tx - u - \overline{s}) = 0, \quad l = 1, \ldots, h \\
&v_i^T(Tx - u - \overline{s}) \geq 0, \quad i = 1, \ldots, k \\
&\qquad\qquad c_i^T x \leq t, \quad i \in I_l \in \{1, \ldots, M\} \\
&\qquad\qquad\quad x \geq 0, \quad u \geq 0.
\end{aligned}$$

If $k = 0$, then ignore the third and fourth sets of constraints as well as the variable t in the objective function. Let (x^k, u^k, t^k) be an optimal solution.

Step 3. For $\beta = Tx^k - u^k - \overline{s}$ solve problem (28). If $\alpha \leq 1$ and $c_i^T x^k \leq t^k$, $i \notin I_l$, then Stop, optimal solution has been obtained.

If $\alpha > 1$ and $c_i^T x^k \leq t^k$, $i \notin I_l$, then introduce the cut

$$v_{k+1}^T(Tx - u - \overline{s}) \geq 0.$$

If $\alpha \leq 1$ but $c_i^T x^k \leq t^k$, $i \in I_l$ does not hold, then identify a j such that $c_j^T x^k > t^k$ and set $I_l \leftarrow I_l \cup \{j\}$.

If $\alpha > 1$ and $c_i^T x^k \leq t^k$, $i \in I_l$ does not hold, then introduce both of the above mentioned two cuts. Set $k \leftarrow k + 1$, $l \leftarrow l + 1$ and go to Step 2.

5.2 Finitely many functions in the multiobjective term and continuous distribution in the probabilistic constraint

In this case we carry out the hybrid algorithm of Section 4.3 with the same modification that we have introduced in connection with the PVB algorithm in Section 5.1. We write the

problem in the following form

$$\begin{aligned}
&\min\{t + q^T y\}\\
&\text{subject to}\\
&\qquad c_i^T x \le t \quad i = 1, \dots, M\\
&Ax + By \ge b\\
&Tx + Wy \in \bigcup_{s \in S} (s + R_+^r)\\
&\qquad x \ge 0, \quad y \ge 0.
\end{aligned} \tag{33}$$

In a similar way, as we have done in Step 3 in the algorithm of Section 5.1, we may take a cut toward the inner approximating convex polyhedron or a cut involving the first set of constraints in problem (33).

The remaining two cases, where C is the multivariate p-quantile of a continuous probability distribution function, can be handled similarly. We disregard the detailed presentation of the relevant algorithms.

6 The case of a random vector ξ with stochastically independent components

The solution of problem (2), where ξ has independent components and each has continuous distribution with logconcave density, is relatively easy.

Let F_i designate the c.d.f. of ξ_i, $i = 1, \dots, r$. Then $F(z_1, \dots, z_r) = F_1(z_1), \dots, F_r(z_r)$, and, since $F(Tx) > 0$ in the set of feasible x vectors, the probabilistic constraint can be written in the form:

$$\begin{aligned}
\log F(T_1 x, \dots, T_r x) &= \log F_1(T_1 x) + \dots + \log F_r(T_r x)\\
&\ge \log p.
\end{aligned}$$

If we introduce the variables $z_i = T_i x$, $i = 1, \dots, r$, then in the probabilistic constraint we have a concave, separable function on the left hand side. The numerical calculation of the function and gradient values of the constraining function in the probabilistic constraint is easy and the solution of the problem, by the use of the supporting hyperplane method, or any other nonlinear programming method does not impose further difficulty, in general.

The case of a discrete ξ, however, remains challenging but to a lesser extent, if its components are independent.

An algorithm to solve problem (2) for the case of a discrete ξ has been given by Dentcheva, Prékopa, Ruszczynski (2000). Even though it solves the problem for a general, integer valued ξ, its numerical performance is significantly better if the components are independent, than if they are dependent. The algorithm makes it possible to generate the p-efficient points simultaneously with the problem solution. To generate a p-efficient point needs the solution of a knapsack problem.

Khachyian et al. (2006) have presented an algorithm to find all p-efficient points in the independent component case. It works in polynomial time. The general algorithm of Prékopa, Vizvári, Badics (1998) that works also in the dependent case is slow if r is large (this means, in this context, $r > 10$).

Below we prove a theorem that allows for the use of independent random variables in the probabilistic constraint, under some conditions.

Theorem 6. *Let $(\eta_1, \ldots, \eta_m)$ be a random vector with independent, 0.1-valued components (their distributions may be different). Let $\xi = D\eta$, where D is an $r \times m$ matrix with nonnegative entries. Then, for any $z \in R^r$ we have the inequalities*

$$P(\xi \le z) \ge P(\xi_1 \le z_1) \ldots P(\xi_r \le z_r) \tag{34}$$

$$P(\xi \ge z) \ge P(\xi_1 \ge z_1) \ldots P(\xi_r \ge z_r). \tag{35}$$

Proof. We need a lemma, the proof of which can be found in Liu, Prékopa (2005).

Lemma. 1 *Let $0 < p < 1$, $q = 1 - p$, $a_0 \ge a_1$, $b_0 \ge b_1, \ldots, t_0 \ge t_1$. Then we have the inequality*

$$\begin{aligned} &pa_0 b_0 \ldots t_0 + qa_1 b_1 \ldots t_1 \\ &\quad \ge (pa_0 + qa_1)(pb_0 + qb_1) \ldots (pt_0 + qt_1). \end{aligned}$$

Remark. 2 Since the role of p and q can be interchanged, the above inequality holds also if $a_0 \le a_1$, $b_0 \le b_1, \ldots, t_0 \le t_1$.

The proof of the theorem goes by induction. We prove (34) but the proof of (35) is the same. First we show that (34) holds for $m = 1$. Let $I = \{i \mid a_{i1} > 0,\ 1 \le i \le r\}$. Then we have

$$\begin{aligned} P(\xi_i \le z_i,\ i = 1, \ldots, r) &= P(a_{i1}\eta_1 \le z_i,\ i \in I) \\ &= P\left(\eta_1 \le \min_{i \in I} \frac{z_i}{a_{i1}}\right) \ge P(a_{11}\eta_1 \le z_1) \ldots P(a_{r1}\eta_1 \le z_r) \\ &= P(\xi_1 \le z_1) \ldots P(\xi_r \le z_r). \end{aligned}$$

Assume that the assertion holds for $m - 1$ and prove that it holds for m, where $m - 1 \ge 1$.

Let us introduce the notations: $p_1 = P(\eta_1 = 0)$, $q_1 = 1 - p_1$. Then, using the Lemma and the inductive hypothesis, we obtain

$$\begin{aligned} &P(\xi_1 \le z_1, \ldots, \xi_r \le z_r) \\ &= P\left(\sum_{k=1}^m a_{1k}\eta_k \le z_1, \ldots, \sum_{k=1}^m a_{rk}\eta_k \le z_r \;\middle|\; \eta_1 = 0\right) P(\eta_1 = 0) \\ &\quad + P\left(\sum_{k=1}^m a_{1k}\eta_k \le z_1, \ldots, \sum_{k=1}^m a_{rk}\eta_k \le z_r \;\middle|\; \eta_1 = 1\right) P(\eta_1 = 1) \end{aligned}$$

$$
\begin{aligned}
&= P\left(\sum_{k=2}^{m} a_{1k}\eta_k \le z_1, \dots, \sum_{k=2}^{m} a_{rk}\eta_k \le z_r\right) p_1 \\
&\quad + P\left(\sum_{k=2}^{m} a_{1k}\eta_k \le z_1 - a_{11}, \dots, \sum_{k=2}^{m} a_{rk}\eta_k \le z_r - a_{r1}\right) q_1 \\
&\ge P\left(\sum_{k=2}^{m} a_{1k}\eta_k \le z_1\right) \dots P\left(\sum_{k=2}^{m} a_{rk}\eta_k \le z_r\right) p_1 \\
&\quad + P\left(\sum_{k=2}^{m} a_{1k}\eta_k \le z_1 - a_{11}\right) \dots P\left(\sum_{k=2}^{m} a_{rk}\eta_k \le z_r - a_{r1}\right) q_1 \\
&\ge \left[P\left(\sum_{k=2}^{m} a_{1k}\eta_k \le z_1\right) p_1 + P\left(\sum_{k=2}^{m} a_{1k}\eta_k \le z_1 - a_{11}\right) q_1\right] \\
&\qquad\qquad \vdots \\
&\quad \left[P\left(\sum_{k=2}^{m} a_{rk}\eta_k \le z_r\right) p_1 + P\left(\sum_{k=2}^{m} a_{rk}\eta_k \le z_r - a_{r1}\right) q_1\right] \\
&= P\left(\sum_{k=1}^{m} a_{1k}\eta_k \le z_1\right) \dots P\left(\sum_{k=1}^{m} a_{rk}\eta_k \le z_r\right) \\
&= P(\xi_1 \le z_1) \dots P(\xi_r \le z_r). \qquad \square
\end{aligned}
$$

Corollary. 1 *The inequalities* (34), (35) *remain valid if* $\eta_i, \dots, \eta_m$ *are sums or limits (in probability) of sums of independent,* $0,1$*-valued random variables. The independence of* $\eta_i, \dots, \eta_m$*, however, should be kept. This implies that in* (34), (35) *we may have binomial, Poisson random variables as well. We may also add constants to the* ξ_i*'s and divide them by positive constants. This shows that the* $\eta_1, \dots, \eta_m$ *random variables in* (34), (35) *may have arbitrarily infinitely divisible distributions.*

If $\eta_1, \dots, \eta_m$ are normally distributed random variables, then $\xi_1, \dots, \xi_r$ are nonnegatively correlated and (34), (35) follow by Slepian's inequalities.

The consequence of the inequality (34) for problem (2) is that if we replace the probabilistic constraint

$$P(Tx \ge \xi) \ge p$$

by the constraint

$$\prod_{i=1}^{r} P(T_i x \ge \xi_i) \ge p,$$

then the set of feasible solutions is reduced and we obtain an upper bound for the optimum value of the original problem (2). Numerical experience shows that if p is large ($p \approx 0.9$, say), then the upper bound gives good approximation to the optimum value of the original problem.

We have seen that the solution of problem (2) is much simpler, if the components of ξ are independent, both in the cases of discrete and continuous random variables.

7 Applications

There are various applications of the new problems formulated in this paper. Below we list a few possibilities.

In stochastic programming sometimes we need penalty terms in the objective function that we do not know exactly. We may then look at several possibilities and then take the maximum of that part of the objective function that is uncertain.

One variant of the model: two-stage programming under uncertainty belongs to our class. The model, introduced by Prékopa (1995) introduces a probabilistic constraint into the original model: the probability of solvability of the second stage problem is prescribed to be greater than or equal to a given probability p. On the other hand, the recourse function that is part of the objective function, is the maximum of finite or infinite linear functions. If the random elements in the problem are discrete with finite supports, then the recourse function is the maximum of a finite number of linear functions. They are generated in the course of the numerical solution by Benders' decomposition.

In financial planning it is quite frequent that we do not know the costs exactly at the outset. For example if the plans include a large real estate project, then typically some of the costs are not completely known at the planning stage.

Goal programming is another example, where the coefficients of the variables in the objective function are not completely known.

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