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ON STOCHASTIC SET FUNCTIONS. III.

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In the present paper we are concerned with two topics: the probability distributions of the random variables consisting of a completely additive stochastic set function and the discontinuities of the realizations by supposing the latters to be completely additive number-valued set functions. Some theorems of Chapter III are analogous to those well-known in the theory of stochastic processes with independent increments. (cf. mainly [13]).

There are questions arising in a natural way in the field discussed in this paper which we did not consider in detail such as the representation of a stochastic set function as a sum of an atomless and an atomic part, etc. These problems are not very difficult to solve with well-known methods or with our results.

We refer to the definitions and notations of the previous papers [19], [20] which are used here without any remark. Some new definitions and notations will also be introduced at the place where they are needed.

I. PRELIMINARY REMARKS ON TOTALS

The notion of the "total", which is a generalization of the Burkill integral, was introduced by M. Cotlar and Y. Frenkel [4]. A special case of this was generalized by the author [22] for functions with values in a Banach algebra. These notions and the theorems proved in connection with them find an extensive application in the present paper.

In this Chapter we formulate some definitions and theorems relative to the totals in a form convenient for our treatment. We do not strive in the definitions for the most general versions and mention only the simplest theorems. To other theorems proved in the paper [22] we refer at the place of their application.

Let H be an abstract set and $\mathcal{K}$ a semi-ring of sets consisting of some subsets of H. This means the following:

Definition 1. [1] A class of sets $\mathcal{K}$ is called a semi-ring if for every pair of sets $A \in \mathcal{K}$, $B \in \mathcal{K}$ for which $A \subseteq B$ there is a finite system of sets $C_k \in \mathcal{K}$ $(k = 1, \dots, r)$ such that

$$B - A = \sum_{k=1}^{r} C_k,$$

[1]This notion of semi-rings is due to Á. Császár.

further with every $E \in \mathcal{K}$ and $F \in \mathcal{K}$ we have also $EF \in \mathcal{K}$.

The elements of $\mathcal{K}$ can obviously always be decomposed as sums of disjoint sets belonging to $\mathcal{K}$. If $A \in \mathcal{K}$ and

$$A = \sum_{k=1}^{r} A_k,$$

where $A_k \in \mathcal{K}$ $(k = 1, \ldots, r)$, $A_i A_k = 0$ for $i \neq k$, then the system of sets $A_1, \ldots, A_r$ will be called a *subdivision* of the set A. The subdivisions will be denoted by simple or signed letters $\mathfrak{z}$ as $\mathfrak{z} = \{A_1, \ldots, A_r\}$.

If $\mathfrak{z}' = \{A'_1, \ldots, A'_{r'}\}$ and $\mathfrak{z}'' = \{A''_1, \ldots, A''_{r''}\}$ are two subdivisions of the set $A \in \mathcal{K}$ and every set A''_k is a part of a set $A'_j \in \mathcal{K}$, then we write $\mathfrak{z}' \sqsubset \mathfrak{z}''$. Every "total" is in a closed connection with a partial ordering relation $<$ which orders (partially) the space of the subdivisions. This relation is supposed to be fulfilling the requirement that $\mathfrak{z}' \sqsubset \mathfrak{z}''$ implies $\mathfrak{z}' < \mathfrak{z}''$. In § 1 of Chapter II we consider the case when the relation $<$ reduces to the relation $\sqsubset$. If H is a metric space, then a possible partial ordering relation, which we denote by the sign $\prec$, is the following: $\mathfrak{z}' \prec \mathfrak{z}''$ if $\max_k d(A''_k) \leq \max_k d(A'_k)$.[2]

Let $f(A)$ $(A \in \mathcal{K})$ be a set function the values of which lie in a commutative Banach algebra $\mathcal{B}$. The additive total of this function over a set $A \in \mathcal{K}$ is defined by

DEFINITION 2. If there is a $g \in \mathcal{B}$ such that to every $\varepsilon > 0$ a subdivision $\mathfrak{z}_\varepsilon$ of the set A can be found with the property

$$\left\| \sum_{k=1}^{r} f(A_k) - g \right\| < \varepsilon \tag{1.1}$$

for every $\mathfrak{z} = \{A_1, \ldots, A_r\} > \mathfrak{z}_\varepsilon$, then the element g will be called the *additive total* of the function f over the set A and will be denoted by

$$\mathrm{S}_A f(dA).$$

For real-valued set functions (i.e. when $\mathcal{B}$ is the Banach algebra of the real numbers) we define the lower and upper totals as follows:

DEFINITION 3. Consider the sums

$$\sum_{k=1}^{r} f(A_k),$$

where $\mathfrak{z} = \{A_1, \ldots, A_r\}$ is a subdivision of the set $A \in \mathcal{K}$ and take the suprema and infima

$$\sup_{\mathfrak{z} : \mathfrak{z} > \mathfrak{z}'} \sum_{k=1}^{r} f(A_k), \qquad \inf_{\mathfrak{z} : \mathfrak{z} > \mathfrak{z}'} \sum_{k=1}^{r} f(A_k).$$

[2] $d(A)$ denotes the diameter of the set A, i.e. $d(A) = \sup_{h_1 \in A,\ h_2 \in A} d(h_1, h_2)$ where $d(h_1, h_2)$ is the distance between the points h_1, h_2.

The infimum of the suprema and the supremum of the infima – while $\mathfrak{z}'$ runs over the subdivisions of the set A – will be called *the upper and lower totals*, resp., of the function f over the set A. For these values we introduce the notations

$$\overline{\mathrm{S}}_a f(dA) \qquad \text{and} \qquad \underline{\mathrm{S}}_A f(dA),$$

respectively.

The following theorems can be proved by well-known arguments:

THEOREM 1.1. *There is at most one $g \in \mathcal{B}$ fulfilling the requirements in Definition 1.*

THEOREM 1.2. *The additive total of a function $f(B)$ $(B \in \mathcal{K})$ exists over the set $A \in \mathcal{K}$ if and only if for every $\varepsilon > 0$ there is a subdivision $\mathfrak{z}_\varepsilon = \{A_1, \ldots, A_r\}$ of the set A such that*

$$\left\| \sum_{k=1}^{r} f(A_k) - \sum_{k=1}^{r'} f(A'_k) \right\| < \varepsilon, \tag{1.2}$$

provided $\mathfrak{z}' = \{A'_1, \ldots, A'_{r'}\} > \mathfrak{z}_\varepsilon$.

THEOREM 1.3. *If $A = A_1 + A_2$ where $A_1 \in \mathcal{K}$, $A_2 \in \mathcal{K}$, $A \in \mathcal{K}$, $A_1A_2 = 0$ and the function $f(B)$ $(B \in \mathcal{K})$ is totalizable over the set A, then this holds also for the sets A_1, A_2 and*

$$\mathrm{S}_A f(dA) = \mathrm{S}_{A_1} f(dA) + \mathrm{S}_{A_2} f(dA). \tag{1.3}$$

THEOREM 1.4. *If $f(B)$ $(B \in \mathcal{K})$ is real-valued and subadditive, i.e.*

$$f(A_1 + A_2) \leq f(A_1) + f(A_2)$$

whenever $A_1 \in \mathcal{K}$, $A_2 \in \mathcal{K}$, $A_1 + A_2 \in \mathcal{K}$, then the total of f exists over every set $A \in \mathcal{K}$ where f has a finite variation and

$$\mathrm{S}_A f(dA) = \sup_{\{A_1, \ldots, A_r\}} \sum_{k=1}^{r} f(A_k), \tag{1.4}$$

where $\{A_1, \ldots, A_r\}$ runs over the subdivisions of the set A.

In the sequel we suppose that the Banach algebra $\mathcal{B}$ has a unity element which will be denoted by e. We introduce the notion of the multiplicative total.

DEFINITION 4. If there is a $g \in \mathcal{B}$ such that to every $\varepsilon > 0$ a subdivision $\mathfrak{z}_\varepsilon$ of the set $A \in \mathcal{K}$ can be found with the property

$$\left\| \prod_{k=1}^{r} f(A_k) - g \right\| < \varepsilon$$

for every $\mathfrak{z} = \{A_1, \ldots, A_r\} > \mathfrak{z}_\varepsilon$, then the element g will be called the *multiplicative total* of the function f over the set A and will be denoted by

$$\prod_A f(dA).$$

The following theorems are almost trivial:

THEOREM 1.5. *There is at most one $g \in \mathcal{B}$ fulfilling the requirements in Definition 4.*

THEOREM 1.6. *The multiplicative total of a function $f(B)$ ($f(B) \in \mathcal{B}$, $B \in \mathcal{K}$) exists over the set A if and only if for every ε there is a subdivision $\mathfrak{z}_\varepsilon = \{A_1, \dots, A_r\}$ of the set A such that*

$$\left\| \prod_{k=1}^{r} f(A_k) - \prod_{k=1}^{r'} f(A'_k) \right\| < \varepsilon,$$

provided $\mathfrak{z}' = \{A'_1, \dots, A'_{r'}\} > \mathfrak{z}_\varepsilon$.

In the present paper multiplicative totals are used only in the case when H is a compact metric space, further $\mathcal{K}$ has the additional properties:

α) If $h \in H$, then $\{h\} \in \mathcal{K}$.

β) If $A \in \mathcal{K}$, then for every $\varepsilon > 0$ there exists a subdivision $\mathfrak{z} = \{A_1, \dots, A_r\}$ of the set A such that

$$\max_k d(A_k) \le \varepsilon.$$

γ) Relation $<$ reduces to $\prec$.

If the (partial) ordering relation $<$ reduces to the relation $\prec$ and $\mathcal{K}$ is the semi-ring of the subintervals of an interval $[a, b]$ (permitting open, closed and semi-closed intervals equally), finally if $\mathcal{B}$ is the Banach algebra of the real numbers, then the additive total introduced by Definition 2 coincides with the Burkill integral.

II. THE PROBABILITY DISTRIBUTION OF A COMPLETELY ADDITIVE STOCHASTIC SET FUNCTION

§ 1. Atomless set functions

One proves in the theory of stochastic processes with independent increments that if some continuity conditions are fulfilled, e.g. if for every $\varepsilon > 0$

$$\mathbf{P}(|\xi_{s+\Delta s} - \xi_s| > \varepsilon) \to 0 \qquad \text{if} \qquad \Delta s \to 0$$

uniformly in s; then the differences $\xi_{s_2} - \xi_{s_1}$ are distributed according to an infinitely divisible probability distribution. This is the starting point of the considerations on other questions concerning the distributions of the process ξ_s. As the space H is abstract, an analogous continuity condition for stochastic set functions cannot be formulated. There is, however, a property which is at the same time of probabilistic and set-theoretic nature and which enables the proof of some theorems concerning the probability distributions of a completely additive stochastic set function $\xi(A)$: the atomlessness. Its definition is the following:

Definition 5. Let $\xi(B)$ be a completely additive set function defined on a σ-ring $\mathcal{S}$. A set $A \in \mathcal{S}$ will be called an *atom* relative to the set function ξ if for every $C \in A\mathcal{S}$ we have either $\xi(C) = 0$ or $\xi(C) = \xi(A)$.

Definition 6. The completely additive set function $\xi(B)$ $(B \in \mathcal{S})$ will be called *atomless* if for every atom A we have $\xi(A) = 0$.

The set function $\xi(B)$ $(B \in \mathcal{K})$ is atomless if and only if for every $A \in \mathcal{S}$ satisfying

$$\mathbf{P}(\xi(A) \neq 0) > 0$$

there exist sets $A_1 \in A\mathcal{S}$, $A_2 \in A\mathcal{S}$, $A_1A_2 = 0$ such that

$$\mathbf{P}(\xi(A_1) \neq 0) > 0, \qquad \mathbf{P}(\xi(A_2) \neq 0) > 0. \tag{2.1}$$

This shows that the values of an atomless set function in a set $A \in \mathcal{S}$ are not concentrated with probability 1, i.e. they are well-distributed in "every part" of the set A. If for every $A \in \mathcal{S}$ the random variable $\xi(A)$ is constant, then our definitions 5–6 reduce to the analogous definitions formulated for number-valued set functions.[3]

The following two theorems have a fundamental role in our discussion:

Theorem 2.1. *Let $\xi(A)$ be a completely additive set function defined on a σ-ring $\mathcal{S}$. If there is a $T > 0$ such that the measure $W(T, A)$ $(A \in \mathcal{S})$*[4] *is atomless, then the same holds for $\xi(A)$ $(A \in \mathcal{S})$.*

Conversely, if $\xi(A)$ $(A \in \mathcal{S})$ is atomless, then for every $T > 0$ the measure $W(T, A)$ $(A \in \mathcal{S})$ has also this property.

Proof. If for some $T > 0$ and $X \in \mathcal{S}$ we have $W(T, X) = 0$, then $\xi(X) = 0$ which proves the first assertion.

Let us consider the second statement. If $\xi(B)$ $(B \in \mathcal{S})$ is atomless and $A \in \mathcal{S}$ is an arbitrary set, then there can be found sets $A_1 \in \mathcal{S}$, $A_2 \in \mathcal{S}$ such that $A_1A_2 = 0$ and (2.1) holds. In this case for every T we have

$$W(T, A_1) > 0, \qquad W(T, A_2) > 0, \qquad \square$$

Theorem 2.2. *If $\xi(B)$ $(B \in \mathcal{S})$ is an atomless completely additive set function, then for every set $A \in \mathcal{S}$ the distribution of the random variables $\xi(A)$ is infinitely divisible.*

Proof. Let T be a fixed positive number. Since the measure $W(T, B)$ $(B \in \mathcal{S})$ is atomless, to every $\varepsilon > 0$ there is a subdivision of the set A into pairwise disjoint sets $B_1, \ldots, B_r$ belonging to $\mathcal{S}$ such that

$$W(T, B_k) \leq \varepsilon \qquad (k = 1, \ldots, r).$$

Hence

[3] Cf. [10], p. 48.

[4] See e.g. [20], p. 12.

$$\sup_{|t|\le T} |1 - f(t, B_k)| \le \varepsilon \qquad (k = 1, \dots, r). \tag{2.2}$$

It follows from the inequality

$$\frac{1}{2T}\int_{-T}^{T} |1 - f(t, B_k)|\, dt \ge \frac{1}{10}\mathbf{P}\left(|\xi(B_k)| > \frac{1}{T}\right)$$

(cf. [19], p. 220) that there exists a sequence of subdivisions $\left\{B_1^{(n)}, \dots, B_{k_n}^{(n)}\right\}$ of the set A such that the random variables in the double sequence

$$\xi\left(B_1^{(n)}\right), \dots, \xi\left(B_{k_n}^{(n)}\right) \qquad (n = 1, 2, \dots)$$

are infinitesimal (cf. [9], § 20). But for every n

$$\xi(A) = \sum_{k=1}^{k_n} \xi\left(B_k^{(n)}\right),$$

hence the distribution of the random variable $\xi(A)$ is a limiting distribution of sums of infinitesimal independent random variables. According to Theorem 2 of [9], § 24, this proves our statement. □

Let us consider the Lévi's canonical form of the characteristic function of $\xi(A)$:

$$\begin{aligned} \log f(t, A) &= i\gamma(A)t - \frac{\sigma^2(A)t^2}{2} \\ &+ \int_{(-\infty,0)} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dM(x, A) + \int_{(0,\infty)} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dN(x, A). \end{aligned} \tag{2.3}$$

A slightly modified form of this is the following:

$$\begin{aligned} \log f(t, A) &= i\gamma(\tau, A)t - \frac{\sigma^2(A)t^2}{2} \\ &+ \int_{(-\infty,-\tau)} (e^{itx} - 1)\, dM(x, A) + \int_{(\tau,\infty)} (e^{itx} - 1)\, dN(x, A) \\ &+ \int_{[-\tau,0)} (e^{itx} - 1 - itx)\, dM(x, A) + \int_{(0,\tau]} (e^{itx} - 1 - itx)\, dN(x, A), \end{aligned} \tag{2.4}$$

where τ is an arbitrary positive number.

The numbers $\gamma(A)$, $\sigma^2(A)$, $\gamma(\tau, A)$ are uniquely determined by the probability distribution of $\xi(A)$. The same holds for the functions $M(x, A)$, $N(x, A)$ if in the points of discontinuity we make some convention. We shall suppose $M(x, A)$ and $N(x, A)$ to be continuous to the left. From what has been said above and from the convergence theorems of infinitely divisible distributions it follows

Theorem 2.3. *Let $\xi(A)$ $(A \in \mathcal{S})$ be an atomless completely additive set function and τ a fixed positive number. Then the number-valued set functions $\gamma(A)$, $\gamma(\tau, A)$ are completely additive, and $\sigma^2(A)$, $M(x, A)$ (for fixed $x < 0$), $-N(x, A)$ (for fixed $x > 0$) are finite measures on the σ-ring $\mathcal{S}$. Each of these number-valued set functions is atomless.*

PROOF. The preceding set functions are obviously additive. Let $B_1, B_2, \ldots$ be a non-increasing sequence of sets of $\mathcal{S}$ for which $\prod_{k=1}^{\infty} B_k = 0$. Since

$$f(t, B_k) \Rightarrow 1 \qquad \text{if} \qquad k \to \infty,$$

it follows that

$$\gamma(B_k) \to 0, \qquad \sigma^2(B_k) \to 0,$$
$$M(x, B_k) \to 0 \quad (x < 0), \quad N(x, B_k) \to 0 \quad (x > 0)$$

if $k \to \infty$.

As for the sequence $\gamma(\tau, B_k)$, we proceed in the following way. The functions $M(x, B_k)$, $N(x, B_k)$ $(k = 1, 2, \ldots)$ are monotone, hence the set of their points of discontinuity is countable. Thus there exists a $\tau_1 > 0$ so that $\tau_1 < \tau$, moreover the points $-\tau_1$ and τ_1 are points of continuity of the functions $M(x, B_k)$ $(k = 1, 2, \ldots)$ and $N(x, B_k)$ $(k = 1, 2, \ldots)$, respectively. But we know that $\lim_{k\to\infty} \gamma(\tau_1, B_k) = 0$ (cf. [9], § 19, Theorem 2), hence the relation

$$\gamma(\tau, B_k) = \gamma(\tau_1, B_k) - \int_{\tau_1 < x \le \tau} x \, dN(x, B_k) - \int_{-\tau \le x < -\tau_1} x \, dM(x, B_k)$$

proves the statement. □

We prove finally that the mentioned set functions are atomless. Let $A \in \mathcal{S}$ and choose a sequence of subdivisions $\mathfrak{z}_n = \left\{ B_1^{(n)}, \ldots, B_{k_n}^{(n)} \right\}$ for which

$$W\left(T, B_k^{(n)}\right) \le \frac{1}{n} \qquad (k = 1, \ldots, k_n;\ n = 1, 2, \ldots). \tag{2.5}$$

If one of the above set functions had an atom, then in view of (2.5) it would be a contradiction. In fact, (2.5) implies that for any sequence $k^{(n)}$ $(1 \le k^{(n)} \le k_n)$

$$\xi\left(B_{k^{(n)}}^{(n)}\right) \Rightarrow 0 \qquad \text{if} \qquad n \to \infty$$

and hence

$$\gamma\left(B_{k^{(n)}}^{(n)}\right) \to 0, \qquad \sigma^2\left(B_{k^{(n)}}^{(n)}\right) \to 0, \qquad \gamma\left(\tau, B_{k^{(n)}}^{(n)}\right) \to 0 \qquad (\text{for } \tau > 0),$$
$$M\left(x, B_{k^{(n)}}^{(n)}\right) \to 0 \qquad (\text{for } x < 0), \qquad N\left(x, B_{k^{(n)}}^{(n)}\right) \to 0 \qquad (\text{for } x > 0)$$

if $n \to \infty$. Thus Theorem 2.3 is proved.

In the following theorem we establish some connections between the above set functions and the distributions $F(x, A)$ $(A \in \mathcal{S})$. The totals in this § are taken relative to the (partial) ordering relation $\sqsubset$.

THEOREM 2.4. *If $\xi(B)$ $(B \in \mathcal{S})$ is an atomless completely additive set function, then for every $A \in \mathcal{S}$ we have the following relations:*

$$\log f(t, A) = \mathsf{S}_A(f(t, dB) - 1), \tag{2.6}$$

$$\begin{cases} M(x, A) = \mathsf{S}_A F(x, dB) & (x < 0), \\ N(x, A) = \mathsf{S}_A(F(x, dB) - 1) & (x > 0) \end{cases} \tag{2.7}$$

for every x where the functions on the left-hand side are continuous. If $\tau > 0$ is a number such that $M(x,A)$ is continuous at $-\tau$ and $N(x,A)$ at τ, then

$$\gamma(\tau, A) = \mathrm{S}_A \alpha(\tau, dB) \qquad \textit{where} \qquad \alpha(\tau, B) = \int_{|x|<\tau} x\, dF(x,B), \tag{2.8}$$

finally

$$\sigma^2(A) = \lim_{\varepsilon\to 0} \overline{\mathrm{S}}_A \beta(\varepsilon, dB) = \lim_{\varepsilon\to 0} \underline{\mathrm{S}}_A \beta(\varepsilon, dB) \tag{2.9}$$

where

$$\beta(\varepsilon, B) = \int_{|x|<\varepsilon} x^2\, dF(x,B) - \left(\int_{|x|<\varepsilon} x\, dF(x,B) \right)^2 .$$

Proof. First we prove the relation (2.6). Let $\mathfrak{z} = \{B_1, \dots, B_r\}$ be a subdivision of the set $A \in \mathcal{S}$ for which (2.2) holds $\left(0 < \varepsilon < \frac{1}{2}\right)$. Hence we get

$$\begin{aligned} &\left| \log f(t,A) - \sum_{k=1}^{r} (f(t,B_k) - 1) \right| \\ &\leq \sum_{k=1}^{r} |\log f(t,B_k) - (f(t,B_k)-1)| \leq \sum_{k=1}^{r} |f(t,B_k) - 1|^2 \leq W(T,A)\varepsilon. \end{aligned} \tag{2.10}$$

If we substitute $\mathfrak{z}' \sqsupset \mathfrak{z}$ for $\mathfrak{z}$, then (2.10) remains true, hence (2.6) is proved. It can be seen from (2.10) that the convergence to the total is uniform in every finite t-interval.

Now, we consider the relations (2.7)–(2.8). Let $\mathfrak{z}_1 \sqsubset \mathfrak{z}_2 \sqsubset \dots$ be a sequence of subdivisions of the set A such that

$$\log f(t,A) = \lim_{n\to\infty} \sum_{k=1}^{k_n} \left(f\left(t, B_k^{(n)}\right) - 1 \right), \tag{2.11}$$

where $\mathfrak{z}_n = \left\{ B_1^{(n)}, \dots, B_{k_n}^{(n)} \right\}$. Suppose that either (2.7) or (2.8) does not hold. Let this be e.g. the first row of (2.7), the others can be treated similarly. By condition there exists an $x < 0$, an $\varepsilon_0 > 0$ and for every n a subdivision $\mathfrak{z}'_n = \left\{ C_1^{(n)}, \dots, C_{S_n}^{(n)} \right\} \sqsupset \mathfrak{z}_n$ such that $M(x,A)$ is continuous at the point x and

$$\left| M(x,A) - \sum_{k=1}^{S_n} F\left(x, C_k^{(n)}\right) \right| \geq \varepsilon_0. \tag{2.12}$$

On account of (2.11) we get

$$\log f(t,A) = \lim_{n\to\infty} \sum_{k=1}^{S_n} \left(f\left(t, C_k^{(n)}\right) - 1 \right),$$

or otherwise expressed

$$\exp \sum_{k=1}^{S_n} \left(f \left(t, C_k^{(n)} \right) - 1 \right) \Rightarrow f(t, A) \qquad \text{if} \qquad n \to \infty. \tag{2.13}$$

On the left-hand side of (2.13) there stands a sequence of infinitely divisible characteristic functions. To the members of this sequence in LÉVI's formula correspond the following functions:

$$M_n(x) = \sum_{k=1}^{S_n} F \left(x, C_k^{(n)} \right). \tag{2.14}$$

Relation (2.13) implies that at every point of continuity of $M(x, A)$

$$\lim_{n \to \infty} M_n(x) = M(x, A)$$

which contradicts (2.12).

Concerning the proof of relation (2.8) we remark that in this case we have to use LÉVY's formula in the form (2.4).

Finally we prove (2.9). Let us construct a sequence of subdivisions $\mathfrak{z}_n = \left\{ B_1^{(n)}, \dots, B_{k_n}^{(n)} \right\}$ for the members of which (2.2) holds and

$$\lim_{n \to \infty} \sum_{k=1}^{k_n} \beta \left(\varepsilon, B_k^{(n)} \right) = \overline{\mathrm{S}}_A \beta(\varepsilon, dB). \tag{2.15}$$

By Theorem 1 of [9], § 22 we have

$$\lim_{\varepsilon \to 0} \overline{\mathrm{S}}_A \beta(\varepsilon, dB) = \sigma^2(A).$$

A similar argument shows the statement relative to the lower total. Thus our theorem is proved. □

§ 2. The probability distribution of a completely additive set function in case of a metric space

In this § we suppose that H is a compact metric space and $\mathcal{K}$ is a semi-ring of some subsets of H satisfying Conditions α), β), γ) on. p. 4. We do not suppose that the stochastic set function $\xi(A)$ is defined on a σ-ring but only that it is defined on every element of $\mathcal{K}$. We assume that $\xi(A)$ is completely additive on $\mathcal{K}$, i.e. besides the independence property[5]

$$\xi(A) = \sum_{k=1}^{\infty} \xi(A_k)$$

[5] To disjoint sets there belong independent random variables.

whenever $A_k \in \mathcal{K}$ $(k = 1, 2, \ldots)$, $A_i A_k = 0$ for $i \neq k$ and $A = \sum_{k=1}^{\infty} A_k \in \mathcal{K}$.

Before turning to the considerations on distributions we prove an auxiliary theorem.

THEOREM 2.5. *If $\xi(A)$ $(A \in \mathcal{K})$ is a completely additive set function defined on the semi-ring $\mathcal{K}$, then there exists a completely additive set function $\overline{\xi}(A)$ defined on $\mathcal{R} = \mathcal{R}(\mathcal{K})$*[6] *for which*

$$\overline{\xi}(A) = \xi(A) \qquad \textit{if} \qquad A \in \mathcal{K}.$$

If $\xi(A)$ $(A \in \mathcal{K})$ satisfies an extension condition formulated in [19], *Chapter III (substituting $\mathcal{K}$ for $\mathcal{R}$), then $\overline{\xi}(A)$ $(A \in \mathcal{R}(\mathcal{K}))$ does also, i.e. the set function $\overline{\xi}$ can be extended to $\mathcal{S}(\mathcal{R})$.*

PROOF. The ring $\mathcal{R}(\mathcal{K})$ consists of finite sums of sets belonging to $\mathcal{K}$. If $A_1, \ldots, A_r$ are disjoint sets of $\mathcal{K}$ and $A = \sum_{k=1}^{r} A_k$, then put

$$\overline{\xi}(A) = \sum_{k=1}^{r} \xi(A_k).$$

It is easy to see that the definition of the set function $\overline{\xi}$ is unique and it is completely additive on $\mathcal{R}$.

We prove the second assertion by the aid of Theorem 3.3 of [19]. If the condition of this theorem holds for $\xi(A)$ $(A \in \mathcal{K})$, then it is obviously holds also for $\overline{\xi}(A)$ $(A \in \mathcal{R})$. On the other hand, if some extension condition holds for $\xi(A)$ $(A \in \mathcal{K})$, then – as it is very easy to see in every special case – the condition of Theorem 3.3 is satisfied too. Thus $\overline{\xi}$ can be extended to $\mathcal{S}(\mathcal{R})$ which implies the fulfilment of all extension conditions for $\overline{\xi}$. □

The fact that for $\xi(A)$ $(A \in \mathcal{K})$ one of the extension conditions of [19], Chapter III holds, will be mentioned simply as follows: $\xi(A)$ $(A \in \mathcal{K})$ can be extended to $\mathcal{S}(\mathcal{R}) = \mathcal{S}(\mathcal{R}(\mathcal{K}))$. In the following theorems the totals are taken with respect to the (partial) ordering relation $\prec$.

THEOREM 2.6. *Let $\xi(A)$ $(A \in \mathcal{K})$ be a completely additive set function and suppose that it can be extended to $\mathcal{S}(\mathcal{R})$. In this case for every $A \in \mathcal{K}$ the following totals exist:*

$$\log g(t, A) = \mathrm{S}_A (f(t, dB) - 1) \tag{2.16}$$

where t is an arbitrary but fixed real number;

$$M(x, A) = \mathrm{S}_A F(x, dB) \qquad (x < 0), \tag{2.17}$$

$$N(x, A) = \mathrm{S}_A (F(x, dB) - 1) \qquad (x > 0) \tag{2.18}$$

except at most a countable x-set and

$$\gamma(\tau, A) = \mathrm{S}_A \alpha(\tau, dB) \qquad \textit{where} \qquad \alpha(\tau, B) = \int_{|x|<\tau} x \, dF(x, B). \tag{2.19}$$

[6] $\mathcal{R}(\mathcal{K})$ denotes the smallest ring containing the semi-ring $\mathcal{K}$.

In (2.19) τ *is a positive number with the property that* $M(x,A)$ *and* $N(x,A)$ *are continuous at* $-\tau$ *and* τ*, respectively.*

The sequence approximating the total (2.16) *converges to its limit uniformly in every finite* t*-interval.*

PROOF. Let $\mathcal{B}_T$ denote the Banach algebra of the continuous complex-valued functions defined in the interval $[-T,T]$. Considering the functions $f(t,B)$ $(B \in \mathcal{K})$ only for $|t| \le T$, we get elements of $\mathcal{B}_T$. By condition $\xi(A)$ $(A \in \mathcal{K})$ can be extended to $\mathcal{S}(\mathcal{R})$. To the extended set function there corresponds a measure $W(T,A)$ $(A \in \mathcal{S}(\mathcal{R}))$. If

$$\delta(T,C) = \sup_{|t|\le T} |1 - f(t,C)|,$$

then obviously

$$\mathrm{Var}_\delta(B) \le W(T,B), \tag{2.20}$$

hence the set function $f(t,C)-1$ $(|t| \le T,\ C \in \mathcal{K})$, the values of which are taken from $\mathcal{B}_T$, is of bounded variation and v-continuous.[7] Hence by Theorem 1 of [22] the total (2.16) exists for every t satisfying $-T \le t \le T$ and the convergence is uniform in the interval $[-T,T]$. Since T was an arbitrary positive number, the statement concerning the totals (2.16) follows.

Let $\mathfrak{z}_n = \left\{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\right\}$ be a sequence of subdivisions of the set A such that $\mathfrak{z}_n \prec \mathfrak{z}_{n+1}$ and $\max_k d\left(B_k^{(n)}\right) \to 0$ if $n \to \infty$. If

$$g_n(t,A) = \prod_{k=1}^{k_n} \exp\left(f\left(t, B_k^{(n)}\right) - 1\right), \tag{2.21}$$

then, by the precedings,

$$g_n(t,A) \Rightarrow g(t,A) = \exp \mathrm{S}_A(f(t,dB) - 1). \tag{2.22}$$

The characteristic functions $g_n(t,A)$ are infinitely divisible. In LÉVI's canonical form (of the type (2.4)) in the place of $M(x)$, $N(x)$ and $\gamma(\tau)$ there stand the following functions:

$$\sum_{k=1}^{k_n} F\left(x, B_k^{(n)}\right), \qquad \sum_{k=1}^{k_n} \left(F\left(x, B_k^{(n)}\right) - 1\right) \tag{2.23}$$

and the constant

$$\sum_{k=1}^{k_n} \int_{|x|<\tau} x\, dF\left(x, B_k^{(n)}\right), \tag{2.24}$$

respectively. The well-known convergence theorems relative to infinitely divisible distributions (cf. [9] , § 19) imply our statements. □

[7] Cf. [22], p. 109, Definition 5.

Remark 1. In the same way as we have proved the relevant assertion of Theorem 2.3, we can prove also here that

$$\sigma^2(A) = \lim_{\varepsilon\to 0} \overline{\mathrm{S}}_A \beta(\varepsilon, dB) = \lim_{\varepsilon\to 0} \underline{\mathrm{S}}_A \beta(\varepsilon, dB), \tag{2.25}$$

where

$$\beta(\varepsilon, B) = \int_{|x|<\varepsilon} x^2\, dF(x,B) - \left(\int_{|x|<\varepsilon} x\, dF(x,B)\right)^2 .$$

Remark 2. If we consider the functions $g(t,B)$, $f(t,B)$ $(B \in \mathcal{K})$ only in the interval $[-T,T]$, then by Theorem 1 of [22]

$$\mathrm{Var}_g(B) \le \mathrm{Var}_{f-1}(B) \qquad (B \in \mathcal{K}) \tag{2.26}$$

whence

$$\sup_{|t|\le T} |\log g(t,B)| \le W(T,B) \qquad (B \in \mathcal{K}), \tag{2.27}$$

where $W(T,B)$ is the same measure as in the proof of Theorem 2.6. Hence by simple arguments follows that for every $\tau > 0$, $\gamma(\tau, A)$ is a bounded, completely additive number-valued set function and $\sigma^2(A)$, $M(x,A)$ $(x<0)$, $-N(x,A)$ $(x>0)$ are bounded measures on $\mathcal{K}$.

Theorem 2.7. *Let us consider the infinitely divisible characteristic function*

$$\begin{aligned} g(t,A) = \exp\Big\{ & i\gamma(\tau,A)t - \frac{\sigma^2(A)t^2}{2} \\ & + \int_{(-\infty,-\tau)} (e^{itx}-1)\, dM(x,A) + \int_{(\tau,\infty)} (e^{itx}-1)\, dN(x,A) \\ & + \int_{[-\tau,0)} (e^{itx}-1-itx)\, dM(x,A) + \int_{(0,\tau]} (e^{itx}-1-itx)\, dN(x,A) \Big\}, \end{aligned} \tag{2.28}$$

where $M(x,A)$, $N(x,A)$, $\gamma(\tau,A)$ $\sigma^2(A)$ are defined by formulae (2.17)–(2.19) *and* (2.25). *Then for every $B \in \mathcal{K}$ and every t we have*

$$f(t,B) = \prod\nolimits_B (1 + \log g(t, dA)) \tag{2.29}$$

and the convergence to the total is uniform in every finite t-interval.

Proof. This theorem follows immediately from formula (2.26) and from Theorem 3 of [22].

Formula (2.29) gives the general form of the probability distributions of the random variables $\xi(A)$ $(A \in \mathcal{K})$.

According to Theorems 3–4 of [22], the correspondence between the set of distributions $\{F(x,A),\ A \in \mathcal{K}\}$ and the set of real-valued set functions $\{M(x,A)$ $(x<0)$, $N(x,A)$ $(x>0)$, $\gamma(\tau,A)$ $(\tau>0)$, $\sigma^2(A)$, $A \in \mathcal{K}\}$ is one-to-one. □

§ 3. The probability distributions in case of a metric space and a weak continuity

In the beginning of this Chapter we have mentioned that in general a weak continuity cannot be formulated for abstract stochastic set functions. It is, however, possible if H is a metric space.

We suppose in this § (such as in § 2) that H is a compact metric space and $\mathcal{K}$ a semi-ring of some subsets of H satisfying Conditions α), β) and γ) on p. 378. We make here a little digression by omitting the condition of complete additivity and supposing only additivity for the set function $\xi(A)$ $(A \in \mathcal{K})$. A remarkable special case of this type of stochastic set functions is that generated by the differences of a stochastic process with independent increments ξ_s. In this case H is an interval $[a, b]$, $\mathcal{K}$ is the semi-ring of all subintervals of $[a, b]$ (permitting closed, open, semi-closed intervals equally) and $\xi(A)$ equals $\xi_{s_2} - \xi_{s_1}$ if $a \leq s_1 \leq s_2 \leq b$ and A equals one of the intervals (s_1, s_2), $[s_1, s_2)$, $(s_1, s_2]$, $[s_1, s_2]$. Our notion of weak continuity (given by Definition 6) reduces in this case to the classical one.

Definition 7. An additive set function $\xi(A)$ defined on the semi-ring $\mathcal{K}$ is said to be weakly continuous if for every sequence of sets $A_1, A_2, \ldots$ satisfying $\lim_{n\to\infty} d(A_n) = 0$ we have

$$\xi(A_n) \Rightarrow 0 \qquad \text{if} \qquad n \to \infty. \tag{2.30}$$

In the following theorems the totals are taken with respect to the (partial) ordering relation $\prec$.

Theorem 2.8. *Let $\xi(A)$ be an additive and weakly continuous set function defined on the semi-ring $\mathcal{K}$. In this case every random variable $\xi(A)$ $(A \in \mathcal{K})$ depends on an infinitely divisible distribution and in Lévy's canonical form the functions and constants are the followings:*

$$M(x, A) = \mathrm{S}_A F(x, dB) \qquad (x < 0), \tag{2.31}$$

$$N(X, A) = \mathrm{S}_A (F(x, dB) - 1) \qquad (x > 0), \tag{2.32}$$

$$\gamma(\tau, A) = \mathrm{S}_A \alpha(\tau, dB) \qquad \textit{where} \qquad \alpha(\tau, B) = \int_{|x|<\tau} x\, dF(x, B), \tag{2.33}$$

finally

$$\sigma^2(A) = \lim_{\varepsilon\to 0} \overline{\mathrm{S}}_A \beta(\varepsilon, dB) = \lim_{\varepsilon\to 0} \underline{\mathrm{S}}_A \beta(\varepsilon, dB), \tag{2.34}$$

where

$$\beta(\varepsilon, B) = \int_{|x|<\varepsilon} x^2\, dF(x, B) - \left(\int_{|x|<\varepsilon} x\, dF(x, B) \right)^2.$$

Relations (2.31) *and* (2.32) *hold at the points of continuity of the functions $M(x, A)$ and $N(x, A)$, respectively. τ denotes a positive number which has the property that $M(x, A)$ and $N(x, A)$ are continuous at $-\tau$ and τ, respectively.*

PROOF. Let $\mathfrak{z}_n = \left\{B_1^{(n)}, \dots, B_{k_n}^{(n)}\right\}$ be a sequence of subdivisions of the set A such that $\mathfrak{z}_n \prec \mathfrak{z}_{n+1}$ and $\lim_{n\to\infty} \max_k d\left(B_k^{(n)}\right) = 0$. The weak continuity of $\xi(B)$ $(B \in \mathcal{K})$ implies that in the double sequence

$$\xi\left(B_1^{(n)}\right), \dots, \xi\left(B_{k_n}^{(n)}\right) \tag{2.35}$$

there stand infinitesimal random variables (cf. [9], § 20). Since for every n

$$\xi(A) = \sum_{k=1}^{k_n} \xi\left(B_k^{(n)}\right),$$

it follows that (cf. [9], § 24, Theorem 2) the probability distribution of $\xi(A)$ is infinitely divisible. The other part of the theorem follows from Theorem 4 of [9], § 25. □

Contrary to Theorem 2.6 of the preceding §, in this case the total

$$\mathrm{S}_A(f(t, dB) - 1) \tag{$*$}$$

does not exist in general. But it can be proved that

$$\log f(t, A) = \mathrm{S}_A(f_1(t, dB) - 1),$$

where

$$f_1(t, B) = f(t, B)e^{-i\alpha(\tau, B)t}, \qquad \alpha(\tau, B) = \int_{|x|<\tau} x\, dF(x, B)$$

(cf. the proof of Theorem 1 of [9], § 24).

If $\xi(A)$ is a set function generated by the differences of a stochastic process with independent increments ξ_s, then the totals in Theorem 2.8 reduce to Burkill integrals. For such set functions $\xi(A)$ simple examples can be given where the total $(*)$ does not exist. We have only to put $\xi_s \equiv \varphi(s)$ where $\varphi(s)$ is a conveniently chosen continuous real function of unbounded variation.

The k-th moment and the k-th semi-invariant of a random variable $\xi(A)$ will in the sequel be denoted by $M_k(A)$ and $x_k(A)$, respectively:

$$M_k(A) = \int_{-\infty}^{\infty} x^k\, dF(x, A), \qquad x_k(A) = \frac{1}{i^k}\frac{d^k}{dt^k} \log f(t, A)\bigg|_{t=0},$$

provided that the integral on the right-hand side of the first relation exists. We prove a theorem concerning their connection. □

THEOREM 2.9. *Let $\xi(A)$ be an additive set function defined on the semi-ring $\mathcal{K}$. Suppose that for every $A \in \mathcal{K}$ the moment $M_k(A)$ exists and the following two conditions hold:*

1. *The set functions $M_i(A)$ $(i = 1, \dots, k-1;\ A \in \mathcal{K})$ are of bounded variation.*

2. *If $B_1, B_2, \ldots$ is a sequence of sets of $\mathcal{K}$ such that $\lim_{n\to\infty} d(B_n) = 0$, then*

$$\lim_{n\to\infty} M_i(B_n) = 0 \qquad (i = 1, \ldots, k-1).$$

In this case for every $A \in \mathcal{K}$ we have

$$x_k(A) = \mathrm{S}_A M_k(dB). \tag{2.36}$$

Proof. It can easily be verified that $x_k(B)$ equals the sum of $M_k(B)$ and a finite number of products of the form $M_i(B)\ M_j(B)$ $(1 \le i \le k-1,\ 1 \le j \le k-1)$. Hence the theorem follows by a simple argument.

We remark that the results of this chapter contain implicitly the solution of the functional equations

$$F(x, A_1 + A_2) = F(x, A_1) * F(x, A_2)$$

and

$$F\left(x, \sum_{k=1}^{\infty} A_k\right) = \mathop{*}_{k=1}^{\infty} F(x, A_k),$$

accordingly as $\xi(A)$ is only additive or also completely additive. The sets $A_1, A_2, \ldots, A_1 + A_2$ and $\sum_{k=1}^{\infty} A_k$ belong to $\mathcal{K}$ or $\mathcal{S}$ according to the problem. The properties of the set function $\xi(A)$ used in this chapter can be formulated in terms of the distribution functions. Thus we need not at all random variables, it is quite sufficient to consider the distributions and the solutions of the above functional equations follow from the modified form of our theorems.

III. CONSIDERATIONS ON THE REALIZATIONS

§ 1. Preliminaries

In this chapter we assume that the space H and the σ-ring $\mathcal{S}$ consisting of some subsets of H satisfy the following condition:

α_1) Every set $A \in \mathcal{S}$ has a sequence of subdivisions $\mathfrak{z}_n = \left\{B_1^{(n)}, \ldots, B_{k_n}^{(n)}\right\}$ with the property that $\mathfrak{z}_n \sqsubset \mathfrak{z}_{n+1}$ and if $h_1 \in H$, $h_2 \in H$, $h_1 \ne h_2$, then for some positive integers i, k, N we have

$$h_1 \in B_i^{(N)}, \qquad h_2 \in B_k^{(n)}, \qquad i \ne k. \tag{3.1}$$

This condition holds e.g. for countable-dimensional Euclidean spaces.

We start from a completely additive set function $\xi(A) = \xi(\omega, A)$ $(\omega \in \Omega)$ which is supposed to be defined on the σ-ring $\mathcal{S}$. If ω is fixed, we get a number-valued set function. This will be called a *realization* of the set function ξ. The complete additivity of ξ does not imply in general that of the realizations as it was shown in [19] (Chapter III, § 1). But in this chapter we need the complete additivity of the realizations. Therefore we introduce the condition:

β_1) There exists a set $\Omega_1 \subseteq \Omega$ with $\mathbf{P}(\Omega_1) = 1$ such that for every fixed $\omega \in \Omega_1$ the number-valued set function $\xi(\omega, A)$ $(A \in \mathcal{S})$ is completely additive.

Besides α_1) and β_1) we suppose the condition:

γ_1) The stochastic set function $\xi(A)$ $(A \in \mathcal{S})$ is atomless.

Condition α_1) implies that if $h \in H$, then $\{h\} \in \mathcal{S}$. In fact, every set $\{h\}$ can be obtained as a limit of a non-decreasing sequence of sets of $\mathcal{S}$. Thus every realization can be decomposed as a sum of a continuous and a purely discontinuous part. We shall show in § 3 that under general conditions the continuous part must be identically a constant number-valued set function with probability 1. This is perhaps the most interesting result of this chapter.

§ 2. Qualitative discussion of the discontinuities

If μ is a completely additive set function defined on the σ-ring $\mathcal{S}$ and for an $h \in H$ we have $\mu(\{h\}) \neq 0$, then we call h a *discontinuity point* relative to μ. The number $\mu(\{h\})$ will be called the *magnitude of the discontinuity.*

Let I be a one-dimensional interval with a positive distance from the point 0 and define the functions $\chi_0(I, A)$, $\chi_1(I, A)$ of the sample elements as follows: $\chi_0(I, A)$ is the number and $\chi_1(I, A)$ the sum of those discontinuities in the set A the magnitudes of which lie in the interval I. First of all we prove a theorem relative to this functions.

Theorem 3.1. *If $I_1, \dots, I_r$ are disjoint intervals with positive distances from 0, then $\chi_0(I_1, A), \dots, \chi_0(I_r, A)$ and similarly $\chi_1(I_1, A), \dots, \chi_1(I_r, A)$ are independent random variables.*

In the general case when $I_1, \dots, I_r$ are arbitrary disjoint intervals, the assertion remains true for $\chi_1(I_1, A), \dots, \chi_1(I_r, A)$ and holds also for $\chi_0(I_1, A), \dots, \chi_0(I_r, A)$ if the realizations belonging to Ω_1 have a finite number of discontinuities.

Proof. Here and in the sequel we shall frequently use the following remark:

There exists a sequence of subdivisions $\mathfrak{z}_n = \left\{B_1^{(n)}, \dots, B_{k_n}^{(n)}\right\}$ of the set A such that $\mathfrak{z}_n \sqsubset \mathfrak{z}_{n+1}$, Condition α_1) holds and finally the random variables

$$\xi\left(B_1^{(n)}\right), \dots, \xi\left(B_{k_n}^{(n)}\right) \qquad (n = 1, 2, \dots) \tag{3.2}$$

are infinitesimal. In fact, the proof of Theorem 3.2 shows that the last property can be fulfilled with a sequence $\mathfrak{z}_n'$. Then, if $\mathfrak{z}_n''$ satisfies α_1), the superposition of $\mathfrak{z}_n'$ and $\mathfrak{z}_n''$ satisfies both requirements.

First we consider the case when the intervals I_k have positive distances from 0. Let us define the functions

$$f_k^{(0)}(x) = \begin{cases} 1 & \text{if} \quad x \in I_k, \\ 0 & \text{if} \quad x \notin I_k, \end{cases} \tag{3.3}$$

$$f_k^{(1)}(x) = \begin{cases} x & \text{if} \quad x \in I_k \\ 0 & \text{if} \quad x \notin I_k \end{cases} \qquad (k = 1, \dots, r), \tag{3.4}$$

and consider the sums

$$\chi_{0s}^{(n)} = \sum_{k=1}^{k_n} f_s^{(0)}\left(\xi\left(B_k^{(n)}\right)\right), \qquad \chi_{1s}^{(n)} = \sum_{k=1}^{k_n} f_s^{(1)}\left(\xi\left(B_k^{(n)}\right)\right), \qquad (s = 1, \dots, r). \tag{3.5}$$

Condition α_1) implies

$$\lim_{n\to\infty} \chi_{0s}^{(n)} = \chi_0(I_s, A), \qquad \lim_{n\to\infty} \chi_{1s}^{(n)} = \chi_1(I_s, A), \tag{3.6}$$

hence $\chi_0(I_s, A)$, $\chi_1(I_s, A)$ $(s = 1, \dots, r)$ are random variables.

The assertion relative to the independence will be proved by the aid of Theorem 1b of [21]. Let us consider the double sequence (3.4) of independent random variables and apply the functions $f_1^{(0)}(x), \dots, f_r^{(0)}(x)$ (and $f_1^{(1)}(x), \dots, f_r^{(1)}(x)$, resp.) to its members. Clearly

$$f_i^{(0)}(x) f_k^{(0)}(x) \equiv 0 \qquad \left(f_i^{(1)}(x) f_k^{(1)}(x) \equiv 0\right) \qquad \text{if} \qquad i \neq k.$$

Now we shall verify the fulfilment of the conditions of Theorem 1b of [21]. Relation (3.6) implies d). Condition b) follows immediately by choosing τ so that $\tau < \delta$, where δ is the minimal distance of the intervals I_s $(s = 1, \dots, r)$ from the point 0. As for Condition c), there exists a $K > 0$ such that

$$f_s^{(0)}(x) \leq K|x| \qquad\qquad (s = 1, \dots, r),$$

moreover clearly

$$f_s^{(1)}(x) \leq |x| \qquad\qquad (s = 1, \dots, r),$$

hence for every s $(1 \leq s \leq r)$ and $\varepsilon > 0$ we have

$$\sup_{1\leq k\leq k_n} \mathbf{P}\left(f_s^{(0)}\left(\xi\left(B_k^{(n)}\right)\right) > \varepsilon\right) \leq \sup_{1\leq k\leq k_n} \mathbf{P}\left(\left|\xi\left(B_k^{(n)}\right)\right| > \frac{\varepsilon}{K}\right) \to 0,$$

$$\sup_{1\leq k\leq k_n} \mathbf{P}\left(f_s^{(1)}\left(\xi\left(B_k^{(n)}\right)\right) > \varepsilon\right) \leq \sup_{1\leq k\leq k_n} \mathbf{P}\left(\left|\xi\left(B_k^{(n)}\right)\right| > \varepsilon\right) \to 0$$

if $n \to \infty$.

The remaining part of the theorem can be proved by the aid of the precedings by substituting for those intervals I_s which have the point 0 inside or as a limit point, intervals having a distance ε from 0 and then taking the limit $\varepsilon \to 0$. This completes the proof. □

Let I be an interval with a positive distance from 0. From the proof of Theorem 3.1 it is clear that if $A_1, \dots, A_m$ are disjoint sets of $\mathcal{S}$, then the random variables $\chi_0(I, A_1), \dots,$

$\chi_0(I, A_m)$ (and $\chi_1(I, A_1), \ldots, \chi_1(I, A_m)$, resp.) are independent (cf. the relations (3.5) and (3.6)). If $A_1, A_2, \ldots$ is a sequence of disjoint sets of $\mathcal{S}$, then in view of β_1)

$$\chi_0(I, A) = \sum_{k=1}^{\infty} \chi_0(I, A_k), \tag{3.7}$$

$$\chi_1(I, A) = \sum_{k=1}^{\infty} \chi_1(I, A_k), \tag{3.8}$$

where $A = \sum_{k=1}^{\infty} A_k$. The relations (3.7) and (3.8) hold not only with probability 1 but they are satisfied in the ordinary sense if $\omega \in \Omega_1$.

Thus for fixed I, $\chi_0(I, A)$ and $\chi_1(I, A)$ are completely additive (stochastic) set functions having completely additive realizations with probability **1**.

The same holds for $\chi_1(I, A)$ without any restriction on the interval I and also for $\chi_0(I, A)$ if almost all realizations have a finite number of discontinuities. We formulate the foregoing statements in the form of a theorem.

THEOREM 3.2. *For every fixed interval I, $\chi_1(I, A)$ $(A \in \mathcal{S})$ is a completely additive set function.*

$\chi_0(I, A)$ $(A \in \mathcal{S})$ is also a completely additive set function if I has a positive distance from 0 or almost all realizations of the set function $\xi(A)$ $(A \in \mathcal{S})$ have a finite number of discontinuities.

If $\omega \in \Omega_1$ is fixed, then the corresponding sample function of $\chi_1(I, A)$ (and under the above condition that of $\chi_0(I, A)$) is completely additive.

§ 3. The probability distributions of the random variables $\xi(A)$, $\chi_0(I, A)$, $\chi_1(I, A)$

Now we fix the set A, the interval I and determine the probability distributions of the random variables $\xi(A)$, $\chi_i(I, A)$ $(i = 1, 2)$. Concerning them we prove three theorems and finally formulate an interesting conclusion (Theorem 3.6). Let us consider first $\xi(A)$.

THEOREM 3.3. *If $A \in \mathcal{S}$, then the canonical form of* $\log f(t, A)$ *is given by*

$$\log f(t, A) = i\gamma(A)t + \int_{(-\infty,0)} (e^{itx} - 1)\, dM(x, A) + \int_{(0,\infty)} (e^{itx} - 1)\, dN(x, A), \tag{3.9}$$

where $\gamma(A)$, $M(x, A)$, $N(x, A)$ have the properties formulated in Theorem 2.3, but besides these also

$$\int_{[-i,0)} |x|\, dM(x, A) + \int_{(0,1]} x\, dN(x, A) < \infty. \tag{3.10}$$

PROOF. Let $\mathfrak{z}_n = \left\{ B_1^{(n)}, \dots, B_{k_n}^{(n)} \right\}$ be a sequence of subdivisions of the set A such that $\beta_1)$ is fulfilled and the random variables in the double sequence (3.2) are infinitesimal.

On account of the complete additivity of the realizations the non-decreasing sequence of the random variables

$$\xi_n = \sum_{k=1}^{k_n} \left| \xi\left(B_k^{(n)}\right) \right|$$

has a finite limit. But for every A we have

$$\xi(A) = \sum_{k=1}^{k_n} \xi\left(B_k^{(n)}\right),$$

hence by Theorem 1 of [24], (3.9) and (3.10) hold.

The other statements were proved in Theorem 2.3. □

Now we consider the random variable $\chi_0(I, A)$ and prove that it depends on a Poisson distribution. This is expressed more precisely by

THEOREM 3.4. *For every $A \in \mathcal{S}$ and every interval I with a positive distance from the point* 0 *the random variable $\chi_0(I, A)$ depends on a Poisson distribution with the expectation*

$$M(\chi_0(I,A)) = \begin{cases} \left.\begin{array}{ll} M(b,A) - M(a,A) & \text{if } I = [a,b), \\ M(b+0,A) - M(a,A) & \text{if } I = [a,b], \\ M(b,A) - M(a+0,A) & \text{if } I = (a,b), \\ M(b+0,A) - M(a+0,A) & \text{if } I = (a,b] \end{array}\right\} & (b < 0), \\ \left.\begin{array}{ll} N(b,A) - N(a,A) & \text{if } I = [a,b), \\ N(b+0,A) - N(a,A) & \text{if } I = [a,b], \\ N(b,A) - N(a+0,A) & \text{if } I = (a,b) \\ N(b+0,A) - N(a+0,A) & \text{if } I = (a,b] \end{array}\right\} & (0 < a). \end{cases} \tag{3.11}$$

Relation (3.11) *holds also when $a = -\infty$ or $b = \infty$. (In this case only the third, forth, fifth and seventh rows have a meaning and $M(-\infty, A) = N(+\infty, A) = 0$.)*

PROOF. We prove (3.11) for $I = [a, b)$, $a > 0$, supposing that a and b are points of continuity of $N(x, A)$. The case $b < 0$ can be treated similarly and the other assertions follow from these by limit processes.

Let $\mathfrak{z}_n = \left\{ B_1^{(n)}, \dots, B_{k_n}^{(n)} \right\}$ be a sequence of subdivisions of the set A with the property that $\beta_1)$ is fulfilled, the random variables

$$\xi\left(B_1^{(n)}\right), \dots, \xi\left(B_{k_n}^{(n)}\right) \qquad (n = 1, 2, \dots)$$

are infinitesimal and finally[8]

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \left(F\left(b, B_k^{(n)}\right) - F\left(a, B_{k_n}^{(n)}\right) \right) = N(b, A) - N(a, A).$$

[8] Cf. the proof of Theorem 2.4

This last property can be prescribed by Theorem 2.4.

Let us define the random variables

$$\xi'\left(B_k^{(n)}\right) = \begin{cases} 1 & \text{if} \quad \xi\left(B_k^{(n)}\right) \in I, \\ 0 & \text{if} \quad \xi\left(B_k^{(n)}\right) \notin I. \end{cases} \tag{3.12}$$

Since

$$\xi'\left(B_k^{(n)}\right) \leq K \left|\xi\left(B_k^{(n)}\right)\right| \qquad (k = 1, \ldots, k_n; \; n = 1, 2, \ldots),$$

where K is a constant, the random variables in the double sequence

$$\xi'\left(B_1^{(n)}\right), \ldots, \xi'\left(B_{k_n}^{(n)}\right) \qquad (n = 1, 2, \ldots)$$

are infinitesimal. Obviously

$$\mathbf{M}\left(\xi'\left(B_k^{(n)}\right)\right) = F\left(b, B_k^{(n)}\right) - F\left(a, B_k^{(n)}\right),$$

hence the characteristic function of $\xi'\left(B_k^{(n)}\right)$ equals

$$1 + \left(F\left(b, B_k^{(n)}\right) - F\left(a, B_k^{(n)}\right)\right)(e^{it} - 1).$$

On the other hand

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \left(F\left(b, B_k^{(n)}\right) - F\left(a, B_k^{(n)}\right)\right)^2 = 0, \tag{3.13}$$

hence for every t

$$\begin{aligned} &\lim_{n\to\infty} \prod_{k=1}^{k_n} \left\{1 + \left(F\left(b, B_k^{(n)}\right) - F\left(a, B_k^{(n)}\right)\right)(e^{it} - 1)\right\} \\ &= \exp\left\{\lim_{n\to\infty} \sum_{k=1}^{k_n} \left(F\left(b, B_k^{(n)}\right) - F\left(a, B_k^{(n)}\right)\right)(e^{it} - 1)\right\} \\ &= \exp\{(N(b, A) - N(a, A))(e^{it} - 1)\}. \end{aligned} \tag{3.14}$$

Taking into account that

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \xi'\left(B_k^{(n)}\right) = \chi_0(I, A),$$

our theorem follows.

Now we determine the characteristic function of the random variable $\chi_1(I, A)$.

THEOREM 3.5. *For every set $A \in \mathcal{S}$ and every interval I we have*

$$M(e^{i\chi_1(I,A)t}) = \exp\left\{\int_{X'I} (e^{itx} - 1)\, dM(x, A) + \int_{X'I} (e^{itx} - 1)\, dN(x, A)\right\}, \tag{3.15}$$

where X' denotes the real line without the point 0.

Proof. For simplicity we suppose that I lies on the positive half-axis. The general case does not require any new arguments. We suppose even that $I = [a, b)$, and the function $N(x, A)$ is continuous at the points a, b. Clearly

$$\chi_1(I, A) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(a + \frac{b-a}{n}k\right) \chi_0 \left(\left[a + \frac{b-a}{n}k, a + \frac{b-a}{n}(k+1)\right], A\right), \tag{3.16}$$

hence for every t

$$\begin{aligned} M(e^{i\chi_1(I,A)t}) &= \lim_{n\to\infty} \exp\left\{\sum_{k=0}^{n-1} e^{i\left(a+\frac{b-a}{n}k\right)t} \left(N\left(a + \frac{b-a}{n}(k+1)\right) - N\left(a + \frac{b-a}{n}k\right)\right)\right\} \\ &= \exp\left\{\int_{[a,b)} (e^{itx} - 1)\, dN(x, A)\right\}. \end{aligned}$$

For an arbitrary interval I the proof can be carried out by a limit procedure and the theorem follows.

If we denote the sum of discontinuities in the set $A \in \mathcal{S}$ by $\zeta(A)$, then according to Theorem 3.5

$$M(e^{i\zeta(A)t}) = \exp\left\{\int_{(-\infty,0)} (e^{itx} - 1)\, dM(x, A) + \int_{(0,\infty)} (e^{itx} - 1)\, dN(x, A)\right\}. \tag{3.17}$$

Let $\eta(A)$ denote the difference $\xi(A) - \zeta(A)$ and prove the independence of the random variables $\zeta(A)$ and $\eta(A)$.

Define the functions

$$\left.\begin{aligned} f_\varepsilon(x) &= \begin{cases} x & \text{if } |x| \le \varepsilon, \\ 0 & \text{otherwise,} \end{cases} \\ g_\varepsilon(x) &= \begin{cases} 0 & \text{if } |x| \le \varepsilon, \\ x & \text{if } |x| > \varepsilon \end{cases} \end{aligned}\right\} \qquad (\varepsilon > 0),$$

and choose a sequence of subdivisions $\mathfrak{z}_n = \left\{B_1^{(n)}, \dots, B_{k_n}^{(n)}\right\}$ of the set A so that it has the property $\beta_1)$ and the random variables

$$\xi\left(B_1^{(n)}\right), \dots, \xi\left(B_{k_n}^{(n)}\right) \qquad (n = 1, 2, \dots) \tag{**}$$

are infinitesimal. Denoting by I_ε the set of real numbers $(-\infty, \varepsilon) + (\varepsilon, \infty)$, we get

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} f_\varepsilon\left(\xi\left(B_k^{(n)}\right)\right) = \chi_1(I_\varepsilon, A),$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} g_\varepsilon\left(\xi\left(B_k^{(n)}\right)\right) = \xi(A) - \chi_1(I_\varepsilon, A).$$

Now we apply Theorem 1b of [21] to the double sequence $(**)$ and the functions $f_\varepsilon(x)$, $g_\varepsilon(x)$. We have only to verify Condition c) of this theorem (cf. [21], p. 322). If $\tau < \varepsilon$, then

$$\int_{|x|<\tau} x\, d_x \mathbf{P}\left(g_\varepsilon\left(\xi\left(B_k^{(n)}\right)\right) < x\right) = 0 \qquad (k = 1, \ldots, k_n;\ n = 1, 2, \ldots).$$

On the other hand, by Theorem 3.8 of [19], there is a constant K_τ such that

$$\sum_{k=1}^{k_n} \left| \int_{|x|<\tau} x\, d_x \mathbf{P}\left(f_\varepsilon\left(\xi\left(B_k^{(n)}\right)\right) < x\right)\right| = \sum_{k=1}^{k_n} \left| \int_{|x|<\tau} x\, dF\left(x, B_k^{(n)}\right)\right| \leq K_\tau.$$

Hence and from the relation

$$\lim_{n\to\infty} \sup_{1\leq k\leq k_n} \left| \int_{|x|<\tau} x\, dF\left(x, B_k^{(n)}\right)\right| = 0$$

the fulfilment of Condition c) follows.

Thus $\chi_1(I_\varepsilon, A)$ and $\xi(A) - \chi_1(I_\varepsilon, A)$ are independent for every $\varepsilon > 0$. By taking the limit $\varepsilon \to 0$ we conclude that $\zeta(A)$ and $\eta(A)$ are also independent.

In view of the equality

$$\xi(A) = \zeta(A) + \eta(A)$$

and of Theorem 3.3 (the sum of the integrals on the right-hand side of (3.9) is the characteristic function of $\zeta(A)$) we must have

$$\mathbf{P}(\eta(A) = \gamma(A)) = 1.$$

Hence we get the following

THEOREM 3.6. *The completely additive set function ξ equals the sum of a completely additive set function ζ having (completely additive and) purely discontinuous realizations with probability* 1 *and a completely additive set function η the random variables of which are constant with probability* 1 (cf. (3.17)).

If the σ-ring $\mathcal{S}$ has a countable basis,[9] *then there exists a set $\Omega_0 \subseteq \Omega_1$ with $\mathbf{P}(\Omega_0) =$* 1 *and a number-valued completely additive set function $\gamma(A)$ $(A \in \mathcal{S})$ such that every realization of $\xi(A) - \gamma(A)$ $(A \in \mathcal{S})$ belonging to Ω_0 is purely discontinuous.*

PROOF. We have only to prove the second assertion. If $A_1, A_2, \ldots$ is a basis of $\mathcal{S}$ and $\Omega^{(k)}$ is the set of those ω's for which

$$\eta(\omega, A_k) = \gamma(A_k), \tag{3.18}$$

then we define Ω_0 as

$$\Omega_0 = \Omega_1 \Omega^{(1)} \Omega^{(2)} \Omega^{(3)} \ldots .$$

[9] I.e. there exists a sequence $A_1, A_2, \ldots$ of sets of $\mathcal{S}$ such that the smallest σ-ring containing these sets is $\mathcal{S}$.

Let $\omega \in \Omega_0$. In view of

$$\eta(\omega, A_k) = \gamma(A_k) \qquad (k = 1, 2, \ldots)$$

we must have also

$$\eta(\omega, A) = \gamma(A)$$

for an arbitrary set $A \in \mathcal{S}$. Since $\mathbf{P}(\Omega_0) = 1$, the proof is complete. $\square$

§ 4. General characterization of the set function $\xi(A)$

Let X_ε denote the set $(-\infty, \varepsilon) + (\varepsilon, \infty)$ and consider the product spaces $X \times H$, $X_\varepsilon \times H$. We denote by $\mathcal{R}$ and $\mathcal{R}_\varepsilon$ the rings consisting of finite sums of the type

$$Y \times A \qquad \text{and} \qquad Y_1 \times A,$$

respectively, where $Y \subseteq X$ and $Y_1 \subseteq X_\varepsilon$ are intervals and $A \in \mathcal{S}$. Consider the stochastic set functions χ_1 and χ_0 defined on the σ-rings $\mathcal{S}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R}_\varepsilon)$, respectively, as follows. If for an $\omega \in \Omega_1$ (Ω_1 denotes the same set as in § 3), the points of discontinuity of the realization are $h_1, h_2, \ldots$, and their magnitudes are in the same order $x_1, x_2, \ldots$, then let

$$\chi_1(B) = \sum_{(x_k, h_k) \in B} x_k \qquad (B \in \mathcal{S}(\mathcal{R})), \tag{3.19}$$

$$\chi_0(B) = \sum_{(x_k, h_k) \in B} 1 \qquad (B \in \mathcal{S}(\mathcal{R}_\varepsilon)). \tag{3.20}$$

On the set $\Omega - \Omega_1$ we may define the functions of the sample elements (3.19)–(3.20) arbitrarily. We shall prove

THEOREM 3.7. *For every $B \in \mathcal{S}(\mathcal{R})$ and $B \in \mathcal{S}(\mathcal{R}_\varepsilon)$ the functions $\chi_1(B)$ and $\chi_0(B)$ are random variables, respectively. To disjoint sets $B_1, B_2, \ldots$ of $\mathcal{S}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R}_\varepsilon)$ there belong independent random variables $\chi_1(B_1), \chi_1(B_2), \ldots$ and $\chi_0(B_1), \chi_0(B_2), \ldots$, respectively, and if $B = \sum_{k=1}^{\infty} B_k$, then*

$$\chi_1(B) = \sum_{k=1}^{\infty} \chi_1(B_k) \qquad \textit{and} \qquad \chi_0(B) = \sum_{k=1}^{\infty} \chi_0(B_k), \tag{3.21}$$

respectively.

PROOF. The fulfilment of the relations (3.21) is obvious. Let $\mathfrak{M}$ denote the class of those sets B of $\mathcal{S}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R}_\varepsilon)$ for which $\chi_1(B)$ and $\chi_0(B)$ are random variables, respectively. If $B_1, B_2, \ldots$ is a monotone sequence of sets of $\mathfrak{M}$ and $B = \lim_{n\to\infty} B_n$, then (3.21) implies

$$\lim_{n\to\infty} \chi_1(B_n) = \chi_1(B) \qquad \text{and} \qquad \lim_{n\to\infty} \chi_0(B_n) = \chi_0(B),$$

respectively, for $\omega \in \Omega_1$. Hence $B \in \mathfrak{M}$ and $\mathfrak{M}$ is a monotone class of sets. But obviously, $\mathcal{R} \subseteq \mathfrak{M}$ and $\mathcal{R}_\varepsilon \subseteq \mathfrak{M}$, hence by Theorem B of [11], p. 27 we have $\mathcal{S}(\mathcal{R}) = \mathfrak{M}$ and $\mathcal{S}(\mathcal{R}_\varepsilon) = \mathfrak{M}$, respectively.

It remains to prove the assertion relative to the independence. We show that it holds for disjoint sets belonging to $\mathcal{R}$ and $\mathcal{R}_\varepsilon$, respectively. Let us consider the system of sets $B_1, \dots, B_r$ defined by

$$B_k = I_k \times A_k, \tag{3.22}$$

where $A_k \in \mathcal{S}$ $(k = 1, \dots, r)$ and $I_1, \dots, I_r$ are intervals in the spaces X and X_ε, respectively.

The sets A_k and the intervals I_k $(k = 1, \dots, r)$ can be represented as sums of disjoint sets $C_1, \dots, C_M$ $(C_k \in \mathcal{S};\ k = 1, \dots, M)$ and intervals $J_1, \dots, J_N$, respectively. Consider the NM sets

$$\begin{array}{c} J_1 \times C_1, \dots, J_1 \times C_M, \\ \dots\dots\dots\dots\dots\dots \\ J_N \times C_1, \dots, J_N \times C_M. \end{array} \tag{3.23}$$

The sets $B_1, \dots, B_r$ can be represented as sums of disjoint groups of the sets (3.23), hence it suffices to prove the independence of the random variables

$$\begin{array}{ccc} \chi_1(J_1 \times C_1), \dots, \chi_1(J_1 \times C_M), & & \chi_0(J_1 \times C_1), \dots, \chi_0(J_1 \times C_M), \\ \dots\dots\dots\dots\dots\dots\dots\dots & \text{and} & \dots\dots\dots\dots\dots\dots\dots\dots \\ \chi_1(J_N \times C_1), \dots, \chi_1(J_N \times C_M) & & \chi_0(J_N \times C_1), \dots, \chi_0(J_N \times C_M), \end{array} \tag{3.24}$$

respectively. But according to Theorem 3.1 in every column of (3.24) the random variables are independent. On the other hand, the vectors

$$\begin{aligned} \vec{\chi}_1^{(i)} &= (\chi_1(J_1, C_i), \dots, \chi_1(J_N, C_i)), \\ \vec{\chi}_0^{(i)} &= (\chi_0(J_1, C_i), \dots, \chi_0(J_N, C_i)), \end{aligned} \tag{3.25}$$

are constructed by the aid of the random variables $\{\xi(C),\ C \in C_i\mathcal{S}\}$ and the sets $C_1, \dots, C_M$ are disjoint, hence the vectors (3.25) (if i runs over $1, \dots, M$) must be independent.

Thus $\chi_1(B)$ and $\chi_0(B)$ are completely additive stochastic set functions on the rings $\mathcal{R}$ and $\mathcal{R}_\varepsilon$, respectively. The extension of both set functions is obviously possible. The extended set functions must coincide with $\chi_1(B)$ $(B \in \mathcal{S}(\mathcal{R}))$ and $\chi_0(B)$ $(B \in \mathcal{S}(\mathcal{R}_\varepsilon))$, respectively, hence taking into account that the extension process leads to a completely additive set function, the theorem follows.

Finally, we give an integral representation of the random variables $\xi(A)$ $(A \in \mathcal{S})$. If $A \in \mathcal{S}$ is a fixed set and we introduce the notation

$$\overline{\chi}_1(Y) = \chi_1(Y \times A),$$

where $Y \subseteq X_\varepsilon$ $(\varepsilon > 0)$ is a Borel set, then $\overline{\chi}_1(Y)$ is a completely additive set function defined on the σ-ring of the Borel sets of the space X_ε. Thus $\xi(A) - \gamma(A)$ equals the improper integral

$$\xi(A) - \gamma(A) = \lim_{\varepsilon \to 0} \int_{X_\varepsilon} y \overline{\chi}_1(dY). \tag{3.26}$$

For every fixed Y the random variable $\overline{\chi}_1(Y)$ depends on a Poisson distribution. Thus we can formulate:

Every completely additive set function $\xi(A)$ $(A \in \mathcal{S})$ satisfying the requirements $\alpha_1)$, $\beta_1)$ and $\gamma_1)$ of § 1, can be represented as the limit (3.26) *of stochastic integrals* (cf. [20]) *taken relative to completely additive set functions of Poisson type.*

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