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EXTENSION OF MULTIPLICATIVE SET FUNCTIONS WITH VALUES IN A BANACH ALGEBRA

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Introduction

Let X be a given set, and let $\mathcal{R}$, $\mathcal{S}$ be some classes of certain subsets of X. We say that $\mathcal{R}$ is a ring if $A+B \in \mathcal{R}$, $A-B \in \mathcal{R}$ whenever $A \in \mathcal{R}$, $B \in \mathcal{R}$. We say that $\mathcal{S}$ is a σ-ring if it is a ring and if for every sequence $A_1, A_2, \ldots$, where $A_i \in \mathcal{S}$ $(i = 1, 2, \ldots)$, the sum $\sum_{i=1}^{\infty} A_i$ is also contained in $\mathcal{S}$.

Let $\mathcal{B}$ be a commutative Banach algebra with a unity, i.e. a Banach space, where for all pairs $f \in \mathcal{B}$, $g \in \mathcal{B}$ a product $fg = gf$ is defined such that if $h \in \mathcal{B}$, then $(fg)h = f(gh)$, $(f+g)h = fh + gh$, $\|fg\| \le \|f\|\,\|g\|$ and there is an $e \in \mathcal{B}$ such that $ef = fe = f$, $\|e\| = 1$.

I shall consider set functions $f(A)$ defined on the elements of a ring $\mathcal{R}$ such that the values $f(A)$ lie in $\mathcal{B}$; $f(A) \in \mathcal{B}$ for $A \in \mathcal{R}$.

A real-valued set function $\alpha(A)$ defined on $\mathcal{R}$ is called of bounded variation if there is a number K such that for every finite sequence of pairwise disjoint sets $A_1, A_2, \ldots, A_r$, $A_i \in \mathcal{R}$ $(i = 1, 2, \ldots, r)$, we have

$$\sum_{i=1}^{r} |\alpha(A_i)| \le K.$$

Let $g_1, g_2, \ldots$ be a sequence of elements of $\mathcal{B}$. I say that the infinite product $\prod_{i=1}^{\infty} g_i$ converges if the number of the factors which are equal to 0 is finite and if $g_i \neq 0$ for $i \ge n_0$, then there is a $g_0 \in \mathcal{B}$ such that

$$\lim_{n\to\infty} \left\| g_0 - \prod_{i=n_0}^{n} g_i \right\| = 0.$$

I define the value of the infinite product as g_0 if $n_0 = 1$ and as

$$\prod_{i=1}^{\infty} g_i = \left(\prod_{i=1}^{n_0-1} g_i\right) g_0.$$

if $n_0 > 1$.

A set function $f(A)$ is called multiplicative (completely multiplicative) in $\mathcal{R}$ if for every $A_1 \in \mathcal{R}$, $A_2 \in \mathcal{R}$, $A_1A_2 = 0$ ($A_1, A_2, \ldots, A_iA_k = 0$ for $i \neq k$, $A = \sum_{i=1}^{\infty} A_i \in \mathcal{R}$), the relation

$$f(A_1 + A_2) = f(A_1)f(A_2) \qquad \left(f(A) = \prod_{i=1}^{\infty} f(A_i)\right) \tag{1}$$

holds. I suppose in this case that $f(0) = e$.

A real-valued set function $\mu(A)$ is called subadditive (completely subadditive) if for every pair A_1, A_2 of disjoint sets of $\mathcal{R}$ (for every sequence $A_1, A_2, \ldots$ of pairwise disjoint sets of $\mathcal{R}$, for which $A = \sum_{k=1}^{\infty} A_k \in \mathcal{R}$), we have

$$\mu(A) \leq \mu(A_1) + \mu(A_2) \qquad \left(\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)\right).$$

The purpose of this paper is the extension of a completely multiplicative set function defined on the ring $\mathcal{R}$ and satisfying certain conditions, to a completely multiplicative set function defined on $\mathcal{S}(\mathcal{R})$, which is the smallest σ-ring containing $\mathcal{R}$. First I prove four lemmas.

§ 1. Lemmas

LEMMA 1 *Let μ be a real-valued, non-negative, completely subadditive set function defined on a ring $\mathcal{R}$. If μ is of bounded variation, then the set function $Var_\mu(A)$ ($A \in \mathcal{R}$), i.e. the least upper bound of the sums $\sum_{i=1}^{r} \mu(A_i)$, where $A_i \subseteq A$, $A_i \in \mathcal{R}$ ($i = 1, 2, \ldots, r$), $A_iA_k = 0$ for $i \neq k$, is a bounded measure*[1] *on the ring $\mathcal{R}$.*

PROOF. Let $B_1, B_2, \ldots$ be a sequence of pairwise disjoint sets of $\mathcal{R}$, for which $B = \sum_{i=1}^{\infty} B_i \in \mathcal{R}$. Let us choose subsets $A_1, A_2, \ldots, A_r$ of B which are elements of $\mathcal{R}$ and

$$\mathrm{Var}_\mu(B) \leq \sum_{i=1}^{r} \mu(A_i) + \varepsilon,$$

where $\varepsilon > 0$ is a given number. Using the subadditivity of the set function μ, we obtain that

$$\mu(A_i) \leq \sum_{k=1}^{\infty} \mu(A_iB_k)$$

[1] A real-valued, non-negative set function m, defined on a ring $\mathcal{R}$, is called a measure if for every sequence $B_1, B_2, \ldots$ of disjoint sets of $\mathcal{R}$, for which $B = \sum_{i=1}^{\infty} B_i \in \mathcal{R}$, we have $m(B) = \sum_{i=1}^{\infty} m(B_i)$ and $m(0) = 0$.

and thus

$$\mathrm{Var}_\mu(B) \le \sum_{i=1}^{r} \sum_{k=1}^{\infty} \mu(A_i B_k) + \varepsilon = \sum_{k=1}^{\infty} \sum_{i=1}^{r} \mu(A_i B_k) + \varepsilon \le \sum_{k=1}^{\infty} \mathrm{Var}_\mu(B_k) + \varepsilon.$$

This inequality is true for all $\varepsilon > 0$, hence we obtain that

$$\mathrm{Var}_\mu(B) \le \sum_{k=1}^{\infty} \mathrm{Var}_\mu(B_k).$$

It is trivial that

$$\mathrm{Var}_\mu(B) \ge \sum_{k=1}^{\infty} \mathrm{Var}_\mu(B_k)$$

and thus Lemma 1 is proved. □

LEMMA 2 *Let $f_1, f_2, \ldots, f_r$ and $g_1, g_2, \ldots, g_r$ be such finite sequences of a Banach algebra, for which*

$$\left\| \prod_{i=1}^{l} f_i \right\| \le K, \qquad \left\| \prod_{i=l}^{r} g_i \right\| \le K \qquad (l = 1, 2, \ldots, r).$$

Then

$$\left\| \prod_{i=1}^{r} f_i - \prod_{i=1}^{r} g_i \right\| \le K^2 \sum_{i=1}^{r} \|f_i - g_i\|.$$

PROOF. Starting from the identity

$$\prod_{i=1}^{r} f_i - \prod_{i=1}^{r} g_i = \sum_{i=1}^{r} f_1 \ldots f_{i-1}(f_i - g_i) g_{i+1} \ldots g_r$$

and taking the norm in both sides, we obtain the required inequality. □

LEMMA 3 *Let $f_1, f_2, \ldots$ be a sequence of elements of a commutative Banach algebra with a unity. If*

$$\sum_{i=1}^{\infty} \|e - f_i\| < \infty,$$

then the infinite product

$$\prod_{i=1}^{\infty} f_i$$

converges to the same element by every ordering of the factors.

PROOF. By the inequality

$$\sum_{i=1}^{\infty} |1 - \|f_i\|\,| \le \sum_{i=1}^{\infty} \|e - f_i\| < \infty$$

there is an n_0 such that $\|f_i\| > 0$ for $i \geq n_0$, and the infinite product of positive numbers

$$\prod_{i=n_0}^{\infty} \|f_i\|$$

is absolutely convergent. Let

$$K = \prod_{i=n_0}^{\infty} (1 + |1 - \|f_i\| \,|).$$

By Lemma 2, for every pair m, n $(m \geq n_0,\ n \geq n_0)$, we have

$$\left\| \prod_{i=n_0}^{m} f_i - \prod_{i=n_0}^{n} f_i \right\| \leq K^2 \sum_{i=\min(m,n)+1}^{\max(m,n)} \|e - f_i\|.$$

Taking into account our assumption, it follows that there is an f_0 such that $\lim\limits_{n\to\infty} \|f_0 - \prod_{i=n_0}^{n} f_i\| = 0$. Let $i_1, i_2, \ldots$ be a rearrangement of the sequence $n_0, n_0+1, \ldots$ and let N_n be a number such that the set $n_0, n_0+1, \ldots, N_n$ contains the numbers $i_1, i_2, \ldots, i_n$. If A_n is the following set of integers: $A_n = \{n_0, n_0+1, \ldots, N_n\} - \{i_1, i_2, \ldots, i_n\}$, then we have

$$\left\| \prod_{k=1}^{n} f_{i_k} - \prod_{k=n_0}^{N_n} f_k \right\| \leq K^2 \sum_{k \in A_n} \|e - f_k\|.$$

This inequality implies the convergence of the product $\prod_{k=1}^{\infty} f_{i_k}$ to the element f_0; this proves our assertion. □

Lemma 4 *Let $\mathcal{R}$ be a ring and $\mu(A)$ a real- and non-negative valued subadditive set function defined on the elements of $\mathcal{R}$ such that the following conditions hold:*

a) *$\mu(A) \leq K$, $A \in \mathcal{R}$, where K is a constant;*

b) *if $A_1, A_2, \ldots$ is a sequence of pairwise disjoint sets of the ring $\mathcal{R}$, then*

$$\sum_{k=1}^{\infty} \mu(A_k) < \infty.$$

From these conditions it follows that the set function $\mu(A)$ is of bounded variation.

Proof. Let us suppose that $\mu(A)$ is not of bounded variation and choose such disjoint sets of $\mathcal{R}, B_1^{(1)}, B_2^{(1)}, \ldots, B_{k_1}^{(1)}$ $(k_1 > 1)$, for which

$$\sum_{l=1}^{k_1} \mu(B_l^{(1)}) \geq 2K.$$

Since the set function $\mu(A)$ is subadditive, it follows that at least in one of the sets $\sum_{l=1}^{k_1} B_l^{(1)}$ and $\overline{\sum_{l=1}^{k_1} B_l^{(1)}}$ it is not of bounded variation. Let the set $\overline{\sum_{l=1}^{k_1} B_l^{(1)}}$ have this property. Then we can find sets of $\mathcal{R}$, $B_{k+1}^{(1)}, B_{k_1+2}^{(1)}, \ldots, B_{k_2}^{(1)}$, which are subsets of $\overline{\sum_{l=1}^{k_1} B_l^{(1)}}$ and

$$\sum_{l=k_1+1}^{k_2} \mu(B_l^{(1)}) \geq K.$$

In the same way as before it can be seen that the set function μ is not of bounded variation at least in one of the sets $\sum_{l=1}^{k_2} B_l^{(1)}$ and $\overline{\sum_{l=1}^{k_2} B_l^{(1)}}$ and so on. After a finite number of steps the chain will terminate; indeed, if $\mu(A)$ is not of bounded variation in any of the sets $\overline{\sum_{l=1}^{k_r} B_l^{(1)}}$, then from the inequality

$$\sum_{l=1}^{k_r} \mu(B_l^{(1)}) \geq (r+1)K$$

it would follow that the sequence of disjoint sets, $B_1^{(1)}, B_2^{(1)}, \ldots$ would have the property

$$\sum_{l=1}^{\infty} \mu(B_l^{(1)}) = \infty,$$

but this is impossible because of Condition b). Consequently, there is a number n_1 among the numbers $k_1, k_2, \ldots$ such that $\mu(A)$ is not of bounded variation in the set $\sum_{l=1}^{n_1} B_l^{(1)}$. From the subadditivity of $\mu(A)$ it follows that the variation will be infinite at least in one of the sets $B_1^{(1)}, B_2^{(1)}, \ldots, B_{n_1}^{(1)}$. This may be the set $B_{n_1}^{(1)}$. Then, since

$$\sum_{l=1}^{n_1} \mu(B_l^{(1)}) \geq \sum_{l=1}^{k_1} \mu(B_l^{(1)}) \geq 2K$$

holds, we have in view of Condition a)

$$\sum_{l=1}^{n_1-1} \mu(B_l^{(1)}) \geq K.$$

A repetition of the preceding consideration shows that in the set $B_{n_1}^{(1)}$ there are disjoint sets of $\mathcal{R}$, $B_1^{(2)}, B_2^{(2)}, \ldots, B_{n_2}^{(2)}$, such that the variation is infinite in the set $B_{n_2}^{(2)}$ and

$$\sum_{l=1}^{n_2-1} \mu(B_l^{(2)}) \geq K.$$

Carrying on this procedure we may choose a sequence

$$B_1^{(1)}, B_2^{(1)}, \ldots, B_{n_1-1}^{(1)}, B_1^{(2)}, B_2^{(2)}, \ldots, B_{n_2-1}^{(2)}, \ldots$$

of disjoint sets of $\mathcal{R}$ such that

$$\sum_{k=1}^{\infty} \sum_{l=1}^{n_k-1} \mu(B_l^{(k)}) = \infty,$$

but, by Condition b), this is a contradiction. This completes the proof of Lemma 4. $\square$

§ 2. Extension of completely multiplicative set functions

Theorem 1 *Let $f(A)$ be a completely multiplicative set function, defined on the ring $\mathcal{R}$, for which $\|f(A)\| \leq 1$ $(A \in \mathcal{R})$. If for every sequence $A_1, A_2, \dots$ of disjoint sets of $\mathcal{R}$ the relation*

$$\sum_{k=1}^{\infty} \|e - f(A_k)\| < \infty \tag{2}$$

holds, then there is one and only one completely multiplicative set function $f^*(A)$, defined on the σ-ring $\mathcal{S}(\mathcal{R})$, for which $f^*(A) = f(A)$ if $(A \in \mathcal{R})$.

If $A_1, A_2, \dots$ is a convergent sequence of sets of $\mathcal{S}(\mathcal{R})$, $\lim\limits_{k\to\infty} A_k = A$, then

$$\lim_{k\to\infty} f^*(A_k) = f^*(A).$$

Proof. Let $A_1, A_2, \dots$ be a sequence of disjoint sets of $\mathcal{R}$, for which $A = \sum_{i=1}^{\infty} A_i \in \mathcal{R}$. From the inequality

$$\begin{aligned}
\|e - f(A_1 + \cdots + A_{n+1})\| &= \|e - f(A_1 + \cdots + A_n) f(A_{n+1})\| \\
&= \|e - f(A_{n+1}) + f(A_{n+1}) - f(A_1 + \cdots + A_n) f(A_n + 1)\| \\
&\leq \|e - f(A_{n+1})\| + \|f(A_{n+1})\| \cdot \|e - f(A_1 + \cdots + A_n)\| \\
&\leq \|e - f(A_{n+1})\| + \|e - f(A_1 + \cdots + A_n)\|
\end{aligned}$$

it follows that

$$\left\| e - f\left(\sum_{k=1}^{n+1} A_k \right) \right\| \leq \sum_{k=1}^{n+1} \|e - f(A_k)\|. \tag{3}$$

Thus the conditions in Lemma 4 for the set function $\|e - f(A)\|$ are fulfilled, hence $\|e - f(A)\|$ is of bounded variation. Taking the limit $n \to \infty$ in the relation (3), we obtain

$$\|e - f(A)\| \leq \sum_{i=1}^{\infty} \|e - f(A_i)\|. \tag{4}$$

Hence $\mu(A) = \|e - f(A)\|$ is a completely subadditive set function. By Lemma 1 $\mathrm{Var}_\mu(A)$ is a bounded measure on $\mathcal{R}$. Let $m(A)$ $(A \in \mathcal{S}(\mathcal{R}))$ denote the extended measure of $\mathrm{Var}_\mu(A)$ to the σ-ring $\mathcal{S}(\mathcal{R})$.

Let us form a sequence of rings in the following manner: $\mathcal{R}_0 = \mathcal{R}$, $\mathcal{R}_1$ is the ring of the sets which are limits of some convergent sequences of $\mathcal{R}_0$ and if $\mathcal{R}_\nu$ is defined for all $\nu < \nu_0 < \omega_1$, then $\mathcal{R}_{\nu_0}$ is the ring of the sets which are limits of some convergent sequences of the ring $\sum_{\nu<\nu_0} \mathcal{R}_\nu$. Clearly, $\sum_r \mathcal{R}_\nu = \mathcal{S}(\mathcal{R})$.

In the sequel we shall use the following remark: If E is a sequence of sets of $\mathcal{R}$ such that $\lim\limits_{n\to\infty} E_n = 0$, then from the inequality

$$\|e - f(E_n)\| \leq m(E_n)$$

we obtain that

$$\lim_{n\to\infty} f(E_n) = e.$$

Let A_n be a convergent sequence of sets of $\mathcal{R}_0$. In this case

$$\lim_{m,n\to\infty}(A_n - A_m) = \lim_{m,n\to\infty}(A_m - A_n) = 0,$$

hence

$$\begin{aligned}
\|f(A_n) - f(A_m)\| &\le \|f(A_n) - f(A_nA_m)\| + \|f(A_m) - f(A_nA_m)\| \\
&= \|f(A_nA_m)f(A_n - A_m) - f(A_nA_m)\| \\
&\quad + \|f(A_nA_m)f(A_m - A_n) - f(A_nA_m)\| \\
&\le \|f(A_nA_m)\| \cdot \|e - f(A_n - A_m)\| \\
&\quad + \|f(A_nA_m)\| \cdot \|e - f(A_m - A_n)\| \\
&\le \|e - f(A_n - A_m)\| + \|e - f(A_m - A_n)\| \to 0 \quad \text{if} \quad m,n\to\infty.
\end{aligned}$$

Thus the sequence $f(A_n)$ is convergent. Let us define the set function f_1 as follows: if $A \in \mathcal{R}_1$, $A = \lim\limits_{n\to\infty} A_n$ where $A_n \in \mathcal{R}_0$ $(n = 1, 2, \dots)$, then

$$f_1(A) = \lim_{n\to\infty} f(A_n).$$

We prove that $f_1(A)$ is uniquely determined. Let $A_n \in \mathcal{R}_0$, $A_n' \in \mathcal{R}_0$ $(n = 1, 2, \dots)$ be two sequences such that

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} A_n' = A.$$

Clearly,

$$\begin{aligned}
A_n &= A_nA_n' + (A_n - A_n'), & A_n' &= A_nA_n' + (A_n' - A_n), \\
f(A_n) &= f(A_nA_n')f(A_n - A_n'), & f(A_n') &= f(A_nA_n')f(A_n' - A_n).
\end{aligned}$$

Since $A_nA_n' \to A$ and

$$\lim_{n\to\infty} f(A_n - A_n') = \lim_{n\to\infty} f(A_n' - A_n) = e,$$

it follows that the sequence $f(A_nA_n')$ converges and

$$\lim_{n\to\infty} f(A_n) = \lim_{n\to\infty} f(A_n') = \lim_{n\to\infty} f(A_nA_n').$$

The uniqueness of the set function f_1 implies that $f_1(A) = f(A)$ for $A \in \mathcal{R}_0$.

$f_1(A)$ is a multiplicative set function on $\mathcal{R}_1$. In fact, if A and B are disjoint sets of $\mathcal{R}_1$, $A = \lim\limits_{n\to\infty} A_n$, $B = \lim\limits_{n\to\infty} B_n$, $A_n \in \mathcal{R}_0$, $B_n \in \mathcal{R}_0$ $(n = 1, 2, \dots)$, then since $f(A)$ is a multiplicative set function on $\mathcal{R}_0$, we have

$$f_1(A + B) = \lim_{n\to\infty} f(A_n + (B_n - A_n)) = \lim_{n\to\infty} f(A_n)f(B_n - A_n) = f_1(A)f_1(B).$$

We prove that $f_1(A)$ is a completely multiplicative set function on $\mathcal{R}_1$. First we prove that if $\mu_1(A) = \|e - f_1(A)\|$, $A \in \mathcal{R}_1$ and if $B \in \mathcal{R}_1$, then $\mathrm{Var}_{\mu_1}(B) \leq m(B)$. In fact, if $B_1, B_2, \ldots, B_r$ are disjoint sets of $\mathcal{R}_1$ for which $B_i \subseteq B$ $(i = 1, 2, \ldots, r)$ and $B_1^{(n)}, B_2^{(n)}, \ldots, B_r^{(n)}$ are sequences of sets of R_0 for which $B_i = \lim\limits_{n\to\infty} B_i^{(n)}$ $(i = 1, 2, \ldots, r)$, $B_i^{(n)} B_k^{(n)} = 0$ for $i \neq k$, $n = 1, 2, \ldots$, then

$$\sum_{i=1}^{r} \|e - f(B_i^{(n)})\| \leq m\left(\sum_{i=1}^{r} B_i^{(n)}\right) \qquad (n = 1, 2, \ldots).$$

If $n \to \infty$, we obtain

$$\sum_{i=1}^{r} \|e - f_1(B_i)\| \leq m\left(\sum_{i=1}^{r} B_i\right) \leq m(B),$$

which proves the assertion.

If $B_1, B_2, \ldots$ is a sequence of disjoint sets of $\mathcal{R}_1$, $B = \sum_{k=1}^{\infty} B_k \in \mathcal{R}_1$, then from the inequality

$$\sum_{k=1}^{\infty} \|e - f_1(B_k)\| \leq m(B)$$

it follows the convergence of the infinite product $\prod_{k=1}^{\infty} f_1(B_k)$. As $f_1(B)$ is multiplicative on $\mathcal{R}_1$, if $C_n = \sum_{k=n}^{\infty} B_k$, we obtain

$$\begin{aligned}\left\| f_1(B) - \prod_{k=1}^{n} f_1(B_k) \right\| &= \left\| \left(\prod_{k=1}^{n} f_1(B_k)\right) f_1(C_{n+1}) - \prod_{k=1}^{n} f_1(B_k) \right\| \\ &\leq \|e - f_1(C_{n+1})\| \leq m(C_{n+1}) \to 0 \text{ if } n \to \infty,\end{aligned}$$

hence $f_1(B)$ is a completely multiplicative set function on the ring $\mathcal{R}_1$.

Such as for $\mathcal{R}_0$, we can prove also for $\mathcal{R}_1$ that the set function $\mu_1(A) = \|e - f_1(A)\|$ $(A \in \mathcal{R}_1)$ is completely subadditive and of bounded variation. By Lemma 1, $\mathrm{Var}_{\mu_1}(A)$ is a bounded measure on $\mathcal{R}_1$. We have seen that if $A \in \mathcal{R}_0 \subseteq \mathcal{R}_1$, then

$$\mathrm{Var}_{\mu_1}(A) \leq m(A).$$

On the other hand, since $\mathcal{R}_0 \subseteq \mathcal{R}_1$, we have

$$\mathrm{Var}_{\mu}(A) \leq \mathrm{Var}_{\mu_1}(A) \quad \text{for} \quad A \in \mathcal{R}_0.$$

Thus

$$\mathrm{Var}_{\mu_1}(A) = m(A) \quad \text{if} \quad A \in \mathcal{R}_0,$$

and since the extension of a bounded measure is uniquely determined, we obtain

$$\mathrm{Var}_{\mu_1}(A) = m(A) \quad \text{if} \quad A \in \mathcal{R}_1.$$

Let us suppose that for every ordinal number ν for which $\nu < \nu_0 < \omega_1$, a completely multiplicative set function $f_\nu(A)$ $(A \in \mathcal{R}_\nu)$ is defined satisfying

$$f_\nu(A) = f_{\nu'}(A) \quad \text{if} \quad A \in \mathcal{R}_{\nu'}, \qquad \nu' < \nu$$

and

$$\mathrm{Var}_{\mu_\nu}(A) = m(A) \quad \text{where} \quad \mu_\nu = \|e - f_\nu(A)\|, \qquad A \in \mathcal{R}_\nu.$$

Let us define a set function g_{ν_0} on the ring $\sum_{\nu<\nu_0} \mathcal{R}_\nu$ as follows:

$$g_{\nu_0}(A) = f_\nu(A) \quad \text{if} \quad A \in \mathcal{R}_\nu, \quad \nu < \nu_0.$$

Obviously, g_{ν_0} is a multiplicative set function. We shall prove that it is also completely multiplicative. In fact, if $A_1, A_2, \ldots$ is a sequence of disjoint sets of the ring $\sum_{\nu<\nu_0} \mathcal{R}_\nu$, $A = \sum_{k=1}^\infty A_k \in \sum_{\nu<\nu_0} \mathcal{R}_\nu$, then

$$\sum_{k=1}^\infty \|e - g_{\nu_0}(A_k)\| \leq m(A),$$

hence the infinite product $\prod_{k=1}^\infty g_{\nu_0}(A_k)$ converges. Since

$$g_{\nu_0}(A) = \left(\prod_{k=1}^n g_{\nu_0}(A_k)\right) g_{\nu_0}\left(\sum_{k=n+1}^\infty A_k\right),$$

it follows that

$$\begin{aligned} \left\| g_{\nu_0}(A) - \prod_{k=1}^n g_{\nu_0}(A_k) \right\| &\leq \left\| e - g_{\nu_0}\left(\sum_{k=n+1}^\infty A_k\right)\right\| \\ &\leq m\left(\sum_{k=n+1}^\infty A_k\right) \to 0 \quad \text{if} \quad n \to \infty \end{aligned}$$

whence

$$g_{\nu_0}(A) = \prod_{k=1}^\infty g_{\nu_0}(A_k).$$

Such as we have constructed the set function f_1 we can construct the set function f_{ν_0} defined on $\mathcal{R}_{\nu_0}$ with the aid of the set function g_{ν_0} defined on $\sum_{\nu<\nu_0} \mathcal{R}_\nu$ and it is easy to see that f_{ν_0} has the properties what we have supposed in the transfinite induction.

Let us define the set function $f^*(A)$ $(A \in \mathcal{S}(\mathcal{R}))$ in the following manner:

$$f^*(A) = f_\nu(A) \quad \text{if} \quad A \in \mathcal{R}_\nu.$$

$f^*(A)$ is a completely multiplicative set function. In fact, if $A_1, A_2, \ldots$ is a sequence of disjoint sets of the σ-ring $\mathcal{S}(\mathcal{R}) = \sum_\nu \mathcal{R}_\nu$ and $A_k \in \mathcal{R}_{\nu_k}$ $(k = 1, 2, \ldots)$, then there is a $\nu' < \omega_1$ such that $\nu_k < \nu'$ $(k = 1, 2, \ldots)$. It follows that $A_k \in \mathcal{R}_{\nu'}$ $(k = 1, 2, \ldots)$, $\sum_{k=1}^{\infty} A_k \in \mathcal{R}_{\nu'}$, hence

$$f^* \left(\sum_{k=1}^{\infty} A_k \right) = f_{\nu'} \left(\sum_{k=1}^{\infty} A_k \right) = \prod_{k=1}^{\infty} f_{\nu'}(A_k) = \prod_{k=1}^{\infty} f^*(A_k).$$

By the same way we can prove that if A_n is a convergent sequence of sets of $\mathcal{S}(\mathcal{R})$, $\lim\limits_{n\to\infty} A_n = A$, then

$$\lim_{n\to\infty} f^*(A_n) = f^*(A).$$

Proof of the uniqueness of the extension. Let f^{**} be a completely multiplicative set function defined on the σ-ring $\mathcal{S}(\mathcal{R})$. First we remark that if $A_1, A_2, \ldots$ is a non-decreasing sequence of sets of $\mathcal{S}(\mathcal{R})$, $\lim\limits_{n\to\infty} A_n = A$, then

$$f^{**}(A) = \left(\prod_{k=1}^{\infty} f^{**}(A_{k+1} - A_k) \right) f^{**}(A_1) = f^{**}(A_1) \lim_{n\to\infty} \prod_{k=1}^{n-1} f^{**}(A_{k+1} - A_k),$$

hence

$$f^{**}(A) = \lim_{n\to\infty} f^{**}(A_n). \tag{5}$$

Let us suppose that the set functions f^*, f^{**} coincide on the ring $\sum_{\nu<\nu_0} \mathcal{R}_\nu$,

$$f^*(A) = f^{**}(A) \quad \text{if} \quad A \in \sum_{\nu<\nu_0} \mathcal{R}_\nu.$$

We shall prove that f^* and f^{**} coincide on the ring $\mathcal{R}_\nu$, too.

Let $B_1, B_2, \ldots$ be a non-increasing sequence of sets of $\sum_{\nu<\nu_0} \mathcal{R}_\nu$, $\lim\limits_{n\to\infty} B_n = B$. The sets functions f^*, f^{**} are completely multiplicative, hence

$$f^*(B_n) = f^*(B) \prod_{k=n}^{\infty} f^*(B_k - B_{k+1}),$$
$$f^{**}(B_n) = f^{**}(B) \prod_{k=n}^{\infty} f^{**}(B_k - B_{k+1}).$$

Taking into account our assumption, we obtain

$$f^*(B_n - B) = \prod_{k=n}^{\infty} f^*(B_k - B_{k+1}) = \prod_{k=n}^{\infty} f^{**}(B_k - B_{k+1}),$$
$$f^*(B_n) = f^{**}(B_n),$$

hence

$$f^*(B) f^*(B_n - B) = f^{**}(B) f^*(B_n - B).$$

Since $\lim_{n\to\infty} f^*(B_n - B) = e$, it follows that $f^*(B) = f^{**}(B)$.

Let $C_1, C_2, \ldots$ be a convergent sequence of sets of $\sum_{\nu<\nu_0} \mathcal{R}_\nu$, $\lim_{n\to\infty} C_n = C$. Let

$$A_n = C_n C_{n+1} \ldots, A_{r,n} = C_n C_{n+1} \ldots C_{n+r} \qquad (r, n = 1, 2, \ldots).$$

Obviously, $A_{r,n} \in \sum_{\nu<\nu_0} \mathcal{R}_\nu$, $A_{r,n} \subseteq C_n$, $A_{r,n} \supseteq A_{r+1,n}$. Hence we conclude

$$f^{**}(C_n) = f^{**}(C_n - A_{r,n}) f^{**}(A_{r,n}) = f^*(C_n - A_{r,n}) f^{**}(A_{r,n}). \tag{6}$$

Since $\lim_{r\to\infty} A_{r,n} = A_n$ and for every fixed n the sequence $A_{1,n}, A_{2,n}, A_{3,n} \ldots$ is non-increasing, we have

$$\lim_{r\to\infty} f^{**}(A_{r,n}) = \lim_{r\to\infty} f^*(A_{r,n}) = f^*(A_n) = f^{**}(A_n). \tag{7}$$

On the other hand, we know that

$$\lim_{r\to\infty} f^*(C_n - A_{r,n}) = f^*(C_n - A_n), \tag{8}$$

hence by (6), (7) and (8) we can write

$$f^{**}(C_n) = f^*(C_n - A_n) f^{**}(A_n). \tag{9}$$

The sequence $A_1, A_2, \ldots$ is non-decreasing, $\lim_{n\to\infty} A_n = C$, hence by (5) we obtain

$$\lim_{n\to\infty} f^{**}(A_n) = f^{**}(C). \tag{10}$$

We know furthermore that

$$\lim_{n\to\infty} f^{**}(C_n) = \lim_{n\to\infty} f^*(C_n) = f^*(C), \quad \lim_{n\to\infty} f^*(C_n - A_n) = e. \tag{11}$$

Taking the limit $n \to \infty$ in the relation (9), (10) and (11) imply

$$f^*(C) = f^{**}(C).$$

By the principle of transfinite induction it follows that the set functions f^*, f^{**} coincide on the σ-ring $\mathcal{S}(\mathcal{R}) = \sum_\nu \mathcal{R}_\nu$. Thus Theorem 1 is proved. $\square$

Theorem 2 *Let $\mathcal{R}$ be a ring and $f(A)$ a completely multiplicative set function, defined on the ring $\mathcal{R}$, for which $f(A) \in \mathcal{B}$, $A \in \mathcal{R}$. Let us further suppose that there is a bounded, completely additive, real-valued set function $\varphi(A)$, defined on the ring $\mathcal{R}$, for which*

$$\|f(A)\| \leq 2^{\varphi(A)} \qquad (A \in \mathcal{R}), \tag{12}$$

and that for every sequence A_k of disjoint sets of $\mathcal{R}$ we have

$$\sum_{k=1}^{\infty} \|e - f(A_k)\| < \infty.$$

In this case there is a uniquely determined completely multiplicative set function $f^*(A)$, defined on the σ-ring $\mathcal{S}(\mathcal{R})$, for which $f^*(A) = f(A)$ if $A \in \mathcal{R}$; for every convergent sequence $A_1, A_2, \dots$ of the σ-ring $\mathcal{S}(\mathcal{R})$ we have

$$\lim_{k\to\infty} f^*(A_k) = f^*(A)$$

where $A = \lim_{k\to\infty} A_k$.

Proof. Let us consider the set function $g(A) = 2^{-\varphi(A)} f(A)$. I prove that $g(A)$ satisfies the conditions of Theorem 1. By (12), $\|g(A)\| \le 1$. Furthermore

$$\begin{aligned} \|e - g(A)\| &= \|e - e2^{-\varphi(A)} + e2^{-\varphi(A)} - 2^{-\varphi(A)} f(A)\| \\ &\le |1 - 2^{-\varphi(A)}| + \|e - f(A)\| 2^{-\varphi(A)} \\ &\le K(|\varphi(A)| + \|e - f(A)\|), \end{aligned} \tag{13}$$

where K is a positive constant, hence if $A_1, A_2, \dots$ is a sequence of disjoint sets of $\mathcal{R}$, then by (13) we have

$$\sum_{k=1}^{\infty} \|e - g(A_k)\| \le K \left(\sum_{k=1}^{\infty} |\varphi(A_k)| + \sum_{k=1}^{\infty} \|e - f(A_k)\| \right) < \infty.$$

Let $g^*(A)$ and $\varphi^*(A)$ denote the extended set functions corresponding to the set functions $g(A)$ and $\varphi(A)$, respectively. Clearly, the set function

$$f^*(A) = 2^{\varphi^*(A)} g^*(A) \qquad (A \in \mathcal{S}(\mathcal{R}))$$

is completely multiplicative and $f^*(A) = f(A)$ for $A \in \mathcal{R}$. If $A_1, A_2, \dots$ is a sequence of sets of $\mathcal{S}(\mathcal{R})$, $\lim_{k\to\infty} A_k = A$, then by Theorem 1 we have

$$\lim_{k\to\infty} g^*(A_k) = g^*(A)$$

whence

$$\lim_{k\to\infty} f^*(A_k) = f^*(A).$$

Thus Theorem 2 is proved. □

Remark If for every $A \in \mathcal{R}$ the inequality $0 < \delta_1 \le \|f(A)\| \le \delta_2$ holds, where δ_1, δ_2 are constants, and if

$$\|f(A)f(B)\| = \|f(A)\| \, \|f(B)\|, \tag{14}$$

where $A \in \mathcal{R}$, $B \in \mathcal{R}$, $AB = 0$, then for the set function $f(A)$ the condition (12) of Theorem 2 is fulfilled. In fact, for $\varphi(A) = \log_2 \|f(A)\|$ the relation (12) holds. For instance, (14) holds in the Banach algebra of the complex numbers.

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